Homework 8

due March 23, 2012

Semicontinuous functions We say that a function $f\colon X \to \mathbf{R}$ on a topological space X is *lower-semicontinuous* if, for any $x \in X$ and for for every $\epsilon > 0$, there exists a neighborhood N of x such that

$$f(x) - \epsilon < f(x') \qquad (x' \in N).$$

We say that a function f is *upper-semicontinuous* if, for any $x \in X$ and for for every $\epsilon > 0$, there exists a neighborhood N of x such that

$$f(x') < f(x) + \epsilon \qquad (x' \in N).$$

1. Show that f is lower-semicontinuous if and only if $f^{-1}((a, \infty))$ is open for any $a \in \mathbf{R}$. Formulate the corresponding result for upper-semicontinuous functions.

2. Show that f is lower-semicontinuous if and only if $f^{-1}((-\infty, a])$ is closed for any $a \in \mathbf{R}$. Formulate the corresponding result for upper-semicontinuous functions.

3. Show that the pointwise limit of any nondecreasing net of lower-semicontinuous functions $\{f_i\}_{i \in I}$ is lower-semicontinuous. Formulate the corresponding result for upper-semicontinuous functions.

4. Show that any linear combination $af + bg$ with nonnegative coefficients $a, b \in [0, \infty)$ of lower-semicontinuous is lower-semicontinuous. Formulate the corresponding result for upper-semicontinuous functions.

5. Show that any lower-semicontinuous function $f\colon X \to \mathbf{R}$ attains its minimum value on any compact subset $K \subseteq X$. Formulate the corresponding result for upper-semicontinuous functions.

Lipschitz functions We say that a function $f\colon X \to \mathbf{R}$ on a metric space (X, ρ) satisfies *Lipschitz condition* (with constant $K > 0$) if

$$\mid f(x) - f(y) \mid \leq K\rho(x, y) \qquad (x, y \in X).$$

Lipschitz functions are obviously continuous.

For an arbitrary function $f\colon X \to (0, \infty)$ on a metric space (X, ρ), and $c > 0$, define the function

$$f_c(x) := \inf\{f(x') + c\rho(x, x') \mid x' \in X\}.$$

6. Show that f_c is a Lipschitz function with constant c.

7. Show that $0 < f_c \leq f$ and $f_c \leq f_d$ whenever $c \leq d$.

8. Show that

$$\lim_{c \to \infty} f_c(x) = f(x) \qquad (x \in X)$$

if and only if f is lower-semicontinuous.

9. Show that *any* lower-semicontinuous function $f\colon X \to \mathbf{R}$ is a limit of a nondecreasing sequence of Lipschitz functions.

The functions associated with a *scale* of sets Let $I \subseteq \mathbf{R}$. We say that a family $\{V_i\}_{i\in I}$ of subsets of a set X is an I*-scale* or, simply, a *scale of subsets* of X, if

$$V_i \subseteq V_j \quad \text{whenever} \quad i \le j \qquad \text{and} \qquad \bigcup_{i\in I} V_i = X.$$

Let us associate with every scale of subsets of X, the function

$$f(x) := \inf\{i \in I \mid x \in V_i\}. \tag{1}$$

10. Suppose I is a *dense* subset of $[0,1]$. Show that for any I-scale of *open* subsets of X, the associated function, (1), is upper-semicontinuous.

11. Suppose I is a *dense* subset of $[0,1]$. Show that for any I-scale of *closed* subsets of X, the associated function, (1), is lower-semicontinuous.