Homework 6
March 2, 2012

The fiber relation For a map between sets \( f: X \to Y \) define its fiber equivalence relation on \( X \) by

\[ x \sim_f x' \quad \text{if} \quad f(x) = f(x') \quad (x, x' \in X). \quad (1) \]

We shall denote the corresponding subset of \( X \times Y \) by \( K_f: \]

\[ K_f := \{ (x, x') \in X \times X \mid f(x) = f(x') \}. \]

Factorization of maps We say that a map \( g: X \to Z \) factorizes through a map \( f: X \to Y \), if there exists a map \( e: Y \to Z \), such that \( g = e \circ f \). (2)

We say that \( g \) factorizes through \( f \) uniquely, if there exists a unique map \( e: Y \to Z \) which satisfies (2).

1. Suppose that \( f: X \to Y \) is surjective. Show that there exists no more than one map \( e: Y \to Z \) satisfying (2).

2. Suppose that \( g \) factorizes through \( f \). Show that, if the image of \( g \) has more than one element and \( f \) is not surjective, then factorization in (2) is not unique, i.e., there exist \( e \neq e' \) such that \( e \circ f = g = e' \circ f \).

3. Show that \( g \) factorizes through \( f \) if and only if \( K_f \subseteq K_g \).

A corollary and a few comments As a corollary, we obtain that \( g \) factorizes uniquely through \( f \) if and only if

\[ K_f \subseteq K_g \quad \text{and} \quad f \text{ is surjective.} \]

In this case we also say that map \( g \) passes to \( Y \) or, equivalently, that map \( g \) induces a map \( Y \to Z \). Notation for this unique induced map \( Y \to Z \) is often derived from the notation used for the map \( X \to Z \). If the latter is denoted \( g \), then \( \tilde{g} \), \( \bar{g} \), or \( \hat{g} \), is frequently used to denote the induced map \( Y \to Z \).

4. Consider arbitrary maps \( f: X \to Y \) and \( e: Y \to Z \), and let \( g := e \circ f \).

Show that

\[ K_g \supseteq K_f \quad \text{and} \quad g(X) \subseteq e(Y). \quad (3) \]

5. Deduce from (3) that if \( e \circ f \) is injective, then \( f \) is injective, and that if \( e \circ f \) is surjective, then \( e \) is surjective.

6. Let \( X \) be a set, let \( I \) be a directed set, and \( \{ f_i \}_{i \in I} \) be a net in \([0, \infty]^X \) which converges to a certain function \( f \in [0, \infty]^X \). For a real number \( 0 < c < 1 \) and a function \( g \leq f \), define sets

\[ E_i := \{ x \in X \mid cg(x) \leq f_i(x) \}. \]

Show that

\[ \bigcup_{i \in I} E_i = X. \]

(Hint: to show that each \( x \in X \) belongs to some \( E_i \), consider separately three cases: \( f(x) = 0 \), \( 0 < f(x) < \infty \), and \( f(x) = \infty \).)