Homework 6

March 2, 2012

The fiber relation For a map between sets $f\colon X \to Y$ define its *fiber equivalence relation* on X by

$$x \sim_f x' \qquad \text{if} \qquad f(x) = f\left(x'\right) \qquad \left(x, x' \in X\right). \tag{1}$$

We shall denote the corresponding subset of $X \times Y$ by K_f:

$$K_f := \left\{ \left(x, x'\right) \in X \times X \mid f(x) = f\left(x'\right) \right\}.$$

Factorization of maps We say that a map $g\colon X \to Z$ *factorizates* through a map $f\colon X \to Y$, if there exists a map $e\colon Y \to Z$, such that

$$g = e \circ f. \tag{2}$$

We say that g factorizes through f *uniquely*, if there exists a unique map $e\colon Y \to Z$ which satisfies (2).

1. Suppose that $f\colon X \to Y$ is surjective. Show that there exists no more than one map $e\colon Y \to Z$ satisfying (2).

2. Suppose that g factorizes through f. Show that, if the image of g has more than one element and f is not surjective, then factorization in (2) is not uique, i.e., there exist $e \neq e'$ such that

$$e \circ f = g = e' \circ f.$$

3. Show that g factorizes through f if and only if $K_f \subseteq K_g$.

A corollary and a few comments As a corollary, we obtain that g factorizes uniquely through f if and only if

$$K_f \subseteq K_g \quad \text{and} \quad f \text{ is surjective.}$$

In this case we also say that map g *passes to* Y or, equivalently, that map g *induces* a map $Y \to Z$. Notation for this unique induced map $Y \to Z$ is often derived from the notation used for the map $X \to Z$. If the latter is denoted g, then $\tilde{g}$, $\bar{g}$, or $\hat{g}$, is frequently used to denote the induced map $Y \to Z$.

4. Consider arbitrary maps $f\colon X \to Y$ and $e\colon Y \to Z$, and let $g := e \circ f$.
Show that

$$K_g \supseteq K_f \qquad \text{and} \qquad g(X) \subseteq e(Y). \tag{3}$$

5. Deduce from (3) that if $e \circ f$ is injective, then f is injective, and that if $e \circ f$ is surjective, then e is surjective.

6. Let X be a set, let I be a directed set, and $\{f_i\}_{i \in I}$ be a net in $[0, \infty]^X$ which converges to a certain function $f \in [0, \infty]^X$. For a real number $0 < c < 1$ and a function $g \leq f$, define sets

$$E_i := \{x \in X \mid cg(x) \leq f_i(x)\}.$$

Show that

$$\bigcup_{i \in I} E_i = X.$$

(Hint: to show that each $x \in X$ belongs to some E_i, consider separately three cases: $f(x) = 0$, $0 < f(x) < \infty$, and $f(x) = \infty$.)