Homework 11

due April 20, 2012

One-point compactification Let $(X, \mathscr{T})$ be a topological space. Denote by $\hat{X}$ the disjoint union of X and a single-element set $\{\infty\}$. We will identify X with a subset of $\hat{X}$ and, for any subset $E \subseteq X$, we will denote $E \cup \{\infty\}$ by $\hat{E}$.

Let

$$\mathscr{T}^{\wedge} := \mathscr{T} \cup \{\hat{V} \mid X^c = X \setminus V \text{ is a compact subset of } X\}. \tag{1}$$

1. Show that $\mathscr{T}^{\wedge}$ is a topology on $\hat{X}$ if X is Hausdorff. Under the same hypothesis, show that $\mathscr{T}^{\wedge}$ induces on X the original topology, $\mathscr{T}$.

2 Let $(X, \mathscr{T})$ be a Hausdorff topological space. Show that $\left(\hat{X}, \mathscr{T}^{\wedge}\right)$ is compact. Show that $\left(\hat{X}, \mathscr{T}^{\wedge}\right)$ is Hausdorff if and only if $(X, \mathscr{T})$ is locally compact.

3. Show that any compact subset $K \subseteq X$ of a metric space (X, ρ) is *separable*, i.e. it contains a countable dense subset.

Hint: *For any positive integer n consider the cover of K by the family of open balls of radius $\frac{1}{n}$ with the centers at all points of K.*

For any subset A of a metric space (X, ρ), and any $\epsilon > 0$, let A_ϵ denote its ϵ-neighborhood

$$A_\epsilon := \{x \in X \mid \text{there exists } a \in A \text{ such that } \rho(x, a) < \epsilon\}$$

(put $A_\epsilon = \emptyset$ for $A = \emptyset$). Given a pair of subsets $A, B \subseteq X$, let

$$\varrho(A, B) := \inf\{\epsilon > 0 \mid A \subseteq B_\epsilon \text{ and } B \subseteq A_\epsilon\}. \tag{2}$$

Formula (2) defines a function $\mathscr{P}(X) \times \mathscr{P}(X) \to [0, \infty]$.

4. Suppose that A is a nonempty bounded set. Show that $\varrho(A, B) < \infty$ if and only if B is bounded and nonempty.

5. Show that function ϱ satisfies the Triangle Inequality.

6. Show that $\varrho(A, B) = 0$ if and only if $\bar{A} = \bar{B}$.

It follows that ϱ is a metric on the set $\mathscr{Z}(X) \subseteq \mathscr{P}(X)$ of all bounded nonempty closed subsets of X. This metric is complete if and only if the original metric ρ on X is complete.

Given a pair of finite Borel measures μ and ν on a metric space (X, ρ), let

$$\varrho(\mu, \nu) := := \inf\{\epsilon > 0 \mid \mu(E) \leq \nu(E_\epsilon) + \epsilon \text{ and } \nu(E) \leq \mu(E_\epsilon) + \epsilon \text{ for all } E \in \mathfrak{B}(X)\} \tag{3}$$

where $\mathfrak{B}(X)$ denotes the σ-algebra of Borel subsets of X.

7. Show that the function $\varrho \colon \mathfrak{B}(X) \times \mathfrak{B}(X) \to [0, \infty)$ defined by (3) satisfies the Triangle Inequality.

8. Show that if $\varrho(\mu, \nu) = 0$, then $\mu(Z) = \nu(Z)$ for any closed subset of X. (Hint: *Use the fact that $Z_{\frac{1}{n}} \searrow Z$.*)

Derive from this that $\mu(E) = \nu(E)$ for any Borel subset of X. (Hint: *Use a certain result from the previous homework.*)

It follows that (3) defines a metric on the set of all finite Borel measures on X. If this metric is complete, then the original metric ρ on X is complete. If the topological space X is separable, then completeness of ρ implies completeness of ϱ.

Metric (3) is of particular significance in Probability Theory. This is due to the fact that studying the limits of various sequences of probabilistic measures[1] is among central questions of Probability Theory.

[1]Recall that a measure μ on a σ-algebra $\mathfrak{M} \subseteq \mathscr{P}(X)$ is *probabilistic* if $\mu(X) = 1$.