Dynkin systems and regularity of finite Borel measures
Homework 10

due April 13, 2012

1. Let $p \in X$ be a point of a topological space. Show that the set $\{p\} \subseteq X$ is closed if and only if for any point $q \neq p$, there exists a neighborhood $N \ni q$ such that $p \notin N$.

Derive from this that X is a T_1-space if and only if every singleton subset is closed.

Let $\mathscr{C}, \mathscr{D} \subseteq \mathscr{P}(X)$ be arbitrary families of subsets of a set X. We define the family $\mathscr{D}{:}\mathscr{C}$ as

$$\mathscr{D}{:}\mathscr{C} := \{E \subseteq X \mid C \cap E \in \mathscr{D} \text{ for every } C \in \mathscr{C}\}.$$

2. The Exchange Property Show that, for any families $\mathscr{B}, \mathscr{C}, \mathscr{D} \subseteq \mathscr{P}(X)$, one has

$$\mathscr{B} \subseteq \mathscr{D}{:}\mathscr{C} \qquad \text{if and only if} \qquad \mathscr{C} \subseteq \mathscr{D}{:}\mathscr{B}.$$

Dynkin systems[1] We say that a family of subsets $\mathscr{D} \subseteq \mathscr{P}(X)$ of a set X is a *Dynkin system* (or a *Dynkin class*), if it satisfies the following three conditions:

($\mathbf{D_1}$) if $D \in \mathscr{D}$, then $D^c \in \mathscr{D}$;

($\mathbf{D_2}$) if $\{D_i\}_{i\in I}$ is a countable family of *disjoint* members of $\mathscr{D}$, then $\bigcup_{i\in I} D_i \in \mathscr{D}$;

($\mathbf{D_3}$) $X \in \mathscr{D}$.

3. Show that any Dynkin system satisfies also:

($\mathbf{D_4}$) if $D, D' \in \mathscr{D}$ and $D' \subseteq D$, then $D \setminus D' \in \mathscr{D}$.

4. Show that the intersection, $\bigcap_{i\in I} \mathscr{D}_i$, of any family of Dynkin systems $\{\mathscr{D}_i\}_{i\in I}$ on a set X is a Dynkin system on X.

It follows that, for any family $\mathscr{F} \subseteq \mathscr{P}(X)$, there exists a smallest Dynkin system containing $\mathscr{F}$, namely the intersection of all Dynkin systems containing $\mathscr{F}$. Let us denote it by $\mathscr{F}^{\diamond}$.

5. Show that, for any families $\mathscr{C}, \mathscr{D} \subseteq \mathscr{P}(X)$, one has:

(a) $\mathscr{D}{:}\mathscr{C}$ satisfies ($\mathbf{D_2}$) if $\mathscr{D}$ satisfies ($\mathbf{D_2}$),

(b) $\mathscr{D}{:}\mathscr{C}$ satisfies ($\mathbf{D_4}$) if $\mathscr{D}$ satisfies ($\mathbf{D_4}$),

(c) $X \in \mathscr{D}{:}\mathscr{C}$ if and only if $\mathscr{C} \subseteq \mathscr{D}$,

(d) $\mathscr{D}{:}\mathscr{D}$ is contained in $\mathscr{D}$ if $X \in \mathscr{D}$.

[1] Explicitly this definition was proposed by Evgeniy Dynkin around 1961; implicitly this and many similar "systems" of subsets appear already in articles by Wacław Sierpiński in early 1920-es.

6. Show that $\mathscr{D}{:}\mathscr{C}$ is a Dynkin system if $\mathscr{D}$ is a Dynkin system and $\mathscr{C} \subseteq \mathscr{D}$. Derive from this that $\mathscr{D}{:}\mathscr{D}$ is a Dynkin system if $\mathscr{D}$ is a Dynkin system.

7. Show that a family $\mathscr{C}$ is closed with respect to finite intersections if and anly if $\mathscr{C}$ is contained in $\mathscr{D}{:}\mathscr{C}$ for any family $\mathscr{D}$ which contains $\mathscr{C}$.

8. Show that the Dynkin system $\mathscr{C}^\diamond$ generated by a family that is closed with respect to finite intersections, is a σ-algebra.

Hint: *Show that $\mathscr{C}^\diamond$ is contained in $\mathscr{C}^\diamond{:}\mathscr{C}$. Use that to show that $\mathscr{C}$ is contained in $\mathscr{C}^\diamond{:}\mathscr{C}^\diamond$. Use this, in turn, to show that $\mathscr{C}^\diamond$ is contained in $\mathscr{C}^\diamond{:}\mathscr{C}^\diamond$.*[2]

It is often much easier to show that a family of subsets of a set X is a Dynkin system than a σ-algebra. The above result is freqently used in modern Measure Theory to prove that a given family of sets is a σ-algebra, or that a a certain property holds for all measurable sets. The following exercises provide an example of this.

9. Let $\mathscr{D}$ be a Dynkin system containing a family $\mathscr{C}$ which is closed with respect to finite intersections. Show that $\mathscr{D}$ contains the σ-algebra generated by $\mathscr{C}$.

μ-regular subsets Suppose that $\mu\colon \mathfrak{B}(X) \to [0,\infty]$ is a measure defined on the σ-algebra of all Borel subsets of a topological space X. We say that a Borel set $E \in \mathfrak{B}(X)$ is μ-regular, if

for any $\epsilon > 0$, there exist a closed subset Z and an open subset V such that $Z \subseteq E \subseteq V$ and $\mu(V \setminus Z) < \epsilon$.

Note that a μ-regular subset is a *union* of an F_σ-set and a set of μ-measure zero. It is also a *difference* of a G_δ-set and a set of μ-measure zero.

10. Let $\mu\colon \mathfrak{B}(X) \to [0,\infty]$ be a *finite*[3] Borel measure. Show that the family of all μ-regular Borel sets is a Dynkin system on X.

11. For any subset $E \subseteq X$ of a *metric space* (X,ρ), define the sets

$$E_\epsilon := \{x \in X \mid \rho(x,e) < \epsilon \text{ for some } e \in E\} \qquad (\epsilon > 0).$$

Show that sets E_ϵ are open, $E_\epsilon \subseteq E_\eta$ if $\epsilon \le \eta$, and

$$\bigcap_{\epsilon>0} E_\epsilon = \overline{E} \qquad \text{(the closure of } E\text{)}.$$

Deduce from this that any closed subset of a metric space is a G_δ-set.

12. Let $\mu\colon \mathfrak{B}(X) \to [0,\infty]$ be a *finite* Borel measure on a *metric space* X. Show that the family of μ-regular Borel sets contains all closed subsets. Deduce from this that every Borel subset of X is μ-regular.

[2] I hope you will appreciate the exquisite beauty of this argument.

[3] I.e., $\mu(X) < \infty$.