Final

due May 8, 2012

Write your solutions clearly and in complete sentences. All notation used must be properly introduced. Your arguments besides being correct should be also complete. Pay close attention the hypotheses of each problem: this will spare you from making unwarranted assumptions.

Part I: Riemann Integral revisited

Recall that a *partition* of an interval $I = [a, b]$ of the real line is a *finite* family $\mathscr{P} \subset \mathscr{P}(I)$ of closed *non-overlapping*[1] subintervals which covers I. A *tagging* of partition $\mathscr{P}$ is a function

$$\tau\colon \mathscr{P} \to I, \qquad P \mapsto \tau_P \qquad (P \in \mathscr{P}).$$

We shall say that $(\mathscr{P}, \tau)$ is a *properly* tagged partition if τ_P belongs to P for each $P \in \mathscr{P}$,

$$\tau_P \in P \qquad (P \in \mathscr{P}). \tag{1}$$

The set $\Delta := (0, \infty)^I$ of all functions $\delta\colon I \to (0, \infty)$, partially ordered by the ordering *reverse* to the usual one,

$$\delta \preceq \delta' \qquad \text{if} \qquad \delta(t) \geq \delta'(t) \quad (t \in I),$$

is a directed set.

For a given $\delta \in \Delta$, we shall say that a tagged partition $(\mathscr{P}, \tau)$ is δ*-fine* if, for every subinterval $P \in \mathscr{P}$, the latter is contained in the open interval with center at τ_P and of radius $\delta(\tau_P)$.

Note that, if interval $I = [a, b]$ is the concatenation $I = I_1 \cup \cdots \cup I_n$ of intervals

$$I_j = [c_{j-1}, c_j] \qquad (a = c_0 < c_1 < \cdots < c_n = b),$$

and $(\mathscr{P}_1, \tau_1), \ldots, (\mathscr{P}_n, \tau_n)$ are tagged partitions of those intervals, then one can combine them into a single tagged partition $(\mathscr{P}, \tau)$ of I, and the resulting partition is δ-fine if and only if

$$(\mathscr{P}_1, \tau_1) \text{ is } \delta_{|I_1}\text{-fine}, \quad \ldots \ , \quad (\mathscr{P}_n, \tau_n) \text{ is } \delta_{|I_n}\text{-fine}.$$

1. Using the above observation prove that, for any $\delta \in \Delta$, there exists a δ-fine properly tagged partition $(\mathscr{P}, \tau)$.

Hint: *Assuming that there is no such tagged partition, show that there exists a nested sequence of closed subintervals*

$$I = I_0 \supset I_1 \supset \cdots \supset I_m \supset \cdots$$

of length $l(I_m) = \frac{1}{2^m} l(I)$, such that each I_m does not admit a $\delta_{|I_m}$-fine partition. Show that this leads to a contradiction (use the compactness of I).

Given a function $f\colon I \to \mathbf{R}$ and a tagged partition $(\mathscr{P}, \tau)$ of I, the associated *Riemann sum* is defined as

$$S(f, \mathscr{P}, \tau) := \sum_{P \in \mathscr{P}} f(\tau_P)\, l(P)$$

where $l(P)$ denotes the *length* of a subinterval $P \subseteq I$. We shall say that the Riemann sum is *proper* if the partition is properly tagged.

[1] We shall say that subsets $A, B \subseteq X$ of a topological space *do not overlap* if the intersection of their interiors is empty.

We shall consider now three nonincreasing nets of subsets of $\mathbf{R}$, the first two:

$$\mathcal{S}_\delta(f) := \{S(f,\mathscr{P},\tau) \mid (\mathscr{P},\tau) \text{ is } \delta\text{-fine}\} \qquad (\delta \in \Delta), \tag{2}$$

and

$$\mathcal{S}_\delta^{\text{prop}}(f) := \{S(f,\mathscr{P},\tau) \mid (\mathscr{P},\tau) \text{ is } \delta\text{-fine and properly tagged}\} \qquad (\delta \in \Delta), \tag{3}$$

are indexed by Δ, while the last one,

$$\mathcal{S}_\delta^{\text{prop}}(f) := \{S(f,\mathscr{P},\tau) \mid (\mathscr{P},\tau) \text{ is } \delta\text{-fine and properly tagged}\} \qquad (\delta \in (0,\infty)), \tag{4}$$

is indexed by positive real numbers which can be considered as forming the set of *constant* functions in Δ.

Given a *nonincreasing net* $\{E_i\}_{i\in I}$ of subsets of a topological space X, the set

$$\bigcap_{i\in I} \bar{E}_i \tag{5}$$

consists of points in X which belong to *every* neighborhood of *every* set E_i. Let us call (5), the *limit-point set* of net $\{E_i\}_{i\in I}$. When the limit-point set consists of a single element $p \in X$, we will say that net $\{E_i\}_{i\in I}$ *converges* to p, and call p *the limit* of $\{E_i\}_{i\in I}$.

Net (4) converges if and only if function f is Riemann integrable: its limit is the classical Riemann integral of f.

We say that f is Riemann integrable in the *generalized sense* (or, in the sense of Henstock-Kurzweil), if net (3) converges. Finally, if net (2) converges, we say that f is Riemann integrable in the sense of McShane. The corresponding limits will be referred to as Henstock-Kurzweil and McShane integrals of f.

Due to the following obvious inclusions,

$$\bigcap_{\delta\in\Delta} \overline{\mathcal{S}_\delta(f)} \;\supseteq\; \bigcap_{\delta\in\Delta} \overline{\mathcal{S}_\delta^{\text{prop}}(f)} \;\subseteq\; \bigcap_{\delta\in(0,\infty)} \overline{\mathcal{S}_\delta^{\text{prop}}(f)}, \tag{6}$$

if f is Riemann integrable in the classical sense or in the sense of McShane, then it is automatically Henstock-Kurzweil integrable and the values of the corresponding integrals coincide. In other words, the Henstock-Kurzweil integral is an extension of both the classsical Riemann and McShane integrals.

One can show that if f is McShane integrable, then also $|f|$ is McShane integrable. This is so also for the classical Riemann integral but not for the Henstock-Kurzweil integral.

One can also show that a function $f \geq 0$ is McShane integrable if and only if it is Henstock-Kurzweil integrable. It follows that f is McShane integrable if and only if both f and $|f|$ are Henstock-Kurzweil integrable.

It is not difficult to show that f is McShane integrable if and only if it is Lebesgue integrable. The latter means: f is measurable with respect to the σ-algebra $\mathfrak{M}$ of Lebesgue measurable sets (this is the σ-algebra constructed in the proof of Riesz' Representation Theorem for $\Lambda\colon f \mapsto \int_a^b f(x)\,dx$) and, additionally, $\int |f|\,d\mu < \infty$. It follows that the Lebesgue integral associated with the Lebesgue measure on the real line coincides with a slight generalization of the original definition of the Riemann integral (one can do similarly for multiple integrals over n-dimensional rectangles in $\mathbf{R}^n$).

In comparison, the importance of the Henstock-Kurzweil integral lies in the fact that *every* derivative is Henstock-Kurzweil integrable and the Fundamental Theorem of Calculus holds for *any* differentiable function.

2. Let f be a differentiable function[2] on an interval $I = [a, b]$

a) Show that, for every $\epsilon > 0$ and every $t \in I$, there exists $\delta(t) > 0$, such that for any interval $[\alpha, \beta] \subseteq I$ which contains t and which is contained in $(t - \delta(t), t + \delta(t))$, one has

$$|f(\beta) - f(\alpha) - f'(t)(\beta - \alpha)| < \epsilon(\beta - \alpha).$$

b) Show that for every δ-fine properly tagged partition $(\mathscr{P}, \tau)$, where δ is the function whose existence you proved in Part a), one has the inequality

$$|f(b) - f(a) - S(f', \mathscr{P}, \tau)| < \epsilon l(I).$$

Deduce from it that f' is Henstock-Kurzweil integrable and that the integral equals $f(b) - f(a)$.

Part II

Let $\nu\colon \mathfrak{R} \to [0, \infty]$ be a function on a family of subsets $\mathfrak{R} \subseteq \mathscr{P}(X)$ of a set X. Define the following function on the whole of $\mathscr{P}(X)$:

$$\nu^*(E) := \inf_{\mathscr{R}} \sum_{R \in \mathscr{R}} \nu(R) \qquad (E \subseteq X) \tag{7}$$

where the infimum is taken over all countable (possibly finite) covers $\mathscr{R}$ of E by members of $\mathfrak{R}$, and when no such cover exists, we set $\nu^*(E) = \infty$.

3. Show that ν^* satisfies the following properties:

(**OM₁**) ν^* is ***isotone***, i.e., $\nu^*(E) \leq \nu^*(E')$ whenever $E \subseteq E'$;

(**OM₂**) ν^* is ***σ-subadditive***, i.e., $\nu^*\left(\bigcup_{i \in I} E_i\right) \leq \sum_{i \in I} \nu^*(E_i)$, for any countable family of subsets $\{E_i\}_{i \in I}$ of X;

and, if $\inf_{R \in \mathfrak{R}} \nu^*(R) = 0$, also

(**OM₃**) $\nu^*(\emptyset) = 0$.

Any function $\mathscr{P}(X) \to [0, \infty]$ which satisfies the above three conditions is called an *outer measure* (on a set X).

Let us consider the following *ternary* relation on the set of all subsets of a set X: we say that a subset A is *separated* from a subset B by a subset E if

$$A \subseteq E^c \qquad \text{and} \qquad B \subseteq E,$$

and will be denoting this fact by writing $A \mid_E B$.

Let $\nu\colon \mathscr{P}(X) \to [0, \infty]$ be a function on $\mathscr{P}(X)$. We shall say that $E \subseteq X$ is ν-*measurable* if, for any pair of subsets A and B separated by E, one has

$$\nu(A \cup B) = \nu(A) + \nu(B).$$

Let us denote the family of all ν-measurable subsets by $\mathfrak{C}_\nu$.

[2] Differentiablity at the endpoints, a and b, means that the corresponding *one-sided* limits,

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \qquad \text{and} \qquad \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b},$$

exist.

4. Show that:

(a) family $\mathfrak{C}_\nu$ is nonempty *if and only if* $\emptyset \in \mathfrak{C}_\nu$, *if and only if* either $\nu(\emptyset) = 0$, or $\nu(E) = \infty$ for every $E \subseteq X$;

(b) family $\mathfrak{C}_\nu$ is closed under complement, difference, union, and intersection, i.e., for any pair of ν-measurable sets E and E', the sets

$$E^c, \qquad E \setminus E', \qquad E \cup E', \qquad \text{and} \qquad E \cap E',$$

are ν-measurable;

(c) if ν is *isotone* and *subadditive*, then any subset $E \subseteq X$ with $\nu(E) = 0$ is ν-measurable;

(d) if ν is *isotone* and σ-subadditive, then $\mathfrak{C}_\nu$ is closed under countable unions of *disjont* sets and, for any $A \subseteq X$ and any countable family of disjoint ν-measurable sets $\{E_i\}_{i \in I}$, one has

$$\nu\left(A \cap \bigcup_{i=I} E_i\right) = \sum_{i \in I} \nu(A \cap E_i).$$

Note that for the following outer measure

$$\nu(A) = \begin{cases} 1 & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases}$$

one has $\mathfrak{C}_\nu = \{\emptyset, X\}$, which demonstrates that even an outer measure may have no measurable sets other than $\emptyset$ and X.

5. Show that, if a family $\mathfrak{M}$ of subsets of a set X is closed under difference, union, and countable union of disjoint sets, then it is closed also under countable union of not necessarily disjoint sets.

Deduce that $\mathfrak{C}_\nu$ is a σ-algebra and ν restricted to $\mathfrak{C}_\nu$ is a measure, if ν is an outer measure, i.e., if ν is isotone, σ-subadditive, and $\nu(\emptyset) = 0$.

6. For a function $\nu\colon \mathfrak{R} \to [0, \infty]$, let $\nu^*\colon \mathscr{P}(X) \to [0, \infty]$ be the associated outer measure.

(a) Suppose that $\mathfrak{R}$ is closed under countable unions and ν is σ-subadditive. Show that

$$\nu^*(E) = \inf_{R \supseteq E} \nu(R) \qquad (E \subseteq X)$$

where the infimum is taken over all $R \in \mathfrak{R}$ which contain E.

(b) Suppose that ν^* satisfies the following condition:

$$\nu^*(R \cap R') + \nu^*(R \setminus R') = \nu^*(R) \qquad (R, R' \in \mathfrak{R}).$$

Show that $\mathfrak{R} \subseteq \mathfrak{C}_{\nu^*}$, i.e., every set $R \in \mathfrak{R}$ is ν^*-measurable. In particular, $\mathfrak{C}_{\nu^*}$ being a σ-algebra itself, contains the σ-algebra generated by $\mathfrak{R}$.

As a corollary we obtain that if ν is the n-dimensional *volume* considered as a function on the family of n-dimensional "rectangles" in $\mathbf{R}^n$, then $\mathfrak{C}_{\nu*}$ contains all Borel subsets in $\mathbf{R}^n$. This is a direct construction of the n-dimensional Lebesgue measure.

Part III

Let $\Lambda\colon C_c(X) \to \mathbf{R}$ be a *positive* linear functional on the space of continuous functions with compact support on a locally compact Hausdorff topological space X.

7. Show that, for any $f \in C_c(X)$ and any $h \in C_c(X)$ which is nonnegative and equals 1 on $\operatorname{supp} f$, one has the double inequality

$$m \cdot \Lambda h \leq \Lambda f \leq M \cdot \Lambda h.$$

where $m = \inf_{x \in X} f(x)$ and $M = \sup_{x \in X} f(x)$. Deduce from it the inequality

$$|\Lambda f| \leq \Lambda(|f|) \leq \mu(\operatorname{supp} f)\|f\|_\infty$$

where $\|f\|_\infty := \sup_{x \in X} |f(x)|$ and $\mu = \mu_\Lambda$ is the measure associated with functional Λ by Riesz' Representation Theorem.

8. Suppose that $f_i \nearrow f$ where $f \in C_c(X)$ and $\{f_i\}_{i \in I}$ is a nondecreasing net of nonnegative continuous functions. Show that:

(a) $\{f_i\}_{i \in I}$ converges to f *uniformly*, i.e., for every $\epsilon > 0$, there exists $j \in I$ such that, for every $i \geq j$ and $x \in X$, one has

$$f(x) - f_i(x) < \epsilon;$$

(b) $\Lambda f_i \nearrow \Lambda f$.

As a corollary, we obtain that Λ is a *preintegral* in the sense of Section 5.1 of *Notes on Measure and Integration*. One can then define the integral with respect to the Riesz measure *directly*, bypassing the construction of μ altogether:

$$\int f\,d\mu := \sup_{\substack{g \leq f \\ g \in C_c(X)}} \Lambda g \qquad \text{(if } f \text{ is non-negative and lower semi-continuous)},$$

and

$$\int f\,d\mu := \inf_{h \geq f} \int h\,d\mu \qquad \text{(for general non-negative } f\text{)}$$

where the infimum is taken over all lower semi-continuous functions h greater or equal than f.