Notes on Ordered Sets

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1 Vocabulary

1.1 Definitions

Definition 1.1 *A binary relation $\preccurlyeq$ on a set S is said to be a* ***partial order*** *if it is* ***reflexive****,*

$$x \preccurlyeq x,$$

weakly antisymmetric*,*

$$\textit{if } x \preccurlyeq y \textit{ and } y \preccurlyeq x, \textit{ then } x = y,$$

and ***transitive****,*

$$\textit{if } x \preccurlyeq y \textit{ and } y \preccurlyeq z, \textit{ then } x \preccurlyeq z$$

Above x, y, z, are arbitrary elements of S.

Definition 1.2 *Let $E \subseteq S$. An element $y \in S$ is said to be an* ***upper bound*** *for E if*

$$x \preccurlyeq y \qquad \textit{for any } x \in E. \tag{1}$$

By definition, any element of S is declared to be an upper bound for $\emptyset$, the empty subset.

We shall denote by $U(E)$ the set of all upper bounds for E

$$U(E) := \{y \in S \mid x \preccurlyeq y \text{ for any } x \in E\}.$$

Note that $U(\emptyset) = S$.

Definition 1.3 *We say that a subset $E \subseteq S$ is* ***bounded (from) above****, if $U(E) \neq \emptyset$, i.e., when there exists at least one element $y \in S$ satisfying* (1).

Definition 1.4 *If $y, y' \in U(E) \cap E$, then*

$$y \preccurlyeq y' \quad \textit{and} \quad y' \preccurlyeq y.$$

Thus, $y = y'$, and that unique upper bound of E which belongs to E will be denoted $\max E$ and called the ***greatest element*** *of E.*

It follows that $U(E) \cap E$ is empty when E has no greatest element, and consists of a single element, namely $\max E$, when it does.

1.2 The Principle of Duality

1.2.1 The opposite ordering

Note that the relation defined by

$$x \preccurlyeq^{\mathrm{op}} y \quad \text{if} \quad y \preccurlyeq x$$

is also an order relation on S. We will refer to it as the ordering *opposite* to $\preccurlyeq$.

1.2.2 Dual concepts and dual statements

Every concept and every statement in theory of partially ordered sets, when we apply them to the opposite partailly ordered set

$$(S, \preccurlyeq^{\mathrm{op}}),$$

yields a concept and a statement for the original partially ordered set $(S, \preccurlyeq)$. We shall refer to such concepts and statements as *dual*. We shall provide numerous illustrations of the duality below.

1.2.3 Duality between $\max E$ and $\min E$

An element $e \in E$ is the *greatest* element of E for the ordering $\preccurlyeq$ if and only if it is the *smallest* element for the opposite ordering $\preccurlyeq^{\mathrm{op}}$.

1.2.4 Duality between $L(E)$ and $U(E)$

Thus, an element $s \in S$ is an *upper* bound for $E \subseteq S$ in $(S, \preccurlyeq)$ if and only if it is a *lower* bound for E in $(S, \preccurlyeq^{\mathrm{op}})$. In particular, the set of upper bounds of E in $(S, \preccurlyeq)$ coincides with the set of lower bounds of E in the opposite partially ordered set $(S, \preccurlyeq^{\mathrm{op}})$.

1.3 The upper and the lower-bound-set operations

1.3.1

Given a partially ordered set $(S, \preccurlyeq)$, we shall make a number of basic observations about the operations that assign to a subset $E \subseteq S$, its set of upper and, respectively, lower bounds.

Exercise 1 *Show that, for any subsets E and F, one has*

$$E \subseteq U(F) \qquad \text{if and only if} \qquad L(E) \supseteq F.$$

Dually

$$E \subseteq L(F) \qquad \text{if and only if} \qquad U(E) \supseteq F.$$

Exercise 2 *Show that if $E \subseteq S$ is bounded below and nonempty, then $L(E)$ is bounded above and nonempty.*

Dually, if E is bounded above and nonempty, then $U(E)$ is bounded below and nonempty.

1.3.2

Note that

$$L(\emptyset) = U(\emptyset) = S;$$

thus, the empty subset of S is bounded below, or above, precisely when $S \neq \emptyset$.

In particular, for $E = \emptyset$, the conclusion of the implication in Exercise 2 fails unless S possesses the greatest element.

1.3.3

If $E \subseteq F \subseteq S$, then

$$\max F \in U(E) \tag{2}$$

when $\max F$ exists, and, dually,

$$\min F \in L(E)$$

when $\min F$ exists.

If both $\max E$ and $\max F$ exist, then

$$\max E \preccurlyeq \max F.$$

Dually, if both $\min E$ and $\min F$ exist, then

$$\min F \preccurlyeq \min E.$$

Exercise 3 (Sandwich Lemma for maxima) *Show that if* $E'' \subseteq E \subseteq E'$ *and both* $\max E'$ *and* $\max E''$ *exist and are equal, then* $\max E$ *exists and*

$$\max E'' = \max E = \max E'.$$

Dually, if both $\min E'$ and $\min E''$ exist and are equal, then $\min E$ exists and

$$\min E' = \min E = \min E''.$$

1.3.4 Supremum and infimum

Definition 1.5 *When* $\min U(E)$ *exists it is called the* ***least upper bound*** *of* E*, or the* ***supremum*** *of* E*, and is denoted* $\sup E$*.*

Dually, when $\max L(E)$ *exists it is called the* ***greatest lower bound*** *of* E*, or the* ***infimum*** *of* E*, and is denoted* $\inf E$*.*

For the supremum of E to exist, subset E must be bounded above. The supremum of E may exist for some bounded above subsets of S and may not exist for others.

1.3.5 An example

Let us consider $S = \mathbf{Q}$, the set of rational numbers, with the usual order. Both the following subset $E_1 \subseteq \mathbf{Q}$,

$$E_1 := \{x \in \mathbf{Q} \mid x^2 < 1\}$$

and the subset $E_2 \subseteq \mathbf{Q}$,

$$E_2 := \{x \in \mathbf{Q} \mid x^2 < 2\},$$

are simultaneously bounded above and below. None of them has either the greatest nor the smallest element but

$$\sup E_1 = 1 \qquad \text{and} \qquad \inf E_1 = -1$$

while neither $\sup E_2$ nor $\inf E_2$ exist in $S = \mathbf{Q}$.

Exercise 4 *Show that*

$$\sup \emptyset = \min S \qquad \textit{and} \qquad \inf \emptyset = \max S.$$

1.3.6

In particular, $\sup \varnothing$ exists if and only if S has the smallest element; this occurs precisely when every subset of S is bounded below.

Similarly, $\inf \varnothing$ exists if and only if S has the greatest element; this occurs precisely when every subset of S is bounded above.

1.3.7 Down-intervals and up-intervals

Let $(S, \preccurlyeq)$ be a partially ordered set. For each $s \in S$, we define the *down-interval*

$$\langle s] := \{t \in S \mid t \preccurlyeq s\}.$$

and the *up-interval*

$$[s\rangle := \{t \in S \mid s \preccurlyeq t\}.$$

Exercise 5 *Show that, for $E \subseteq S$, one has*

$$L(E) = \langle s] \qquad \text{for some } \varepsilon \in S$$

if and only if $\inf E$ *exists. In this case,* $\varepsilon = \inf E$.

Dually,

$$U(E) = [\eta\rangle, \qquad \text{for some } \eta \in S,$$

if and only if $\sup E$ exists. In this case, $\eta = \sup E$.

1.3.8 An example: the set of natural numbers oredered by the "m divides n" relation

Consider the set of natural numbers,

$$\mathbf{N} := \{0, 1, 2, \dots\},$$

equipped with the ordering given by

$$m \preccurlyeq n \qquad \text{if} \qquad m \mid n$$

("*m divides n*").

Exercise 6 *Does $(\mathbf{N}, |\,)$ have the maximum? the minimum? If yes, then what are they?*

Exercise 7 *For a given* $n \in \mathbf{N}$*, describe intervals* $\langle n]$ *and* $[n\rangle$ *in* $(\mathbf{N}, |\,)$*.*

Exercise 8 *For given* $m, n \in \mathbf{N}$*, is set* $\{m, n\}$ *bounded below? Does it possess infimum? If yes, then describe* $\inf\{m, n\}$*.*

Exercise 9 *For given* $m, n \in \mathbf{N}$*, is set* $\{m, n\}$ *bounded above? Does it possess supremum? If yes, then describe* $\sup\{m, n\}$*.*

Exercise 10 *Does every subset of* $\mathbf{N}$ *possess infimum in* $(\mathbf{N}, |\,)$*? Does every subset of* $\mathbf{N}$ *possess supremum?*

1.3.9 Totally ordered sets

Definition 1.6 *We say that a partially ordered set* $(S, \preccurlyeq)$ *is* ***totally****, or* ***linearly****, ordered if any two elements* s *and* t *of* S *are* comparable

$$\text{either} \quad s \preccurlyeq t \quad \text{or} \quad t \preccurlyeq s.$$

Totally ordered subsets in any given partially ordered set are called **chains**.

Exercise 11 *Let* $(S, \preccurlyeq)$ *be a totally ordered set and* $E, F \subseteq S$ *be two subsets. Show that*

$$\text{either } L(E) \subseteq L(F) \text{ or } L(F) \subseteq L(E).$$

1.4 Fundamental properties of the upper and the lower-bound-set operations

1.4.1

For any subset $E \subseteq S$, one has

$$E \subseteq LU(E) := L(U(E)) \tag{3}$$

and

$$E \subseteq UL(E) := U(L(E)). \tag{4}$$

1.4.2

If $E \subseteq F \subseteq S$, then

$$U(E) \supseteq U(F) \tag{5}$$

and

$$L(E) \supseteq L(F).$$

Exercise 12 *Show that if* $E \subseteq F$ *and both* $\sup E$ *and* $\sup F$ *exist, then*

$$\sup E \preccurlyeq \sup F.$$

Dually, if both $\inf E$ and $\inf F$ exist, then

$$\inf F \preccurlyeq \inf E.$$

Exercise 13 (Sandwich Lemma for infima) *Show that if* $E'' \subseteq E \subseteq E'$ *and both* $\inf E'$ *and* $\inf E''$ *exist and are equal, then* $\inf E$ *exists and*

$$\inf E'' = \inf E = \inf E'.$$

1.4.3 Sandwich Lemma for suprema

Dually, if both $\sup E'$ and $\sup E''$ exist and are equal, then $\sup E$ exists and

$$\sup E' = \sup E = \sup E''.$$

1.4.4

By applying (5) to the pair of subsets in (3), one obtains

$$U(E) \supseteq ULU(E) := U(L(U(E)))$$

while (4) applied to subset $U(E)$ yields

$$U(E) \subseteq ULU(E).$$

It follows that

$$U(E) = ULU(E). \tag{6}$$

Dually,

$$L(E) = LUL(E). \tag{7}$$

Note that equality (7) is nothing but equality (6) for the *opposite* ordering.

1.4.5

For any subsets $E \subseteq S$ and $F \subseteq S$, one has

$$U(E \cup F) = U(E) \cap U(F)$$

and

$$L(E \cup F) = L(E) \cap L(F).$$

Lemma 1.7 *For any* $E \subseteq S$*,* $\max E$ *exists if and only if* $\sup E$ *exists and belongs to* E*, and they are equal*

$$\sup E = \max E. \tag{8}$$

Dually, $\min E$ *exists if and only if* $\inf E$ *exists and belongs to* E*, and they are equal*

$$\inf E = \min E.$$

Proof. It suffice to prove the first statement. The second one follows the Duality Principle. The greatest element of E is an upper bound of E and every upper bound of E is greater or equal than it.

If $\sup E$ exists and is a member of E, then it belongs to $U(E) \cap E$ which as we established, cf. Definition (1.4), consists of the single element $\max E$ when $U(E) \cap E$ is nonempty. □

1.4.6

For any $E \subseteq S$*,* $\inf U(E)$ *exists if and only if* $\sup E$ *exists, and they are equal*

$$\max LU(E) = \inf U(E) = \min U(E) = \sup E. \tag{9}$$

Indeed,

$$\inf U(E) := \max LU(E) \in U(E),$$

in view of $E \subseteq LU(E)$, cf., (3), combined with (2) where $F = LU(E)$. Thus,

$$\inf U(E) = \min U(E)$$

by (8).

Dually, $\sup L(E)$ *exists if and only if* $\inf E$ *exists, and they are equal*

$$\min UL(E) = \sup L(E) = \max L(E) = \inf E.$$

1.5 An example: the power set as a partially ordered set

1.5.1

Let $S = \mathscr{P}(X)$ be the *power set* of a set X:

$$\mathscr{P}(X) := \text{the set of all subsets of } X.$$

Containment $\subseteq$ is a partial order relation on $\mathscr{P}(X)$.

Subsets $\mathscr{E}$ of $\mathscr{P}(X)$ are the same as *families* of subsets of X. Since $S = \mathscr{P}(X)$ contains the greatest element, namely X, and the smallest element, namely $\emptyset$, every subset of $\mathscr{P}(X)$ is bounded above and below.

The *union* of all members of a family $\mathscr{E}$,

$$\bigcup \mathscr{E} = \bigcup_{E \in \mathscr{E}} E := \{a \in X \mid a \in E \text{ for some } E \in \mathscr{E}\} \tag{10}$$

is the smallest subset of X which *contains every member* of family $\mathscr{E}$. Hence, $\sup \mathscr{E}$ exists and equals (10).

Dually, the *intersection* of all members of family $\mathscr{E}$,

$$\bigcap \mathscr{E} = \bigcap_{E \in \mathscr{E}} E := \{a \in X \mid a \in E \text{ for all } E \in \mathscr{E}\} \tag{11}$$

is the greatest subset of X which is *contained in every member* of family $\mathscr{E}$. Hence, $\inf \mathscr{E}$ exists and equals (11).

1.5.2

The power set is an example of a partially ordered set in which every subset (including the empty set) possesses both suppremum and infimum.

1.6 Completeness

1.6.1

Definition 1.8 *We say that a partially ordered set* $(S, \preccurlyeq)$ *has the* ***greatest-lower-bound property*** *if* $\inf E$ *exists for every subset* $E \subseteq S$ *which is nonempty and bounded below.*

Dually, we say that S *has the* ***least-upper-bound property*** *if* $\sup E$ *exists for subset* $E \subseteq S$ *which is nonempty and bounded above.*

1.6.2 An alternative terminology

Partially ordered sets with the greatest-lower-bound property are said to be $\inf$**-complete**, and those with the least-upper-bound property are said to be $\sup$**-complete**.

Lemma 1.9 *A partially ordered set* S *has the greatest-lower-bound property if and only if it has the least-upper-bound property.*

Proof. Suppose that S is $\inf$-complete. If $E \subseteq S$ is bounded above and nonempty, then the set of upper bounds, $U(E)$ is nonempty. Since

$$L(U(E)) \supseteq E \neq \emptyset$$

subset $U(E)$ is also bounded below. Then $\inf U(E)$ exists in view of our assumption about $(S, \preccurlyeq)$. But then it coincides with $\sup E$ in accordance with (9). This shows that S is $\sup$-complete.

The reverse implication,

$$\sup\text{-completeness} \Rightarrow \inf\text{-completeness}$$

follows by applying the already proven implication

$$\inf\text{-completeness} \Rightarrow \sup\text{-completeness}$$

to the opposite order on S. □

Since $\sup$- and $\inf$-completeness are equivalent we shall simply call such sets *complete*.

1.7 Lattices

1.7.1 Pre-lattices

Definition 1.10 *A partially ordered set* $(S, \preccurlyeq)$ *is called a* ***pre-lattice*** *if every nonempty finite subset* $E \subseteq S$ *has supremum and infimum.*

Exercise 14 *Show that* $(S, \preccurlyeq)$ *is a* ***pre-lattice*** *if and only if , for any* $s, t \in S$*, both* $\sup\{s, t\}$ *and* $\inf\{s, t\}$ *exist.*

1.7.2 Lattices

A partially ordered set $(S, \preccurlyeq)$ is called a **lattice** if every finite subset $E \subseteq S$, including $\emptyset \subseteq S$, has supremum and infimum.

1.7.3 Complete lattices

Complete partially ordered sets with the greatest and the smallest elements are the same as *complete lattices*. Note that in such sets every subset is bounded below and above.

For example, the totally ordered set of rational numbers, $(\mathbf{Q}, \leq)$, is a pre-lattice but not a lattice, and it is not complete.

The power set of an arbitrary set, $(\mathscr{P}(X), \subseteq)$, is an example of a complete lattice. A less obvious example is the subject of the next section.

1.8 Down-sets, up-sets

1.8.1 Down-sets

A subset L of a partially ordered set $(S, \preccurlyeq)$ is called a **down-set** if

$$\textit{for any } s \in L \textit{ and } s' \preccurlyeq s\textit{, also } s' \in L.$$

Exercise 15 *Show that the union and the intersection of any family* $\mathscr{L} \subseteq \mathscr{P}(X)$ *of down-sets is a down-set.*

Exercise 16 *Show that a down-set* L *is a down-interval if and only if* $\sup L$ *exists and belongs to* L*.*

1.8.2 Down-set closure of a subset

The family of down-sets containing a given subset $E \subseteq S$ is nonempty since $E \subseteq S$ and S is a down-set. It follows that the intersection of all down-sets L containing E,

$$\mathrm{Cl}^{\downarrow}(E) := \bigcap_{L \supseteq E} L,$$

is the *smallest* down-set containing E.

1.8.3 Down-set interior of a subset

The family of down-sets contained in a given subset $E \subseteq S$ is nonempty since $\emptyset \subseteq E$ and $\emptyset$ is a down-set. It follows that the union of all down-sets L contained in E,

$$\mathrm{Int}^{\downarrow}(E) := \bigcup_{L \subseteq E} L,$$

is the *greatest* down-set contained in E.

1.8.4

By definition, one has

$$\mathrm{Int}^{\downarrow}(E) \subseteq E \subseteq \mathrm{Cl}^{\downarrow}(E)$$

Exercise 17 *Show that $E \subseteq S$ is a down-set if and only if*

$$\mathrm{Int}^{\downarrow}(E) = E$$

if and only if

$$E = \mathrm{Cl}^{\downarrow}(E).$$

Exercise 18 *Any subset $E \subseteq S$ is contained in $\bigcup_{s \in E} \langle s]$. Show that E is a down-set if and only if*

$$E = \bigcup_{s \in E} \langle s].$$

1.8.5

Let E and F be two subsets of a partially ordered set $(S, \preccurlyeq)$. We may say that F *dominates* E *above*, and express this symbolically with $E \rightfishtail F$, if

for every $s \in E$ *there exists* $t \in F$ *such that* $s \preccurlyeq t$.

Exercise 19 *Show that* $U(E) \supseteq U(F)$ *whenever* $E \rightfishtail F$. *In particular*

$$\sup E \preccurlyeq \sup F$$

when both suprema exist.

Exercise 20 *Show that* $E \rightfishtail F$ *if and only if* $\mathrm{Cl}^{\downarrow}(E) \subseteq \mathrm{Cl}^{\downarrow}(F)$.

1.8.6 Cofinal pairs of subsets

We shall say that subsets E and F are **cofinal** if $E \rightfishtail F$ and $F \rightfishtail E$. For cofinal subsets $U(E) = U(F)$. In particular, $\sup E$ exists if and only if $\sup F$ exists and they are equal (cf. Ex. 19.

1.8.7

It follows that E and F are *cofinal* if and only if $\mathrm{Cl}^{\downarrow}(E) = \mathrm{Cl}^{\downarrow}(F)$.

1.8.8 Up-sets

Up-sets are defined, by duality, as down-sets for the opposite ordering $\preccurlyeq^{\mathrm{op}}$. In particular, a subset E is an up-set if and only if

$$E = \bigcup_{s \in E} [s\rangle.$$

One can also define the corresponding notions of the *up-closure*, $\mathrm{Cl}^{\uparrow}(E)$, and the *up-interior*, $\mathrm{Int}^{\uparrow}(E)$, of a subset E.

Exercise 21 *Define the dual concept*

F dominates E below

(*one can denote this fact by using notation* $F \leftfishtail E$).

1.8.9

If we replace $\preccurlyeq$ by the opposite order, $\preccurlyeq^{\mathrm{op}}$, we obtain another pair of relations between subsets of S:

$$E \mathrel{-\mkern-6mu 3}^{\mathrm{op}} F \qquad \text{and} \qquad F \mathrel{\varepsilon\mkern-6mu -}^{\mathrm{op}} E.$$

Note that

$$F \mathrel{\varepsilon\mkern-6mu -} E \quad \text{if and only if} \quad E \mathrel{-\mkern-6mu 3}^{\mathrm{op}} F \quad (\text{not } E \mathrel{-\mkern-6mu 3} F!).$$

1.8.10

The dual concept to a cofinal pair of subsets is a **coinitial** pair of subsets.

1.9 Partially ordered subsets

1.9.1

Any subset $S \subseteq T$ of a set ordered by a relation $\preccurlyeq$ can be regarded as a partially ordered set in its own right. One has to be cautioned, however, that S with $\preccurlyeq$ restricted to S, may have very different properties from the partially ordered set $(T, \preccurlyeq)$.

1.9.2

For a subset $E \subseteq S$, the sets of upper and lower bounds will generally depend on whether one considers E as a subset of S or T. In particular, E may be not bounded as a subset of S yet be bounded as a subset of T.

In order to avoid confusion, we shall often indicate in which partially ordered set we form the sets of upper and lower bounds by adding subscript S or T. Thus,

$$L_T(E), \quad U_T(E), \quad \inf_T E, \quad \sup_T E,$$

will denote the set of lower bounds, the set of upper bounds, the infimum, and the supremum, when E is viewed as a subset of T.

1.9.3

Note that

$$L_S(E) = L_T(E) \cap S \qquad \text{and} \qquad U_S(E) = U_T(E) \cap S.$$

1.9.4

Similary, for an element $s \in S$, we shall denote by $\langle s]_T$ the corresponding down-interval in T:

$$\langle s]_T = \{t \in T \mid t \preccurlyeq s\}.$$

and by $[s\langle_T$ the corresponding up-interval in T:

$$[s\rangle_T = \{t \in T \mid s \preccurlyeq t\}.$$

Note that

$$\langle s]_S = \langle s]_T \cap S \qquad \text{and} \qquad [s\rangle_S = [s\rangle_T \cap S.$$

Exercise 22 *Show that, for* $E \subseteq S$, *one has*

$$\sup{}_T E \preccurlyeq \sup{}_S E$$

whenever both suprema exist.

1.9.5

Dually, one has

$$\inf{}_S E \preccurlyeq \inf{}_T E$$

whenever both infima exist.

Lemma 1.11 *Given a subset* $E \subseteq S$, *suppose that* $\sup_T E$ *exists and belongs to* S. *Then also* $\sup_S E$ *exists and the two suprema are equal*

$$\sup{}_S E = \sup{}_T E.$$

Proof. The supremum of E in $(T, \preccurlyeq)$ exists if and only if $U_T(E)$ is an interval $[\eta\rangle_T$ for some $\eta \in T$. Moreover, $\sup_T E = \eta$. If $\eta \in S$, then

$$U_S(E) = U_T(E) \cap S = [\eta\rangle_T \cap S = [\eta\rangle_S.$$

In particular, $\sup_S E$ exists and equals η. $\square$

1.9.6

Dually, if $\inf_T E$ exists and belongs to S, then $\inf_S E$ exists and

$$\inf_S E = \inf_T E.$$

Exercise 23 *Given a subset* $E \subseteq S$*, suppose that* $\inf_T E$ *exists and equals* ε*. Show that*

$$L_S(E) = \langle \varepsilon]_T \cap S = \{s \in S \mid s \preccurlyeq \varepsilon\}.$$

1.9.7

Dually, if $\sup_T E$ exists and equals η, then

$$U_S(E) = [\eta\rangle_T \cap S = \{s \in S \mid \eta \preccurlyeq s\}.$$

Exercise 24 *Find examples of pairs* $E \subseteq S$ *of subsets of* $\mathbf{Q}$ *such that:*

(a) E *is unbounded above in* S *yet bounded in* $\mathbf{Q}$*;*

(b) E *is bounded in* S*, and* $\sup_{\mathbf{Q}} E$ *exists but* $\sup_S E$ *does not;*

(c) E *is bounded in* S*, and* $\sup_S E$ *exists but* $\sup_{\mathbf{Q}} E$ *does not;*

(d) both $\sup_S E$ *and* $\sup_{\mathbf{Q}} E$ *exist but* $\sup_S E \neq \sup_{\mathbf{Q}} E$*.*

1.10 Sublattices

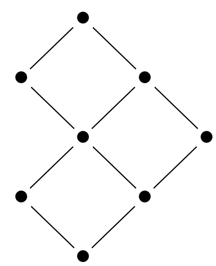
1.10.1

A partially ordered subset $(S, \preccurlyeq)$ of a lattice $(T, \preccurlyeq)$ is said to be a *sublattice* if infima and suprema of arbitrary **finite** subsets $E \subseteq S$ exist and **coincide** with the corresponding infima and suprema in $(T, \preccurlyeq)$,

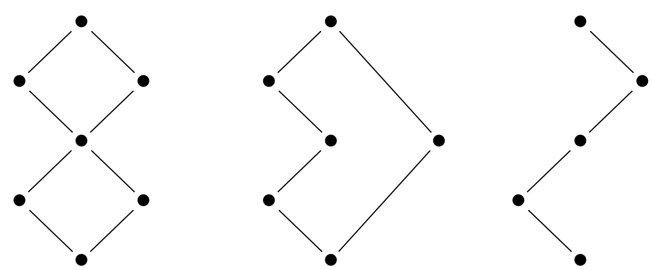
$$\inf_S E = \inf_T E \qquad \text{and} \qquad \sup_S E = \sup_T E.$$

1.10.2 An example

Consider an 8 element lattice $(T, \preccurlyeq)$ whose ordering is represented by the following diagram



Each of the following partially ordered subsets of $(T, \preccurlyeq)$ is a lattice.



Exercise 25 *Which one is a sublattice of $(T, \preccurlyeq)$? Which one is not? Explain your answers.*

1.10.3 Complete-sublattices

A partially ordered subset $(S, \preccurlyeq)$ of a complete lattice $(T, \preccurlyeq)$ is said to be a *sublattice* if infima and suprema of *all*, not just finite, subsets $E \subseteq S$ exist and *coincide* with the corresponding infima and suprema in $(T, \preccurlyeq)$.

1.10.4 An example: the family of all down-sets

According to Exercise 15, the family of all down-sets

$$(\mathscr{P}^{\downarrow}(S, \preccurlyeq), \subseteq)$$

in a partially ordered set $(S, \preccurlyeq)$ is a complete-sublattice of $(\mathscr{P}(S), \subseteq)$.

The same holds for the family of all up-sets

$$(\mathscr{P}^{\uparrow}(S, \preccurlyeq), \subseteq).$$

1.10.5 An (almost) example: the power set of a subset $X \subseteq Y$

If $X \subseteq Y$, then $(\mathscr{P}(X), \subseteq)$ is a complete lattice contained in the complete lattice $(\mathscr{P}(Y), \subseteq)$. The infima in $(\mathscr{P}(X), \subseteq)$ coincide with the infima in $(\mathscr{P}(Y), \subseteq)$, the suprema of all *nonempty* families $\mathscr{E} \subseteq \mathscr{P}(X)$ coincide with the suprema in $(\mathscr{P}(Y), \subseteq)$. In the case of the empty family, however,

$$\inf_{\mathscr{P}(X)} \emptyset = X \qquad \text{while} \qquad \inf_{\mathscr{P}(Y)} \emptyset = Y.$$

1.11 Density

1.11.1

Let $(S, \preccurlyeq)$ be a partially ordered subset of $(T, \preccurlyeq)$.

Lemma 1.12 *If an element $t \in T$ is the supremum of a subset E of S,*

$$t = \sup_T E,$$

then

$$t = \sup_T \big(\langle t]_T \cap S \big).$$

Proof. If $t \in T$ is an upper bound of $E \subseteq S$ in $(T, \preccurlyeq)$, then

$$E \subseteq \langle t]_T \cap S \subseteq \langle t]_T.$$

If $t = \sup_T E$, then

$$\sup_T E = \sup_T \langle t]_T$$

and the Sandwich Lemma for suprema, cf. 1.4.3, yields the assertion.

□

1.11.2

Dually, if $t \in T$ is the infimum of a subset E of S,

$$t = \inf_T E$$

then

$$t = \inf_T \big([t\rangle_T \cap S \big).$$

1.11.3 The sup- and inf-closure of a subset

The set

$$\bar{S}_T := \{ t \in T \mid t = \sup_T(\langle t]_T \cap S) \}$$

will be called the sup-*closure* of S in T.

Dually, the set

$$\underline{S}_T := \{ t \in T \mid t = \inf_T([t\rangle_T \cap S) \}$$

will be called the inf-*closure* of S in T.

1.11.4 sup- and inf-closed subsets

We shall say that $S \subseteq T$ is sup-*closed* if it coincides with its sup-closure,

$$S = \bar{S}_T.$$

Dually, we shall say that $S \subseteq T$ is inf-*closed* if it coincides with its inf-closure,

$$S = \underline{S}_T.$$

1.11.5

Let F be a subset of the sup-closure of S. Every element $t \in F$ is the supremum of a certain subset E_t of S,

$$t = \sup_T E_t.$$

If $u \in T$ is an upper bound of

$$E := \bigcup_{t \in F} E_t,$$

then it belongs to $U_T(E_t) = [t\rangle_T$. In particular, $t \preccurlyeq u$, for every $t \in F$, i.e., u is an upper bound of F. Since an upper bound of F is automatically an upper bound of every subset E_t, it is also an upper bound of their union E.

We conclude that the two sets of upper bounds coincide

$$U_T(F) = U_T(E).$$

In particular, if $u = \sup_T F$, then

$$[u\rangle_T = U_T(F) = U_T(E)$$

and $u = \sup_T E$.

We established the following result.

Proposition 1.13 *The* sup*-closure of a subset* $S \subseteq T$ *is* sup*-closed.*

1.11.6 $\sup$- and $\inf$-dense subsets

If $\bar{S}_T = T$, we shall say that ST is $\sup$-*dense* in T.

Dually, if $\underline{S}_T = T$, we shall say that ST is $\inf$-*dense* in T.

Theorem 1.14 *Let S be a $\sup$-dense subset of T and E be a subset of S. If*

$$\varepsilon = \inf_S E,$$

then the infimum of E in $(T, \preccurlyeq)$ exists and equals ε.

Dually, if S is an $\inf$-dense subset of T and

$$\eta = \sup_S E,$$

then the supremum of E in $(T, \preccurlyeq)$ exists and equals η.

1.11.7 Exactness

Suppose that a partially ordered subset $(S, \preccurlyeq)$ of $(T, \preccurlyeq)$ has the property

for any $E \subseteq S$, if $\sup_S E$ exists, then $\sup_T E$ exists and they are equal.

In this case, we shall say that the inclusion

$$(S, \preccurlyeq) \hookrightarrow (T, \preccurlyeq) \tag{12}$$

is $\sup$-*exact*.

1.11.8

Suppose $(S, \preccurlyeq)$ has the dual property,

for any $E \subseteq S$, if $\inf_S E$ exists, then $\inf_T E$ exists and they are equal.

In this case, we shall say that inclusion (12) is $\inf$-*exact*.

1.11.9

The following important statement is an immediate corollary of Theorem 1.14.

Corollary 1.15 *If S is a $\sup$-dense subset of $(T, \preccurlyeq)$, then inclusion* (12) *is* $\inf$-*exact.*

Dually, if S is $\inf$-*dense, then inclusion* (12) *is* $\sup$-*exact.*

Proof of Theorem 1.14. It suffices to prove the first assertion of the theorem. If $t \in T$ is a lower bound of E, then

$$\langle t]_T \subseteq L_T(E)$$

and, accordingly,

$$\langle t]_T \cap S \subseteq L_T(E) \cap S = L_S(E) = \langle \varepsilon]_S.$$

In other words, ε is an upper bound of the set

$$\langle t]_T \cap S.$$

By the density hypothesis, t is the least upper bound of that set, hence

$$t \preccurlyeq \varepsilon,$$

this way proving that ε is also the greatest lower bound of E in T. □

1.11.10 Examples

Suppose that a partially ordered set $(S, \preccurlyeq)$ is the union of three subsets $S = X \cup Y \cup Z$ such that

$$x \preccurlyeq y \quad \text{and} \quad x \preccurlyeq z \qquad \text{for any } x \in X,\ y \in Y, \text{ and } z \in Z,$$

no $y \in Y$ and $z \in Z$ are comparable, and neither Y nor Z possess the smallest element.

Let us extend the ordering relation to $T = S \cup \{v, \zeta\}$ by setting

$$x \prec v \prec y \qquad \text{for any } x \in X,\ y \in Y,$$

and

$$x \prec \zeta \prec z \qquad \text{for any } x \in X,\ z \in Z.$$

Note that v is not comparable with elements of $Z \cup \{\zeta\}$, nor ζ is comparable with elements of $Y \cup \{v\}$. Finally, denote by T' the subset $S \cup \{v\}$ of T.

a) $(S, \preccurlyeq)$ **as a subset of** $(T, \preccurlyeq)$**.** One has $L_T(Y) = \langle v]$ and $L_T(Z) = \langle \zeta]$. It follows that

$$v = \inf_T Y \qquad \text{and} \qquad \zeta = \inf_T Z,$$

and therefore S is inf-dense in T. In addition,

$$\langle v]_T \cap S = X = \langle \zeta]_T \cap S$$

but $v \neq \zeta$, they are not even comparable.

Exercise 26 *Show that neither v nor ζ equals* $\sup_T E$ *for any $E \subseteq S$. In particular, S is* inf*-dense in T but not* sup*-dense.*

b) $(S,\preccurlyeq)$ **as a subset of** $(T',\preccurlyeq)$**.** One has

$$v = \inf_{T'} Y$$

hence S is inf-dense in T' while $\inf_{T'} Z$ does not exist. Note that

$$L_S(Y) = \langle v]_{T'} \cap S = X = L_S(Z).$$

1.12 The tower $\mathscr{I}^{\downarrow}(S,\preccurlyeq) \subseteq \mathscr{L}(S,\preccurlyeq) \subseteq \mathscr{P}^{\downarrow}(S,\preccurlyeq) \subseteq \mathscr{P}(S)$

1.12.1

We shall consider three subsets of the power set $\mathscr{P}(S)$, the set of all down-intervals

$$\mathscr{I}^{\downarrow}(S,\preccurlyeq) := \big\{ \langle s] \mid s \in S \big\},$$

the set of the lower-bound-sets

$$\mathscr{L}(S,\preccurlyeq) := \big\{ L(E) \mid E \subseteq S \big\},$$

and the set of all down-sets

$$\mathscr{P}^{\downarrow}(S,\preccurlyeq) := \{ D \subseteq S \mid D \text{ is a down-set} \}.$$

1.12.2

Since the union and the intersection of any family of down-sets is a down-set, $\mathscr{P}^{\downarrow}(S,\preccurlyeq)$ is both sup- and inf-closed in $(P(S),\subseteq)$ and the inclusion

$$\big(\mathscr{P}^{\downarrow}(S,\preccurlyeq),\subseteq\big) \hookrightarrow (P(S),\subseteq)$$

is both sup- and inf-exact.

1.12.3

Since a subset of S is a down-set if and only if it is the union of a family of down-intervals, $\mathscr{P}^{\downarrow}(S,\preccurlyeq)$ is the sup-closure of $\mathscr{I}^{\downarrow}(S,\preccurlyeq)$ in $(P(S),\subseteq)$,

$$\mathscr{P}^{\downarrow}(S,\preccurlyeq) = \overline{\mathscr{I}^{\downarrow}(S,\preccurlyeq)}_{\mathscr{P}(S)}.$$

1.12.4

In particular, also $\mathscr{L}(S,\preccurlyeq)$ is sup-dense in $\mathscr{P}^{\downarrow}(S,\preccurlyeq)$ and, in view of Lemma 1.11, $\mathscr{I}^{\downarrow}(S,\preccurlyeq)$ is sup-dense in $\mathscr{L}(S,\preccurlyeq)$.

1.12.5

By Corollary 1.15, both inclusions

$$\big(\mathscr{I}^{\downarrow}(S,\preccurlyeq),\subseteq\big) \hookrightarrow \big(\mathscr{L}(S,\preccurlyeq),\subseteq\big) \hookrightarrow \big(\mathscr{P}^{\downarrow}(S,\preccurlyeq),\subseteq\big)$$

are inf-exact.

1.12.6

In the case of the inclusion

$$\big(\mathscr{L}(S,\preccurlyeq),\subseteq\big) \hookrightarrow \big(\mathscr{P}^{\downarrow}(S,\preccurlyeq),\subseteq\big)$$

this follows also from the fact that any subset of $\mathscr{L}(S,\preccurlyeq)$ is of the form

$$\{L(E) \mid E \in \mathscr{E}\}$$

for some family of subsets $\mathscr{E}$ of S and

$$L\left(\bigcup_{E\in\mathscr{E}} E\right) = \bigcap_{E\in\mathscr{E}} L(E). \tag{13}$$

for any family $\mathscr{E} \subseteq \mathscr{P}(S)$.

Exercise 27 *Prove equality* (13) *directly.*

1.12.7

Dually, one has

$$U\left(\bigcup_{E\in\mathscr{E}} E\right) = \bigcap_{E\in\mathscr{E}} U(E). \tag{14}$$

1.12.8

In the special case, equality (13) yields

$$L(E) = \bigcap_{e\in E} L\big(\{e\}\big) = \bigcap_{e\in E} \langle e] = \inf_{\mathscr{L}(S,\preccurlyeq)}\big\{\langle e] \mid e \in E\big\},$$

meaning that $\mathscr{I}^{\downarrow}(S,\preccurlyeq)$ is inf-dense in $\mathscr{L}(S,\preccurlyeq)$.

1.12.9

The dual assertion of Corollary 1.15 then yields sup-exactness of the inclusion

$$\big(\mathscr{I}^{\downarrow}(S,\preccurlyeq),\subseteq\big) \hookrightarrow \big(\mathscr{L}(S,\preccurlyeq),\subseteq\big).$$

1.12.10

Equality (13) has also another consequence: it implies that any subset of $\mathscr{L}(S,\preccurlyeq)$ has infimum in $\mathscr{L}(S,\preccurlyeq)$, i.e., $\big(\mathscr{L}(S,\preccurlyeq),\subseteq\big)$ is inf-complete.

Indeed,

$$\inf_{\mathscr{L}(S,\preccurlyeq)}\{L(E) \mid E\in\mathscr{E}\} = \bigcap\{L(E) \mid E\in\mathscr{E}\} = L\left(\bigcup_{E\in\mathscr{E}} E\right) \in \mathscr{L}(S,\preccurlyeq).$$

1.12.11

In view of Lemma 1.9, the partially ordered set $\big(\mathscr{L}(S,\preccurlyeq),\subseteq\big)$ is also sup-complete.

1.12.12

Since, for any subset E of S, one has

$$L(S)\subseteq L(E)\subseteq L(\varnothing)=S,$$

S is the greatest and $L(S)$ is the smallest element of $\mathscr{L}(S,\preccurlyeq)$. Note that

$$L(S)=\{s_0\},$$

where s_0 is the smallest element of S, when S is bounded below, and $L(S)$ is empty when S is not bounded below.

1.12.13

All together, we infer that $\big(\mathscr{L}(S,\preccurlyeq),\subseteq\big)$ is a complete lattice. However, it is *not* a complete-sublattice of the lattice of all down-sets $\big(\mathscr{P}^{\downarrow}(S,\preccurlyeq),\subseteq\big)$, if there is at least one down-set D *not* of the form $L(E)$ for some $E\subseteq S$.

Indeed,

$$D=\bigcup_{d\in D}\langle d] = \sup_{\mathscr{P}^{\downarrow}(S,\preccurlyeq)}\big\{\langle d] \mid d\in D\big\}$$

whereas

$$\sup_{\mathscr{L}(S,\preccurlyeq)}\big\{\langle d] \mid d\in D\big\}$$

coincides with the smallest element of $\mathscr{L}(S,\preccurlyeq)$ that contains D.

1.12.14 The LU-closure of a subset $E \subseteq S$

According to (7), members of $\mathscr{L}(S, \preccurlyeq)$ are precisely the subsets $B \subseteq S$ such that

$$LU(B) = B.$$

Thus, if a subset E is contained in a member B of $\mathscr{L}(S, \preccurlyeq)$, then

$$U(E) \supseteq U(B)$$

and, therefore,

$$E \subseteq LU(E) \subseteq LU(B) = B.$$

It follows that $LU(E)$ is the *smallest* member of $\mathscr{L}(S, \preccurlyeq)$ that contains E, i.e.,

$$LU(E) = \inf_{\mathscr{L}(S,\preccurlyeq)} \{B \in \mathscr{L}(S, \preccurlyeq) \mid E \subseteq B\}.$$

1.12.15

In particular, the supremum of any family of subsets $\mathscr{E} \subseteq \mathscr{L}(S, \preccurlyeq)$ coincides with the LU-closure of

$$\sup_{\mathscr{P}^{\downarrow}(S,\preccurlyeq)} \mathscr{E} = \bigcup \mathscr{E}.$$

1.12.16

This concludes our study of the tower of inclusions

$$\left(\mathscr{I}^{\downarrow}(S, \preccurlyeq), \subseteq\right) \hookrightarrow \left(\mathscr{L}(S, \preccurlyeq), \subseteq\right) \hookrightarrow \left(\mathscr{P}^{\downarrow}(S, \preccurlyeq), \subseteq\right) \hookrightarrow \left(\mathscr{P}(S), \subseteq\right). \quad (15)$$

The following theorem collects what we established.

Theorem 1.16 (*a*) *Every inclusion in* (15)*, is* inf*-exact.*

(*b*) *The first and the last inclusions are also* sup*-exact; the middle one is* sup*-exact only when* $\mathscr{L}(S, \preccurlyeq) = \mathscr{P}^{\downarrow}(S, \preccurlyeq)$*, i.e., when every down-set is a lower-bound-set.*

(*c*) $\mathscr{L}(S, \preccurlyeq)$, $\mathscr{P}^{\downarrow}(S, \preccurlyeq)$ *and* $\mathscr{P}(S)$ *are complete lattices.*

(*d*) $\mathscr{P}^{\downarrow}(S, \preccurlyeq)$ *is a complete-sublattice of* $\mathscr{P}(S)$*; in particular, it is both* sup*- and* inf*-closed in* $\mathscr{P}(S)$.

(*e*) $\mathscr{L}(S, \preccurlyeq)$ *is* inf*-closed in* $\mathscr{P}(S)$ *and is* sup*-dense in* $\mathscr{P}^{\downarrow}(S, \preccurlyeq)$.

(*f*) $\mathscr{I}^{\downarrow}(S,\preccurlyeq)$ *is both* sup- *and* inf-*dense in* $\mathscr{L}(S,\preccurlyeq)$.

(*g*) $\mathscr{P}^{\downarrow}(S,\preccurlyeq)$ *is the* sup-*closure of* $\mathscr{I}^{\downarrow}(S,\preccurlyeq)$ *in* $\mathscr{P}(S)$,

$$\mathscr{P}^{\downarrow}(S,\preccurlyeq) \;=\; \overline{\mathscr{I}^{\downarrow}(S,\preccurlyeq)}_{\mathscr{P}(S)}.$$

(*h*) $\mathscr{L}(S,\preccurlyeq)$ *is the* inf-*closure of* $\mathscr{I}^{\downarrow}(S,\preccurlyeq)$ *in* $\mathscr{P}(S)$,

$$\mathscr{L}(S,\preccurlyeq) \;=\; \underline{\mathscr{I}^{\downarrow}(S,\preccurlyeq)}_{\mathscr{P}(S)}.$$

□

2 Mappings between partially ordered sets

2.1 Morphisms

2.1.1

Definition 2.1 *Given two partially ordered sets* $(S,\preccurlyeq)$ *and* $(S',\preccurlyeq')$, *a mapping* $\phi\colon S\longrightarrow S'$ *which preserves order,*

$$\text{if} \quad s \preccurlyeq t, \quad \text{then} \quad \phi(s) \preccurlyeq' \phi(t) \qquad (s,t \in S),$$

is said to be a ***morphism*** $(S,\preccurlyeq)\longrightarrow(S',\preccurlyeq')$.

2.1.2 The opposite morphism

If ϕ is a morphism $(S,\preccurlyeq) \longrightarrow (S',\preccurlyeq')$, then it is also a morphism

$$(S,\preccurlyeq^{\mathrm{op}}) \longrightarrow (S',(\preccurlyeq')^{\mathrm{op}}).$$

To distinguish the two, we shall denote the latter by ϕ^{op}.

Definition 2.2 *A morphism* $\phi\colon (S,\preccurlyeq)\longrightarrow(S',\preccurlyeq')$ *is said to be an* ***isomorphism*** *if it has an inverse, i.e., if there is a morphism* $\psi\colon (S',\preccurlyeq')\longrightarrow(S,\preccurlyeq)$ *such that* $\phi \circ g = \mathrm{id}_{S'}$ *and* $g \circ f = \mathrm{id}_S$.

2.1.3 Order embeddings

Definition 2.3 *A mapping* $\iota\colon S\longrightarrow S'$ *is said to be an* ***order embedding***,

$$(S,\preccurlyeq) \hookrightarrow (S',\preccurlyeq'), \tag{16}$$

if it satisfies a stronger condition

$$s \preccurlyeq t \quad \textit{if and only if} \quad \iota(s) \preccurlyeq' \iota(t) \qquad (s, t \in S).$$

Exercise 28 *Show that an order embedding is injective.*

Exercise 29 *Show that an order embedding,* (16)*, is an isomorphism onto its image,* $(\iota(S), \preccurlyeq')$.

2.2 Morphisms between the power sets associated with $f\colon X \longrightarrow Y$

2.2.1 Notation

Consider a mapping between arbitrary sets

$$f\colon X \longrightarrow Y.$$

We shall adopt the following notation throughout. By A we shall denote an arbitrary subset of the source set X, and by B we shall denote an arbitrary subset of the target set Y.

Similarly, by $\mathscr{A}$ we shall denote an arbitrary family of subsets of X, and by $\mathscr{B}$ an arbitrary family of subsets of Y.

2.2.2

In order to study the structure of a mapping, we introduce a number of related concepts. Each of them becomes an indispensible tool of modern Mathematics.

2.2.3 The fiber at $y \in Y$

The *fiber* of f at $y \in Y$ is the subset of the source-set

$$\mathrm{Fib}_y f := \{x \in X \mid f(x) = y\}$$

2.2.4 The preimage of a subset $B \subseteq Y$

The *preimage* (under f) of a subset $B \subseteq Y$ is the subset of the source-set

$$f^*B := \{x \in X \mid f(x) \in B\} = \bigcup_{y \in B} \mathrm{Fib}_y f.$$

Note that the fiber of f at y is the preimage of $B = \{y\}$.

2.2.5 The image of a subset $A \subseteq X$

The *image* (under f) of a subset $A \subseteq X$ is the subset of the target set

$$f_* A := \{y \in Y \mid y = f(x) \text{ for some } x \in A\},$$

Exercise 30 *Show that*

$$f_* A = \{y \in Y \mid \mathrm{Fib}_y f \cap A \neq \emptyset\}.$$

2.2.6 The fiber-image of a subset $A \subseteq X$

The *fiber-image* (under f) of a subset $A \subseteq X$ is the subset of the target set

$$f_! A := \{y \in Y \mid \mathrm{Fib}_y f \subseteq A\}.$$

Exercise 31 *Show that*

$$X \setminus f^* B = f^*(Y \setminus B) \tag{17}$$

while

$$Y \setminus f_* A = f_!(X \setminus A) \qquad \textit{and} \qquad Y \setminus f_! A = f_*(X \setminus A) \tag{18}$$

2.2.7 Notation

Undoubtedly, you must have encountered before the concepts of the 'image' and the 'preimage' when studying mapping between sets. The usual notation for the image of A under f,

$$f(A),$$

and for the preimage of B,

$$f^{-1}(B),$$

have, however, a serious disadvantage when one deals not just with subsets but also with *families* of subsets. This is why we adopt the notation that is unambiguous.

2.2.8 The associated power-set mappings

The image, preimage and fiber-image preserve the $\subseteq$ relation between subsets, thus they yield morphisms between the corresponding power sets

$$\begin{array}{c} (\mathscr{P}(X), \subseteq) \\ f_* \Big\downarrow \quad \Big\uparrow f^* \quad \Big\downarrow f_! \\ (\mathscr{P}(Y), \subseteq) \end{array}$$

2.2.9

If we denote by $(\ )^c$ the complement-of-the-subset operation on the power set, then identity (17) can be expressed in as the commutativity of the square

$$\begin{array}{ccc} \mathscr{P}(X) & \xleftrightarrow{(\ )^c} & \mathscr{P}(X) \\ {\scriptstyle f^*}\Big\uparrow & & \Big\uparrow{\scriptstyle f^*} \\ \mathscr{P}(Y) & \xleftrightarrow{(\ )^c} & \mathscr{P}(Y) \end{array} \tag{19}$$

whereas the pair of identities (18) is equivalent to the commutativity of the square

$$\begin{array}{ccc} \mathscr{P}(X) & \xleftrightarrow{(\ )^c} & \mathscr{P}(X) \\ {\scriptstyle f_*}\Big\downarrow & & \Big\downarrow{\scriptstyle f_!} \\ \mathscr{P}(Y) & \xleftrightarrow{(\ )^c} & \mathscr{P}(Y) \end{array} \tag{20}$$

Note that the vertical arrows in diagrams (19) and (20) are *morphisms* while the horizontal ones are *anti-morphisms*, i.e., they *reverse* the order.

2.2.10 Fundamental properties of the associated power-set mappings

We are ready to make several fundamental observations about the associated power-set mappings. They form a sequence of exercises. You *cannot* skip doing these exercises if you intend to continue beyond this point.

Exercise 32 *Show that the preimage preserves the unions*

$$f^*\left(\bigcup_{B\in\mathscr{B}} B\right) = \bigcup_{B\in\mathscr{B}} f^*B$$

and the intersections

$$f^*\left(\bigcap_{B\in\mathscr{B}} B\right) = \bigcap_{B\in\mathscr{B}} f^*B$$

Exercise 33 *Show that the image preserves the unions*

$$f_*\left(\bigcup_{A\in\mathscr{A}} A\right) = \bigcup_{A\in\mathscr{A}} f_*A$$

while, for intersections, one has only the inclusion

$$f_*\left(\bigcap_{A\in\mathscr{A}} A\right) \subseteq \bigcap_{A\in\mathscr{A}} f_*A. \tag{21}$$

Explain why symbol $\subseteq$ *in* (21) *cannot be, in general, replaced by the equality sign* $=$. *Provide an example when both sides of* (21) *are not equal.*

Exercise 34 *Show that the fiber-image preserves the intersections*

$$f_!\left(\bigcap_{A\in\mathscr{A}} A\right) = \bigcap_{A\in\mathscr{A}} f_!A$$

while, for unions, one has only the inclusion

$$\bigcup_{A\in\mathscr{A}} f_!A \subseteq f_!\left(\bigcup_{A\in\mathscr{A}} A\right). \tag{22}$$

Explain why symbol $\subseteq$ *in* (22) *cannot be, in general, replaced by the equality sign* $=$. *Provide an example when both sides of* (22) *are not equal.*

2.2.11 The adjunction properties of the associated power-set mappings

The next two properties are particularly important. We remind you that $A \subseteq X$ and $B \subseteq Y$ stand for arbitrary subsets of X and Y, respectively.

Exercise 35 *Show that*

$$A \subseteq f^*B \qquad \text{if and only if} \qquad f_*A \subseteq B.$$

Exercise 36 *Show that*

$$f^* B \subseteq A \qquad \text{if and only if} \qquad B \subseteq f_! A.$$

2.3 A characterization of morphisms

2.3.1

Let $\phi\colon S \longrightarrow S'$ be a mapping between the underlying sets of partially ordered sets. Note that, for any $t \in S$,

$$\phi^* \langle \phi(t)] = \{ s \in S \mid \phi(s) \preccurlyeq' \phi(t) \}.$$

The preimage of the down-interval $\langle \phi(t)]$ is a down-set if and only if

$$\forall_{s \in S} \big(s \preccurlyeq t \;\Rightarrow\; \phi(s) \preccurlyeq' \phi(t) \big).$$

It follows that ϕ is a morphism of partially ordered sets if and only if

for any $t \in S$, the preimage of $\langle \phi(t)]$ is a down-set.

Exercise 37 *Suppose $\phi\colon S \longrightarrow S'$ is a morphism of partially ordered sets*

$$(S, \preccurlyeq) \longrightarrow (S', \preccurlyeq') \tag{23}$$

and $D' \subseteq S'$ be a down-set in the target set. Show that its preimage $\phi^* D'$ is a down-set in the source set.

2.3.2

We obtain the following characterization of morphisms between partially ordered sets.

Proposition 2.4 *Let $\phi\colon S \longrightarrow S'$ be a mapping between the underlying sets of partially ordered sets. The following conditions are equivalent:*

(a) *for any $t \in S$, the preimage $\phi^* \langle \phi(t)]$ is a down-set;*

(b) *for any $s' \in S'$, the preimage $\phi^* \langle s']$ is a down-set;*

(c) *for any down-set $D' \subseteq S'$, the preimage $\phi^* D'$ is a down-set;*

(d) *ϕ is a morphism of partially ordered sets.*

2.3.3

Implication $(a) \Rightarrow (d)$ was established in 2.3.1, implication $(d) \Rightarrow (c)$ is the subject of Exercise 37, implications $(c) \Rightarrow (b) \Rightarrow (a)$ are trivial.

2.3.4

Note that ϕ is a morphism (23) if and only if it is a morphism between the opposite partially ordered sets

$$(S, \preccurlyeq^{\mathrm{op}}) \longrightarrow (S', \preccurlyeq'^{\mathrm{op}}).$$

In particular, one can replace in Proposition 2.4 down-intervals and down-sets by up-intervals and, respectively, up-sets.

2.4 Exact mappings between partially ordered sets

2.4.1 sup-exact mappings

We say that a mapping $\phi\colon S \longrightarrow S'$ is sup*-exact* if ϕ preserves the suprema, i.e., if it has the following property

$$\forall_{E \subseteq S} \big(\sup E \text{ exists} \quad \Rightarrow \quad \sup \phi_* E \text{ exists and } \sup \phi_* E = \phi(\sup E)\big).$$

2.4.2 inf-exact morphisms

Exercise 38 *State the dual definition of an* inf***-exact*** *mapping.*

2.4.3 Exact mappings are necessarily morphisms

Given a pair of elements $s \preccurlyeq t$ in S, consider the set $E \coloneqq \{s, t\}$. One has

$$s = \inf E \qquad \text{and} \qquad t = \sup E.$$

If ϕ is inf-exact, then $\phi(s)$ is a lower bound of $\phi_* E = \{\phi(s), \phi(t)\}$, hence

$$\phi(s) \preccurlyeq' \phi(t). \tag{24}$$

Dually, if ϕ is sup-exact, then $\phi(t)$ is an upper bound, which yields the same inequality (24). In particular, inf- and sup-exact mappings are automatically morphisms of partially ordered sets.

2.4.4

General morphisms are neither inf- nor sup-exact, they however preserve the greatest and the least elements of subsets.

Exercise 39 *Let ϕ be a morphism $(S,\preccurlyeq)\longrightarrow(S',\preccurlyeq')$. Show that, for any subset $E\subseteq S$, one has*

$$\min f(E) = f(\min E) \qquad \textit{and} \qquad \max f(E) = f(\max E)$$

whenever $\min E$ or $\max E$ exists.

Exercise 40 *Let ϕ be a morphism $(S,\preccurlyeq)\longrightarrow(S',\preccurlyeq')$. Show that, for any nonempty subset $E\subseteq S$, one has*

$$\phi_* U(E) \subseteq U(\phi_* E) \qquad \textit{and} \qquad \phi_* L(E) \subseteq L(\phi_* E).$$

Deduce from this the inequalities

$$\phi(\inf E) \preccurlyeq' \inf \phi(E) \preccurlyeq' \sup \phi(E) \preccurlyeq' \phi(\sup E)$$

whenever the corresponding infima and suprema exist.

2.5 The lower- and the upper-bound-set morphisms

2.5.1

Given a partially ordered set $(S,\preccurlyeq)$, the correspondences

$$E \longmapsto L(U) \qquad \text{and} \qquad E \longmapsto U(E)$$

define morphisms

$$(\mathscr{P}(S),\subseteq) \longrightarrow (\mathscr{P}(S),\supseteq).$$

We shall refer to them as the *lower* and, respectively, the *upper-bound-set morphisms*. We shall denote them L and, respectively, U.

2.5.2

In view of equalities (13) and (14), both the lower and the upper-bound-set morphisms are sup-exact.[1]

[1] Note that the supremum of a family $\mathscr{E}$ of subsets in $(\mathscr{P}(S),\supseteq)$ is its intersection $\bigcap\mathscr{E}$.

2.6 The canonical embedding $(S, \preccurlyeq) \hookrightarrow (\mathscr{P}(S), \subseteq)$

2.6.1

For any pair of elements s and t in a partially ordered set $(S, \preccurlyeq)$, one has

$$s \preccurlyeq t \qquad \text{if and only if} \qquad \langle s] \subseteq \langle t].$$

Thus, the correspondence

$$\langle\ ]\colon S \longrightarrow \mathscr{P}(S), \qquad s \longmapsto \langle s], \tag{25}$$

is an order embedding of $(S, \preccurlyeq)$ into $(\mathscr{P}(S), \subseteq)$.

2.6.2

Embedding (25) identifies $(S, \preccurlyeq)$ with the partially ordered set of down-intervals $(\mathscr{I}^{\downarrow}(S), \subseteq)$.

2.6.3 The canonical completion of a partially ordered set

The canonical embedding of $(S, \preccurlyeq)$ into $(\mathscr{P}(S), \subseteq)$ has precisely the same behavior, regading the suprema and the infima, as the inclusion of $\mathscr{I}^{\downarrow}(S)$ into $\mathscr{P}(S)$. Thus it is inf-exact and, usually, not sup-exact. It is however both inf- and sup-exact if we consider it as an embedding of

$$(S, \preccurlyeq) \hookrightarrow \big(\mathscr{L}(S, \preccurlyeq), \subseteq\big). \tag{26}$$

Embedding (26) provides an explicitly constructed *completion* of $(S, \preccurlyeq)$, i.e., an *exact* order embedding onto a *dense* subset of a complete lattice.

2.7 Preimages of intervals

2.7.1

According to Proposition 2.4, a mapping $\phi\colon S \longrightarrow S'$ between the underlying sets of partially ordered sets is a morphism precisely when the preimages of down-intervals of $(S', \preccurlyeq')$ are down-sets of $(S, \preccurlyeq)$.

2.7.2

Let ϕ be a morphism, s' be an element of S' and t be an element of S.

2.7.3

Note that

$$\phi(t) \preccurlyeq' s' \iff \phi_*\langle t] \subseteq \langle s'] \iff \langle t] \subseteq \phi^*\langle s'].$$

2.7.4

Note that

$$t \in U(\phi^*\langle s']) \iff \phi^*\langle s'] \subseteq \langle t].$$

2.7.5

Thus,

$$\phi^*\langle s'] = \langle t] \iff t \text{ is an upper bound for } \phi^*\langle s'] \text{ and } \phi(t) \preccurlyeq' s'.$$

If $\phi^*\langle s'] = \langle t]$, then t is, of course, also the greatest element of $\phi^*\langle s']$. In particular, $\sup \phi^*\langle s']$ exists and equals t.

2.7.6

For an element $s \in$, the inequality

$$s \preccurlyeq t$$

describes membership $s \in \langle t]$ whereas the inequality

$$\phi(s) \preccurlyeq' s'$$

describes membership in $s \in \phi^*\langle t]$. Therefore equality $\phi^*\langle s'] = \langle t]$ can be expressed as the statement

$$\forall_{s \in S} \left(s \preccurlyeq t \iff \phi(s) \preccurlyeq' s'\right).$$

The following lemma collects the observations we made.

Lemma 2.5 *The following statements are equivalent*

(*a*) $\phi^*\langle s'] = \langle t]$;

(*b*) $\forall_{s \in S} \left(s \preccurlyeq t \iff \phi(s) \preccurlyeq' s'\right)$;

(*c*) *t is an upper bound for $\phi^*\langle s']$ and $\phi(t) \preccurlyeq' s'$*,

(*d*) *$t = \sup \phi^*\langle s']$ and $\phi(t) \preccurlyeq' s'$*;

(*e*) *$t = \max \phi^*\langle s']$ and $\phi(t) \preccurlyeq' s'$*.

2.7.7

Let $E = \phi^*\langle s']$. Since

$$\phi_* E = \phi_*\phi^*\langle s'] \subseteq \langle s'],$$

element s' is an upper bound of

$$\phi_* E = \phi_*\phi^*\langle s'].$$

If both $\sup E$ and $\sup \phi_* E$ exist and

$$\phi(\sup E) = \sup \phi_* E, \tag{27}$$

then

$$\phi(\sup E) \preccurlyeq' s'$$

and, according to Lemma 2.5, E is a down-interval. This yields the following corollary of Lemma 2.5.

Corollary 2.6 *Let $E = \phi^*\langle s']$. If both $\sup E$ and $\sup \phi_* E$ exist and equality* (27) *holds, then $E = \phi^*\langle s']$ is a down-interval.*

2.7.8

Let E be an arbitrary subset of S. Note that

$$s' \in U(\phi_* E) \iff \phi_* E \subseteq \langle s'] \iff E \subseteq \phi^*\langle s'].$$

If $\phi^*\langle s'] = \langle t]$, this yields

$$s' \in U(\phi_* E) \iff \phi_* E \subseteq \langle s'] \iff E \subseteq \langle t] \iff t \in U(E),$$

i.e.,

$$s' \text{ is an upper bound of } \phi_* E \iff t \text{ is an upper bound of } E.$$

2.7.9

If $\eta = \sup E$, then $\eta \preccurlyeq t$. Hence,

$$\phi(\eta) \preccurlyeq' \phi(t) \preccurlyeq' s'.$$

Since ϕ is a morphism, $\phi(\eta)$ is itself an upper bound of $\phi_* E$. Let us record the observations we made in our next lemma.

Lemma 2.7 *If $\eta = \sup E$, then $\phi(\eta)$ is an upper bound of $\phi_* E$. Moreover, for any upper bound s' of $\phi_* E$, such that $\phi^* \langle s']$ is a down-interval, one has*

$$\phi(\eta) \preccurlyeq' s'.$$

.

2.7.10 Residuated mappings

Mappings $\phi \colon S \longrightarrow S'$ which have the property that the preimage of any down-interval $\langle s']$ in $(S', \preccurlyeq')$ is a down-interval in $(S, \preccurlyeq)$ are said to be *residuated*. Residuated mappings are automatically morphisms of the corresponding partially ordered sets.

Corollary 2.8 *Every residuated mapping is* sup*-exact.*

2.7.11

In view of Corollary 2.6, under the additional hypothesis that the preimages of down-intervals of $(S', \preccurlyeq')$ have suprema in $(S, \preccurlyeq)$, the reverse implication holds.

Proposition 2.9 *The following statements are equivalent*

(*a*) *a mapping ϕ is residuated;*

(*b*) *a mapping ϕ is* sup*-exact and the preimages of down-intervals of $(S', \preccurlyeq')$ have suprema in $(S, \preccurlyeq)$.*

2.7.12

In particular, ϕ preserves suprema precisely when ϕ^* preserves the family of down-intervals, whenever $(S, \preccurlyeq)$ is a complete lattice.

2.7.13

By replacing everywhere down-intervals by up-intervals, suprema by infima, and sup-exact by inf-exact, we obtain the dual versions of the above results.

2.7.14 Residual mappings

In theory of ordered sets, a mapping ϕ is said to be *residual*, if the preimage of any up-interval is an up-interval.

2.7.15 The residual of a residuated mapping

A residuated mapping $\phi\colon S \longrightarrow S'$ defines a mapping $\psi\colon S' \longrightarrow S$ by setting

$$\psi(s') := t \qquad \text{where} \qquad \phi^*\langle s'] = \langle t].$$

Lemma 2.5 yields the following statement

$$\forall_{\substack{s \in S \\ s' \in S'}} \big(s \preccurlyeq \psi(s') \iff \phi(s) \preccurlyeq' s'\big). \tag{28}$$

2.7.16

Note that

$$\forall_{s' \in S'} \big(s \preccurlyeq \psi(s') \iff \phi(s) \preccurlyeq' s'\big).$$

expresses the fact that $\psi^*[s\rangle$ is the up-interval $[\phi s\rangle$. In particular, (28) means that the preimage under ψ of any up-interval of $(S, \preccurlyeq)$ is an up-interval of $(S', \preccurlyeq')$. In other words, $\psi\colon S' \longrightarrow S$ is a residual mapping. In theory of ordered sets, it is referred to as *the residual* of ϕ.

2.7.17 Galois connections

A pair of mappings

$$\begin{array}{c} S \\ \phi \downarrow \uparrow \psi \\ S' \end{array}$$

satisfying condition (28) is called a **Galois connection** between partially ordered sets $(S, \preccurlyeq)$ and $(S', \preccurlyeq')$.

2.7.18 Terminology

If (ϕ, ψ) forms a Galois connection, ϕ is referred to as the *lower* (or, *left*) *adjoint* of ψ, and ψ is called the *upper* (or, *right*) *adjoint* of ϕ. This reflects the fact that ϕ occurs on the "lower", i.e., left, side of one of the two inequalities while ψ occurs on the "upper", i.e., right, side of the other inequality.

Exercise 41 *Let (ϕ, ψ) be a Galois connection between $(S, \preccurlyeq)$ and $(S', \preccurlyeq')$, and (v, χ) be a Galois connection between $(S', \preccurlyeq')$ and $(S'', \preccurlyeq'')$. Show that $(v \circ \phi, \psi \circ \chi)$ is a Galois connection between $(S, \preccurlyeq)$ and $(S'', \preccurlyeq'')$.*

2.7.19 Duality between residuated and residual mappings

A pair (ϕ, ψ) is a Galois connection between $(S, \preccurlyeq)$ and $(S', \preccurlyeq')$ if and only if (ψ, ϕ) is a Galois connection between $(S', (\preccurlyeq')^{\mathrm{op}})$ and $(S, \preccurlyeq^{\mathrm{op}})$. Reversing simultaneously the orderings on S and S' exchanges the roles of the residual and the residuated mapping.

Exercise 42 *Show that*

$$\mathrm{id}_S \preccurlyeq \psi \circ \phi \qquad \textit{and} \qquad \phi \circ \psi \preccurlyeq' \mathrm{id}_{S'}$$

Exercise 43 *Show that if ϕ is an isomorphism between $(S, \preccurlyeq)$ and $(S', \preccurlyeq')$, then (ϕ, ϕ^{-1}) is a Galois connection between these sets.*

Exercise 44 *Show that if (ϕ, ψ) is a Galois connection between $(S, \preccurlyeq)$ and $(S', \preccurlyeq')$, and (ψ, ϕ) is a Galois connection between $(S', \preccurlyeq')$ and $(S, \preccurlyeq)$, then $\psi = \phi^{-1}$. In particular, ϕ and ψ are isomorphisms of partially ordered sets.*

Exercise 45 *Show that if (ϕ, ψ) is a Galois connection, then mapping ϕ is residuated and ψ is the residual of ϕ.*

2.7.20

By combining 2.7.15 with Exercise 45, we obtain the following proposition.

Proposition 2.10 *A mapping $\phi\colon S \longrightarrow S'$ is residuated if and only if there exists $\psi\colon S' \longrightarrow S$ such that (ϕ, ψ) is a Galois connection. In particular, there is a natural correspondence*

$$\left\{ \begin{array}{c} \textit{residuated mappings} \\ \phi\colon (S, \preccurlyeq) \longrightarrow (S', \preccurlyeq') \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Galois connections } (\phi, \psi) \\ \textit{between } (S, \preccurlyeq) \textit{ and } (S', \preccurlyeq') \end{array} \right\}.$$

2.7.21

Dually, there is a natural correspondence

$$\left\{\begin{matrix} \textit{Galois connections } (\phi, \psi) \\ \textit{between } (S, \preccurlyeq) \textit{ and } (S', \preccurlyeq') \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{residual mappings} \\ \psi\colon (S', \preccurlyeq') \longrightarrow (S, \preccurlyeq) \end{matrix}\right\}.$$

Proposition 2.11 *Let* (ϕ, ψ) *be a Galois connection between partially ordered sets* $(S, \preccurlyeq)$ *and* $(S', \preccurlyeq')$*. Then, for any* $E' \subseteq S'$*,*

$$\textit{sets } \ \psi_* L(E') \ \textit{ and } \ L(\psi_* E') \textit{ are cofinal} \tag{29}$$

and, for any $E \subseteq S$*,*

$$\textit{sets } \ \phi_* U(E) \textit{ and } \ U(\phi_* E) \textit{ are coinitial.} \tag{30}$$

Proof. Let $s \in L(g(E'))$. Since ψ is a morphism from $(S', \preccurlyeq')$ to $(S, \preccurlyeq)$, one one has $\phi(s) \in L(E')$. Noting that $g(\phi(s)) \in L(g(E'))$ and combining this observation with inequality $s \preccurlyeq g(\phi(s))$ shows that the set $g(L(E'))$ dominates the set $L(g(E'))$ from above. Since the latter set contains the former, the two sets are cofinal.

Statement (30) is statement (29) for the opposite ordering. □

3 Galois connections between power-sets

3.1 Multivalued functions

3.1.1 Multimaps $\varphi\colon X \multimap Y$

We shall think of a *multivalued function* from a set X to a set Y as a function $\varphi\colon X \longrightarrow \mathscr{P}(Y)$. We shall refer to it as a *multimap* and use the notation $\varphi\colon X \multimap Y$.

3.1.2 The canonical multimap $\iota_X\colon X \multimap X$

The canonical embedding

$$X \longrightarrow \mathscr{P}(X), \qquad x \longmapsto \{x\} \qquad (x \in X)$$

defines a multimap $\iota_X\colon X \multimap X$.

3.1.3 The opposite multimap $\varphi^{\mathbf{op}}\colon Y \multimap X$

Given a multimap $\varphi\colon X \multimap Y$, the *opposite* multimap $\varphi^{\mathrm{op}}\colon Y \multimap X$ is defined by

$$\varphi^{\mathrm{op}}(y) := \{x \in X \mid y \in \varphi(x)\}.$$

Exercise 46 *Show that*

$$\left(\varphi^{\mathrm{op}}\right)^{\mathrm{op}} = \varphi.$$

3.2 The Galois connection associated with a multimap

3.2.1 $\varphi_\bullet$ and $\varphi^\bullet$

A multimap $\varphi\colon X \multimap Y$ induces a pair of morphisms

$$\begin{array}{c} (\mathscr{P}(X), \subseteq) \\ \varphi_\bullet \Big\downarrow \quad \Big\uparrow \varphi^\bullet \\ (P(Y), \subseteq) \end{array}$$

where

$$\varphi_\bullet(A) := \bigcup_{x \in A} \varphi(x) = \{y \in Y \mid y \in \phi(x) \text{ for some } x \in A\} \qquad (A \subseteq X)$$

and

$$\varphi^\bullet(B) := \{x \in X \mid \varphi(x) \subseteq B\} \qquad (B \subseteq Y).$$

3.2.2

We shall refer to $\varphi_\bullet$ as the *direct image map* and to $\varphi^\bullet$ as the *inverse image map*, induced by φ.

Exercise 47 *Show that*

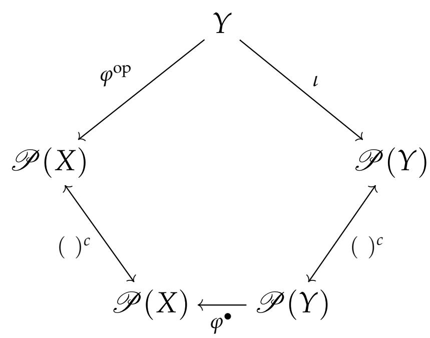
$$\varphi_\bullet\varnothing = \varnothing \qquad \textit{and} \qquad \varphi^\bullet Y = X.$$

Exercise 48 *Show that*

$$\varphi^\bullet\left(\{y\}^c\right) = \{x \in X \mid y \notin \varphi(x)\} = \left(\varphi^{\mathrm{op}}\right)^c. \tag{31}$$

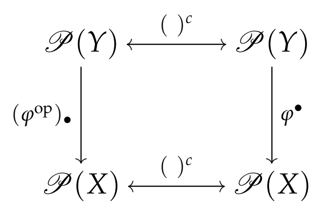
3.2.3

Equality (31) can be expressed as the commutativity of the following pentagon diagram



(32)

Exercise 49 *Show that the square*



commutes.

Exercise 50 *Show that*

$$Y = \bigcup_{x \in X} \varphi(x) \qquad \textit{and} \qquad \forall_{x \neq x'}\ \varphi(x) \cap \varphi(x') = \emptyset \tag{33}$$

if and only if

$$\varphi(x) = \mathrm{Fib}_x g$$

for a certain function $g \colon Y \longrightarrow X$.

3.2.4

In other words, the double condition (33) characterizes "the fiber-of-a-function" multimaps.

3.2.5

Note that, for any $A \subseteq X$ and $B \subseteq Y$,

$$A \subseteq \varphi^{\bullet} B \iff \forall_{x \in A} \{\varphi(x) \subseteq B\} \iff \varphi_{\bullet} A \subseteq B.$$

In other words, $(\varphi_{\bullet}, \varphi^{\bullet})$ is a Galois connection between $(\mathscr{P}(X), \subseteq)$ and $(\mathscr{P}(Y), \subseteq)$. We shall refer to it as *the Galois connection associated with* φ.

3.2.6

In particular, the direct image map is sup-exact while the inverse image is inf-exact.

3.2.7

Vice-versa, any sup-exact morphism

$$F\colon (\mathscr{P}(X), \subseteq) \longrightarrow (\mathscr{P}(Y), \subseteq)$$

is of the form $F = \varphi_{\bullet}$ for a unique multimap φ.

Indeed, any $A \subseteq X$ is the union of the family

$$\mathscr{A} := \iota_* A = \big\{\{x\} \mid x \in A\big\}.$$

Hence,

$$F(A) = F\Big(\bigcup \mathscr{A}\Big) = \bigcup_{x \in A} F\big(\{x\}\big) = \varphi_{\bullet} A$$

where φ is the multimap $X \multimap Y$ represented by the function

$$X \longrightarrow \mathscr{P}(Y), \qquad x \longmapsto F\big(\{x\}\big) \qquad (x \in X).$$

3.2.8

Similarly, any inf-exact morphism

$$G\colon \mathscr{P}(Y, \subseteq) \longrightarrow (\mathscr{P}(X), \subseteq)$$

is of the form $G = \varphi^{\bullet}$ for a unique multimap φ.

Indeed, G forms a Galois connection (F, G) with an appropriate sup-exact morphism F and the latter coincides with $\varphi_{\bullet}$ for a unique multimap

φ. Accordingly, $G = \varphi^\bullet$. The commutative pentagon diagram (32) then yields

$$G\big(\{y\}^c\big)^c = \varphi^{\text{op}}(y)$$

which allows one to express $\varphi^{\text{op}}(y)$ directly in terms of G evaluated on the complement of the singletn set $\{y\}$.

Exercise 51 *Show that, if $G = \varphi^\bullet$, then*

$$\varphi(x) = \big\{y \in Y \mid y \notin G(\{y\}^c)\big\}.$$

3.2.9

We record the results of our investigation in following proposition.

Proposition 3.1 *Every Galois connection (F, G) between $(\mathscr{P}(X), \subseteq)$ and $(\mathscr{P}(Y), \subseteq)$ is of the form $(\varphi_\bullet, \varphi^\bullet)$ for a unique multimap $\varphi\colon X \multimap Y$. One has*

$$\begin{aligned} \varphi(x) &= F\big(\{x\}\big) \\ &= \big\{y \in Y \mid y \notin G(\{y\}^c)\big\} \end{aligned}$$

and

$$\varphi^{\text{op}}(y) = \big(G(\{y\}^c)\big)^c.$$

□

3.2.10 (*Conjunction* and *Implication* form a Galois connection

Given a subset P of an arbitrary set X, let

$$F\colon A \longmapsto A \cap P \qquad \text{and} \qquad G\colon B \longmapsto P \Rightarrow B \qquad (A, B \subseteq X)$$

where

$$P \Rightarrow B := P^c \cup B.$$

Note that membership in $A \cap P$ is expressed as the *conjunction* of two statements

$$x \in A \textit{ and } x \in P$$

while membership in $P \Rightarrow B$ is expressed as the *implication* statement

$$\textit{if } x \in P, \textit{ then } x \in B.$$

Exercise 52 *Show that (F, G) is a Galois connection on $(\mathscr{P}(X), \subseteq)$.*

3.2.11

The above property of the operations $(\ )\cap P$ and $P\Rightarrow(\ )$ on $(\mathscr{P}(X),\subseteq)$ is interpreted in Logic of Sentences as:

Implication is a right-adjoint operation to Conjunction.

3.2.12 The canonical embedding $\langle\ ]$ viewed as a multimap $S \multimap S$

The canonical embedding, (25), of a partially ordered set $(S,\preccurlyeq)$ into its power set $(\mathscr{P}(S),\subseteq)$ defines a multimap $\langle\ ]\colon S \multimap S$.

Exercise 53 *Show that the direct and the inverse image maps associated with $\langle\ ]\colon S \multimap S$ coincide with the down-closure and, respectively, down-interior operations,*

$$\langle\ ]_\bullet A = \mathrm{Cl}^{\downarrow}(A) \qquad \text{and} \qquad \langle\ ]^\bullet B = \mathrm{Int}^{\downarrow}(B).$$

3.2.13 Composition of multimaps

Given multimaps $\chi\colon Y \multimap Z$ and $\varphi\colon X \multimap Y$, their composition is the multimap $X \multimap Z$, represented by the composite function

$$\chi_\bullet \circ \varphi :\ X \longrightarrow \mathscr{P}(Z).$$

We shall denote it $\chi \diamond \varphi$.

Exercise 54 *Show that*

$$\iota_Y \diamond \varphi = \varphi = \varphi \diamond \iota_X.$$

Exercise 55 *Show that $\diamond$ is associative, i.e.,*

$$(\psi \diamond \chi) \diamond \varphi = \psi \diamond (\chi \diamond \varphi).$$

3.3 Functions as special multimaps

3.3.1 The multimap associated with a function $f\colon X \longrightarrow Y$

Given a function $f\colon X \longrightarrow Y$, the composition $\iota_Y \circ f$ defines the multimap

$$X \longrightarrow \mathscr{P}(Y), \qquad x \longmapsto \{f(x)\} \qquad (x \in X).$$

We shall refer to it as the *multimap associated with f*.

3.3.2

A multimap $\varphi\colon X \multimap Y$ is associated with a function $X \longrightarrow Y$ if and only if $\varphi(x)$ is a singleton set for each $x \in X$,

$$\forall_{x \in X}\ |\varphi(x)| = 1. \tag{34}$$

Injectivity of ι_Y implies that the function f such that $\varphi = \iota_Y \circ f$ is unique.

3.3.3

This allows us to identify functions $X \longrightarrow Y$ with multimaps $X \multimap Y$ satisfying condition (34). In particular, we may informally refer to such multimaps as 'functions'.

Exercise 56 *Show that*

$$f_* = (\iota \circ f)_\bullet \qquad \text{and} \qquad f^* = (\iota \circ f)^\bullet.$$

Exercise 57 *Show that*

$$(\iota \circ f)^{\mathrm{op}}(y) = \mathrm{Fib}_y f.$$

Exercise 58 *Show that*

$$\iota \circ (g \circ f) = (\iota \circ g) \diamond (\iota \circ f).$$

3.3.4

In particular, composition of functions corresponds to composition of multimaps.

3.4 Exact morphisms between power sets

3.4.1

Acording to Proposition 3.1, an exact morphism

$$F\colon (\mathscr{P}(X), \subseteq) \longrightarrow (\mathscr{P}(Y), \subseteq)$$

equals $\varphi_\bullet$, for some multimap $\varphi\colon X \multimap Y$, and also equals $\chi^\bullet$, for some multimap $\chi\colon Y \multimap X$.

3.4.2

If $\varphi_\bullet$ is inf-exact, then

$$\varnothing = \varphi_\bullet \varnothing = \varphi_\bullet\big(\{x\} \cap \{x'\}\big) = \varphi(x) \cap \varphi(x')$$

whenever $x \neq x'$.

3.4.3

If $\chi_\bullet$ is sup-exact, then

$$Y = \chi^\bullet X = \chi^\bullet\left(\bigcup_{x \in X} \{x\}\right) = \bigcup_{x \in X} \chi^\bullet \{x\} = \bigcup_{x \in X} \{y \in Y \mid \chi(y) \subseteq \{x\}\} = \big\{y \in Y \,\big|\, |\chi(y)| \leq 1\big\}. \tag{35}$$

3.4.4

In view of

$$\varnothing = \varphi_\bullet \varnothing = \chi^\bullet \varnothing = \{y \in Y \mid \chi(y) = \varnothing\}, \tag{36}$$

$\chi(y) \neq \varnothing$ for every $y \in Y$.

3.4.5

By combining (36) with (35), we obtain

$$Y = \big\{y \in Y \,\big|\, |\chi(y)| = 1\big\},$$

i.e., χ is a function $g\colon Y \longrightarrow X$. More precisely, $\chi = \iota \circ g$.

3.4.6

Alternatively, we could observe that

$$Y = \chi^\bullet X = \chi^\bullet\left(\bigcup_{x \in X} \{x\}\right) = \varphi_\bullet\left(\bigcup_{x \in X} \{x\}\right) = \bigcup_{x \in X} \varphi(x),$$

i.e., Y is the union of disjoint subsets $\varphi(x)$ which, in view of 3.2.4, means that φ is the fiber-of-a-function multimap.

3.4.7

We arrive at the following characterization of exact morphisms between $(\mathscr{P}(X), \subseteq)$ and $(\mathscr{P}(Y), \subseteq)$.

Proposition 3.2 *A morphism* $F\colon (\mathscr{P}(X), \subseteq) \longrightarrow (\mathscr{P}(Y), \subseteq)$ *is exact if and only if* $F = g^*$ *for a certain function* $g\colon Y \longrightarrow X$.