Lifting stably dominated types

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February 26, 2014

1 Introduction

Fix some C-minimal theory T expanding ACVF. We've shown that for global invariant types, the following are equivalent:

- Generic stability
- Orthogonality to Γ
- Stable domination. (In fact, after naming parameters, one gets domination by the generic type of k^n for some n.)

Let $X \to Y$ be a definable surjection of interpretable sets. We will show that the induced map $\hat{X} \to \hat{Y}$ on the stable completions (space of generically stable types) is surjective. Using this, we can mimic the proof from Hrushovski and Loeser that $\hat{X}$ is *strictly pro-definable*, not just pro-definable (as it would be in any NIP theory).

Throughout $\mathbb{U}^{eq}$ will be a monster model of T^{eq}, K will be the home sort, k will be the residue sort, and Γ will be the value group. By default, acl and dcl will mean acl^{eq} and dcl^{eq}.

2 Lifting stably dominated types

First we recall some assorted facts about chaining together definable, generically stable, and algebraic types.

Remark 2.1. *Recall that if M is a model and $\mathrm{tp}(a/M)$ is definable (resp. generically stable), then $\mathrm{tp}(a'/M)$ is definable (resp. generically stable), for any $a' \in \mathrm{acl}(aM)$. If $\mathrm{tp}(a/M)$ is definable (resp. generically stable) and $\mathrm{tp}(b/aM)$ has an aM-definable extension (resp.* is *generically stable), then $\mathrm{tp}(ab/M)$ is definable (resp. is generically stable[1]).*

[1] I don't remember whether transitivity of generic stability holds in general [actually, it does], but note that in our case, it suffices to show that $\Gamma(Mab) = \Gamma(M)$. Since $\mathrm{tp}(a/M)$ is generically stable, $\Gamma(Ma) = \Gamma(M)$. Then since p is orthogonal to Γ, $\Gamma(Mab) = \Gamma(Ma)$.

Lemma 2.2. *Let M be a model (an elementary substructure of $\mathbb{U}$) and suppose* $\mathrm{tp}(a/M)$ *is generically stable. Let $D \subset K^1$ be aM-definable and non-empty. Then there is some $b \in D$ such that* $\mathrm{tp}(ab/M)$ *is generically stable.*

Proof. Let c be a code for one of the swiss cheeses in the canonical decomposition of D into swiss cheeses. Then $c \in \mathrm{acl}(aM)$, so by Remark 2.1, $\mathrm{tp}(ac/M)$ is generically stable. Replacing a with ac and D with the swiss cheese coded by c, we may assume that D is a swiss cheese.

Suppose that there is a closed ball B' which is $\mathrm{acl}(aM)$-definable, such that the generic type of B' is in D. Then $\mathrm{tp}(\ulcorner B' \urcorner a/M)$ is generically stable (by Remark 2.1) and if b realizes the generic type of B', then $\mathrm{tp}(b \ulcorner B' \urcorner a/M)$ is generically stable, by Remark 2.1 again, so we are done.

So assume that there is no closed ball B' which is $\mathrm{acl}(aM)$-definable, such that the generic type of B' is in D.

As a swiss cheese, we can write D as $B_0 \setminus (B_1 \cup \cdots \cup B_n)$, where each B_i is a ball (open or closed, possibly K or a singleton), where $n \geq 0$, and where things are nested in the correct way. If B_0 is a closed ball (or a singleton), then the generic type of B_0 is in D, and B_0 is aM-definable, contradicting our assumption.

Next suppose that B_0 is all of K. If $n = 0$, then $D = K$, and we can take some $b \in D \cap \mathrm{dcl}(M)$ because M is a model. Otherwise, let B be the smallest closed ball containing $B_1, \ldots, B_n$. Since M is a model, there is some $\delta < 0$ in $\Gamma(M)$. Let B' be the closed ball around B of radius δ plus the radius of B. Then the generic type of B' is in D, and B' is algebraic (in fact, definable) over aM, a contradiction.

We are left with the case that B_0 is an open ball. If $n = 1$, then we can take a closed ball between B_0 and B_1 whose (valuative) radius is halfway between the radii of B_0 and B_1 (or δ plus the radius of B_0, in the case where B_1 is a singleton). If $n > 1$, let B be the smallest closed ball containing $B_1, \ldots, B_n$. Then B is strictly smaller than B_0, and strictly bigger than each of the B_i's, so its generic type is in D, a contradiction.

So we are left with the case that $D = B_0$ is an open ball. Suppose that there is some $\mathrm{acl}(aM)$-definable subball $B \subset B_0$ (open or closed or singleton) of D. Then as above, we can take a closed ball halfway between B and B_0 (or use δ), and find an $\mathrm{acl}(aM)$-definable closed subball of D, a contradiction.

So we may assume that not only is D an open ball, but that no proper subball of D is $\mathrm{acl}(aM)$-definable. From the swiss cheese decomposition, it follows that the only $\mathrm{acl}(aM)$-definable subsets of D are D and $\emptyset$.

Let b realize the generic type of D. This type is aM-definable, so $\mathrm{tp}(ab/M)$ is definable. It remains to show that $\mathrm{tp}(ab/M)$ is orthogonal to Γ. Suppose not. Then there is some aM-definable function $f : K^1 \to \Gamma^1$ such that $f(b) \notin \Gamma(M)$.

The set $f(D)$ is a definable subset of Γ. By o-minimality of Γ, it is a finite union of intervals. The endpoints of these intervals are in $\Gamma(Ma) = \Gamma(M)$. If $f(D)$ is finite, then $f(b) \in f(D) \subset \Gamma(M)$, a contradiction. So $f(D)$ contains an infinite interval. As M is a model, this infinite interval contains at least three M-definable points $\gamma_1, \ldots, \gamma_3$. Then $f^{-1}(\gamma_1)$ and $f^{-1}(\gamma_2)$ and $f^{-1}(\gamma_3)$ are three distinct aM-definable subsets of D, a contradiction. ☐

(By being more careful, one could perhaps arrange that $\text{tp}(ab/M)$ is definable over the algebraic closure of whatever $\text{tp}(a/M)$ is definable over…)

Lemma 2.3. *Let M be a model (an elementary substructure of $\mathbb{U}$) and suppose* $\text{tp}(a/M)$ *is generically stable. Let $D \subset K^n$ be aM-definable and non-empty. Then there is some $b \in D$ such that* $\text{tp}(ab/M)$ *is generically stable.*

Proof. By induction on n. The $n = 1$ case was Lemma 2.2. Suppose $n > 1$. Let π be the projection $K^n \to K^{n-1}$ coordinates. By induction there is some $b_0 \in \pi(D)$ such that $\text{tp}(ab_0/M)$ is generically stable. Then $\pi^{-1}(b_0) \cap D = \{b_0\} \times D'$ for some non-empty ab_0M-definable $D' \subset K^1$. By Lemma 2.2, there is some $c \in D'$ such that $\text{tp}(ab_0c/M)$ is generically stable. Take $b = (b_0, c)$. ☐

Theorem 2.4. *Let $f : X \to Y$ be a definable surjection. Let p be a generically stable type in Y. Then there is a generically stable type q in X such that $f_*q = p$. In fact, if p, f, X, and Y are defined over a model M, we can take q to be defined over the same model.*

Proof. Let X' be some (M-)definable subset of K^n such that there is an (M-)definable surjection from X' onto X. If we can lift p to X' (along the composition $X' \to X \to Y$), then we can certainly lift it to X. Replacing X with X', we may assume that X is a definable subset of K^n for some n.

Let a realize $p|M$. Let D be $f^{-1}(a)$, a non-empty aM-definable set. By Lemma 2.3, there is some $b \in D$ such that $\text{tp}(b/M)$ is generically stable. Take q to be the canonical global extension of $\text{tp}(b/M)$. ☐

3 Strict pro-definability

In any NIP theory, one has uniform definability of generically stable types. That is, for every formula $\phi(x; y)$ there is some formula $\psi(y; z)$ such that for every generically stable type p, $(d_p x)\phi(x; y)$ is of the form $\psi(y; c)$ for some $c \in M$. This follows from the fact that generically stable types are definable by voting in Morley sequences. So in fact we can take $\psi(y; z)$ to be of the form

$$\psi(y; z_1, \ldots, z_N) \leftrightarrow \bigvee_{S \subset \{1,\ldots,2N-1\},\ |S| \geq N} \bigwedge_{i \in S} \psi(z_i; y)$$

where N is something like one or two times the alternation number of $\phi(x; y)$.

From this it follows that if X is a definable set in *any* NIP theory, the space $\hat{X}$ of generically stable types in X is pro-definable (in T^{eq}).

Proof. For each formula $\phi(x; y)$, choose some formula $\psi_\phi(y; z)$ which gives uniform definitions. Since we're working in T^{eq}, we may arrange that $\psi_\phi(\mathbb{U}; z) \neq \psi_\phi(\mathbb{U}; z')$ for $z \neq z'$. Let V_ϕ be the sort where z lives. So if p is a generically stable type, then the code for the ϕ-definition of p is an element of V_ϕ.

So we have a map from generically stable types to $\prod_{\phi \in \mathcal{L}} V_\phi$. It remains to show that the range of this map is $*$-definable.

A tuple $\langle c_\phi \rangle_{\phi \in \mathcal{L}}$ will define a consistent global type if and only if it defines a type which is finitely satisfiable. This can be expressed as follows: for every $\phi_1(x; y_1), \ldots, \phi_n(x; y_n)$ in the language, the following must hold:

$$\mathbb{U} \models \forall y_1, \ldots, y_n \exists x : \bigwedge_{i=1}^{n} (\phi_i(x; y_i) \leftrightarrow \psi_{\phi_i}(y_i; c_{\phi_i}))$$

So the set of tuples $\langle c_\phi \rangle_\phi$ for which we get a consistent type is $*$-definable.

Now a definable type p is generically stable iff $p(x_1) \otimes p(x_2)$. Equivalently, for every formula $\phi(x_1; x_2; y)$,

$$(d_p x_1)(d_p x_2)\phi(x_1, x_2; y) = (d_p x_2)(d_p x_1)\phi(x_1; x_2; y). \tag{1}$$

We can express this as a condition in terms of the c_ϕ's. Let $\phi_1(x_2; x_1, y)$ be $\phi(x_1; x_2; y)$. Let $\phi_2(x_1; y, z)$ be $\psi_{\phi_1}(x_1, y; z)$. Let $\phi_3(y, z, w)$ be $\psi_{\phi_2}(y, z, w)$. If p is the definable type defined by the c_ϕ's, then

$$(d_p x_1)(d_p x_2)\phi(x_1, x_2; y) = (d_p x_1)(d_p x_2)\phi_1(x_2; x_1, y) = (d_p x_1)\phi_2(x_1; y, c_{\phi_1}) = \phi_3(y, c_{\phi_1}, c_{\phi_2}).$$

Similarly, we can find some formulas ϕ_4, ϕ_5, ϕ_6 such that

$$(d_p x_2)(d_p x_1)\phi(x_1, x_2; y) = \phi_4(y, c_{\phi_5}, c_{\phi_6}).$$

Then (1) is basically just the assertion that

$$\forall y : \phi_3(y, c_{\phi_1}, c_{\phi_2}) \leftrightarrow \phi_4(y, c_{\phi_5}, c_{\phi_6})$$

Doing this for each $\phi(x_1, x_2, y)$ in the language, we get a small family of first order statements about the c_ϕ whose conjunction is equivalent to the condition that the resulting type is generically stable. □

If X is a definable set, the space $\hat{X}$ of generically stable types in X is pro-definable.

The part where we must use C-minimality is to show that we get *strict* pro-definability. For reasons explained by Hrushovski and Loeser (or Kamensky), it suffices to show that the image of the generically stable types in V_ϕ is definable (rather than just type-definable) for each ϕ. Or perhaps one needs to show this for products $V_{\phi_1} \times \cdots \times V_{\phi_n}$. But note that the map

$$\hat{X} \to \prod_{i=1}^{n} V_{\phi_i}$$

factors through $\hat{X} \to V_\psi$ for some ψ, so we really can reduce to the case of one V_ϕ. This will work if ψ has the property that every ϕ_i-formula for $1 \leq i \leq n$ is a ψ-formula, because then the ψ-definition of a definable type will determine the ϕ_i-definition, for each i. It is well-known how to find such a ψ.

So we are reduced to proving the following:

Theorem 3.1. *(Assuming we are in a C-minimal expansion of ACVF.) Let X be a definable set. Let $\phi(x;y)$ be a formula. The set of ϕ-definitions of generically stable types in X is a small union of definable sets. (Since it is also type-definable, this implies that it is definable.)*

Proof. We are basically going to use the same argument as Hrushovski and Loeser, except using the previous section instead of metastability (which may or may not work in this setting).

Let $\psi(y;z)$ be the formula that uniformly ϕ-defines generically stable types. Let g be the generic type of k, so that $g^{\otimes n}$ is the generic type of k^n.

For each definable map $f : X \times K^m \to k^n$, and $w \in K^m$, let f_w denote the map $f(-,w) : X \to k^n$. Let W_f be the (definable) set of $w \in K^m$ such that $(d_{g^{\otimes n}} s)(s \in f_w(X))$, i.e., such that $f_w(X)$ hits the generic type of k^n. For $w \in W_f$, let $Z_{f,w}$ be the set of z such that

$$\forall y d_{g^{\otimes n}} s \forall (x \in f_w^{-1}(s)) : (\phi(x;y) \leftrightarrow \psi(y;z))$$

Note that $Z_{f,w}$ is definable uniformly in w. So the union of all the $Z_{f,w}$'s is a small union of definable sets.

We claim that this union is exactly the set of c such that $\psi(y;c)$ is the ϕ-definition of a generically stable type.

Suppose first that $\psi(y;c)$ is the ϕ-definition of some generically stable type $p(x)$. Since p is generically stable, there is some set of parameters C over which p is defined, and some C-definable map $f_0 : X \to k^n$ such that $f_{0,*}p$ is $g^{\otimes n}$ and p is "dominated" along f_0.

That is, $(p|C)(x) \cup g^{\otimes n}(f_0(x)) \vdash p(x)$.

Claim 3.2. *There is some finite subtype $\Sigma(x)$ of $(p|C)(x)$ such that $\Sigma(x) \cup g^{\otimes n}(f_0(x))$ implies the restriction of p to a ϕ-type.*

Proof. For each subtype $\Sigma(x)$ of $p|C$, let S_Σ denote the set of b such that

$$\Sigma(x) \cup g^{\otimes n}(f_0(x)) \vdash \phi(x;b)$$

and let S'_Σ denote the set of b such that

$$\Sigma(x) \cup g^{\otimes n}(f_0(x)) \vdash \phi'(x;b).$$

The set of formulas in $g^{\otimes n}(f_0(x))$ is ind-definable because $g^{\otimes n}$ is a definable type. So each of S_Σ and S'_Σ is small union of definable sets. Note that $S_{p|C} = \psi(\mathbb{U};c)$ and $S'_{p|C} = \psi(\mathbb{U};c)$. In particular, $S_{p|C}$ and $S'_{p|C}$ are definable. By the most basic form of compactness,

$$S_{p|C} = \bigcup_{\Sigma \subset_f p|C} S_\Sigma$$

$$S'_{p|C} = \bigcup_{\Sigma \subset_f p|C} S'_\Sigma$$

By saturation of the monster model, it follows that $S_{p|C} = S_\Sigma$ and $S_{p|C} = S'_\Sigma$ for some $\Sigma \subset_f p|C$. Then for every b, if $\phi(x;b) \in p(x)$, then $b \in S_{p|C} = S_\Sigma$, so $\Sigma(x) \cup g^{\otimes n}(f_0(x)) \vdash \phi(x;b)$. And similarly, if $\neg\phi(x;b) \in p(x)$, then $\Sigma(x) \cup g^{\otimes n}(f_0(x)) \vdash \neg\phi(x;b)$. So $\Sigma(x) \cup g^{\otimes n}(f_0(x))$ implies the restriction of p to a ϕ-type. □

Let f be f_0 on $\Sigma(\mathbb{U})$, and $0 \in k^n$ off of $\Sigma(\mathbb{U})$. Since Σ is a finite type, f is still a (C-)definable function. If $a \models p|C$, then $f(a) = f_0(a) \models g^{\otimes n}|C$. So $f(X)$ still hits the generic type of k^n.

Write f as f_w. We claim that $c \in Z_{f,w}$. Let s realize $g^{\otimes n}|\mathbb{U}$, outside the monster. Suppose $a \in f^{-1}(s)$. Then $\Sigma(a)$ holds, by definition of f. Also, $f(a) = f_0(a)$ realizes $g^{\otimes n}$. By the claim, the ϕ-type of a over $\mathbb{U}$ is the restriction of p to a ϕ-type. That is, for every $b \in \mathbb{U}$, $\phi(a; b)$ holds if and only if $\psi(b; c)$ holds.

So we have shown that

$$\forall (y \in \mathbb{U}) \forall x \in f^{-1}(s) : (\phi(x; y) \leftrightarrow \psi(y; c))$$

Since $\mathrm{tp}(s/\mathbb{U}) = g^{\otimes n}$, this is equivalent to saying

$$\forall (y \in \mathbb{U})(d_{g^{\otimes n}} s) \forall x \in f^{-1}(s) : \phi(x; y) \leftrightarrow \psi(y; c)$$

This means that $c \in Z_{f,w}$, by definition of $Z_{f,w}$.

Conversely, suppose that $c \in Z_{f,w}$ for some f and $w \in W_f$. Then $g^{\otimes n}$ is in $f(X)$, hence is an element of the stable completion of $f(X)$. By the previous section, there is a generically stable type p in X such that $f_* p = g^{\otimes n}$. Let C be a set over which everything so far is defined.

We claim that $\psi(y; c)$ is the ϕ-definition of p. Let b be arbitrary, and let a realize $p|bC$. Then $f(a)$ realizes $g^{\otimes n}|bC$. As $c \in Z_{f,w}$, we know that

$$\models d_{g^{\otimes n}} s \forall x \in f^{-1}(s) : \phi(x, b) \leftrightarrow \psi(b; c).$$

Everything inside the $d_{g^{\otimes} n} s$ is bC-definable, and $f(a)$ realizes $g^{\otimes n}|bC$, so we can take $s = f(a)$, yielding

$$\forall x \in f^{-1}(f(a)) : \phi(x, b) \leftrightarrow \psi(b; c)$$

In particular, taking $x = a$, we see that $\phi(a, b) \iff \psi(b; c)$. As b was arbitrary, $\psi(-; c)$ is the ϕ-definition of p. $\square$