The so-called Stabilizer Theorem

Will Johnson

February 21, 2015

1 Introduction

Last time, we left off with the statement of the following theorem, misleadingly called the "stabilizer theorem" for historical reasons.

Theorem 1.1 (Hrushovski's Theorem 3.5). *Let G be a definable group, X a definable subset, and $\tilde{G}$ be the $\vee$-definable group generated by X. Suppose all this data is defined over a model M. Suppose μ is an M-invariant translation-invariant ideal on subsets of $\tilde{G}$. Let q be a wide type over M, contained in $\tilde{G}$. Identify q with its $\wedge$-definable set of realizations. Suppose*

> *(F) there exist two realizations $a, b \models q$ with $a \mathop{\smile\hskip-0.9em\vert\hskip0.5em}_M b$ and $b \mathop{\smile\hskip-0.9em\vert\hskip0.5em}_M a$.*

Then there is a wide $\wedge$-definable subgroup $S < G$ defined over M, with the following properties:

- *$S = (q^{-1}q)^2$. In particular, S is the group generated by $q^{-1}q$.*
- *$qq^{-1}q$ is a coset of S*
- *$S \lhd \tilde{G}$.*
- *(Some technical/motivational condition: $S \setminus q^{-1}q$ is contained in a small union of non-wide definable sets.)*

In fact, S is $\tilde{G}^{00}_M$, the smallest $\wedge$-definable subgroup of $\tilde{G}$ of bounded index defined over M.

There are various weakenings of the hypotheses for which the theorem can still be proven. We'll disregard these.

The "motivation" for the theorem is that this is supposed to look like something that happens in stable theories: if G is a stable group, and q is a generic complete type in G, then $q^{-1}q$ is a group, in fact, it's $\mathrm{Stab}(q)$.

We can also think of this theorem as something akin to Zilber's indecomposability theorem, because it asserts that some set generates a ($\wedge$-)definable subgroup, rather than an ind-definable one as one would a priori expect.

1.1 Why this matters

As I understand it, the point of the theorem is that it guarantees that $\tilde{G}^{00}_M$ exists, i.e., that there is *any* $\wedge$-definable subgroup of bounded index. As we discussed last time, subgroups of bounded index are the same thing as wide subgroups, or generic subgroups.

For example, we get

Corollary 1.2 (Hrushovski's Corollary 3.6)**.** *Let μ be an invariant S1-ideal on definable subsets of $\tilde{G}$, invariant under translations by elements of $\tilde{G}$. Then there is a wide $\wedge$-definable subgroup S of $\tilde{G}$, with $\tilde{G}/S$ bounded. (And S is of the form $(q^{-1}q)^2$, for some complete type over some model M.)*

(The proof of this corollary boils down to figuring out how to satisfy condition (F). This is where we previously used skolemization and Erdos-Rado. See Hrushovski's Lemma 2.16)

Once we have a subgroup of bounded index, we can form the locally compact group $\tilde{G}/\tilde{G}^{00}_M$, and begin applying the structure theory of locally compact groups.

This ends up being useful in the asymptotic setting, where μ comes from the counting measure. Usually, G is some ambient infinite group, X is some finite set (or pseudo-finite set, in the limit), and $\tilde{G}$ is the group generated by X. Relatively speaking, it doesn't take much work to deduce the following from Hrushovski's corollary 3.6

Corollary 1.3 (Hrushovski's Corollary 1.2)**.** *For any $k, \ell, m \in \mathbb{N}$, for some $p < 1$ and $K \in \mathbb{N}$ we have the following statement: Let G be a group, X_0 be a finite subset, $X = X_0^{-1}X_0$ and suppose $|X_0X| \leq k|X_0|$. Also assume that with probability p, an ℓ-tuple $(a_1, \ldots, a_\ell) \in X^\ell$ satisfies $|a_1^X \cdots a_\ell^X| \geq |X|/m$. Then there is a subgroup S of G, with $S \subseteq X^2$, such that X is contained in $\leq K$ cosets of S.*

2 Tools

The following machinery from previous sections will be used.

First, some basic properties of forking and invariant types:

- All types over M can be extended to global M-invariant types.
- Global A-invariant types don't fork over A.
- In particular, if p and q are global A-invariant types and $(a, b) \models p \otimes q$ then $a \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.75em\raise-0.3ex\hbox{$|$}}_A b$.
- Left transitivity of forking: if $b \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.75em\raise-0.3ex\hbox{$|$}}_A c$ and $a \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.75em\raise-0.3ex\hbox{$|$}}_{Ab} c$, then $ab \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.75em\raise-0.3ex\hbox{$|$}}_A c$.

Also, there's the relation of wideness to forking:

Lemma 2.1. *If μ is an A-invariant S1 ideal, any wide type doesn't fork over A.*

Proof. It suffices to show that any formula $\phi(x;b)$ which divides over A is in the ideal. By definition of dividing, there's an A-indiscernible sequence b_i in $\mathrm{tp}(b/A)$ with $\bigwedge_i \phi(x;b_i)$ inconsistent. Suppose for the sake of contradiction that $\mu(\phi(x;b_i)) > 0$. Let m be maximal such that

$$\mu\left(\bigwedge_{i=1}^{m} \phi(x;b_i)\right) > 0.$$

Then if

$$X_n = \bigwedge_{i=1}^{m-1} \phi(x;b_i) \wedge \phi(x;b_{m+n}),$$

the X_n violate the S1 property, because $\mu(X_n) > 0$ but $\mu(X_n \cap X_m) = 0$. $\square$

The notion of a stable relation also comes up:

Definition 2.2. *An A-invariant relation R is* unstable *if there's an A-indiscernible sequence $\langle a_i; b_i\rangle$ such that*

$$i < j \iff \models R(a_i; b_j)$$

Otherwise R is stable*.*

If R is stable, so is $\neg R$.

The key fact about stable relations is the following:

Lemma 2.3 (Part of Hrushovski's Lemma 2.3)**.** *Let p and q be two complete types over a model M, and let R be a stable M-invariant relation. Then the truth value of $R(a,b)$ is the same across all independent realizations $a \models p$ and $b \models q$. More specifically, the truth value of $R(a,b)$ is constant on the set*

$$\{(a,b) : a \models p,\ b \models q,\ \text{and } (a \mathop{\smile\!\!\!\!\!\!\;\vert}_{M} b \text{ or } b \mathop{\smile\!\!\!\!\!\!\;\vert}_{M} a)\}.$$

Note: this is going to be the main technical tool in the proof of the stabilizer theorem thing.

Proof. Choose global M-invariant extensions of p and q. Replacing R with $\neg R$, we may assume $R(b,a)$ holds for $(b,a) \models q \otimes p|M$. We claim that $R(a',b')$ whenever $a' \mathop{\smile\!\!\!\!\!\!\;\vert}_M b'$ and (a' and b' realize the correct types.)

Suppose not. Let $r(x,y)$ be $\mathrm{tp}(a'b'/M)$. Then $r(x;b')$ doesn't divide over M, and any realization of r fails to satisfy R. Inductively build a sequence $a_1, b_2, a_3, b_4, \ldots$ as follows:

- $b_i \models q|M(a_{<i}b_{<i})$
- a_i realizes $\bigwedge_{j<i} r(x;b_j)$. This is doable because $r(x;b')$ doesn't divide over M and $b_{<i}$ are an indiscernible sequence in $\mathrm{tp}(b'/M)$.

Note that for $i < j$, $\operatorname{tp}(b_j a_i/M) = q \otimes p$ so $R(a_i; b_j)$ holds. And if $i > j$ then $\operatorname{tp}(a_i; b_j/M) = r$, so $R(a_i; b_j)$ is false. Extracting an indiscernible sequence in the usual way, the types of $(a_i; b_j)$ remain the same, and we contradict the stability of R.

So now we know that R holds on the entire set

$$\{(a, b) : a \models p,\ b \models q,\ \text{and } a \mathop{\smile\!\!\!\!\!\!|}_{M} b\}.$$

In particular, it holds for $(a, b) \models p \otimes q|M$. Now, repeating the same argument with the roles of p and q swapped, it follows that R also holds on the entirety of

$$\{(a, b) : a \models p,\ b \models q,\ \text{and } b \mathop{\smile\!\!\!\!\!\!|}_{M} a\}.$$

$\square$

The only example of a stable relation we care about is the following:

Lemma 2.4 (An instance of Hrushovski's 2.10)**.** *In the setup of the stabilizer theorem, the relation*

$$R(a, b) \iff (qa^{-1} \cap qb^{-1} \text{ is wide})$$

is stable. (Here, we are identifying q with its set of realizations.)

Proof. Suppose not. Then we can find an M-indiscernible sequence $\langle a_i b_i \rangle_{i<\omega}$ witnessing the failure, so that

$$qa_i^{-1} \cap qb_j^{-1} \text{ is wide iff } i \leq j.$$

In particular,

$$(qa_i^{-1} \cap qb_i^{-1}) \cap (qa_j^{-1} \cap qb_j^{-1}) \text{ is wide iff } i = j$$

which basically contradicts the S1 condition (after replacing q with a finite subtype). $\square$

We maybe also care about the following fact:

Lemma 2.5. *In the setup of the stabilizer theorem (an ind-definable group with a left- and right-translation invariant S1 ideal μ, nontrivial), the following are equivalent for a $\wedge$-definable subgroup S of $\tilde{G}$:*

- *S has bounded index in $\tilde{G}$*
- *S is wide*
- *S is generic, in the sense that for any definable neighborhood U of S, a small number of translates of U cover $\tilde{G}$.*

If one such group exists, there is a unique minimal M-definable one, which is normal.

Proof. Bounded index to generic is trivial(!). Generic implies wide because *some* set is wide, and it's covered by finitely many translates of any generic set. If S is wide, then S has bounded index. Otherwise, by Erdös-Rado or whatever, we may find an indiscernible sequence $a_1, \ldots, a_n$ such that the cosets Sa_i are distinct. This essentially contradicts the S1 condition.

The existence of a unique minimal M-definable one is easy—just take the intersection of all the M-definable subgroups of bounded index.

If N is the unique minimal one, then the normalizer of N in $\tilde{G}$ is bigger than N, so it has bounded index, meaning that N has boundedly many conjugates. The intersection of all these still has bounded index, and is M-invariant, hence type-definable over M. So it must be N. $\square$

3 Proof of the stabilizer theorem

We have a $\vee$-definable group $\tilde{G}$ with a bi-invariant M-invariant proper ideal μ on definable subsets of $\tilde{G}$, and a μ-wide complete M-type q. We're trying to show that $\tilde{G}^{00}_M$ exists and is $(q^{-1}q)^2$, with $qq^{-1}q$ a coset, and some other condition (whose proof we'll skip).

Proof. Recall that we are identifying q with its set of realizations. The first main goal is to show that $S = (q^{-1}q)^2$ is a group. This is the most difficult and technical part of the proof, making extensive use of Lemmas 2.3+2.4 (2.3+2.10). We'll use the following two sets:

$$Q = \{a^{-1}b : a, b \in q \text{ and } b \mathop{\downarrow}_M a\}$$
$$Q' = \{a^{-1}b : a, b \in q \text{ and } \operatorname{tp}(b/Ma) \text{ is wide}\}$$

Each of these sets is type-definable, though we won't use this fact (since we're skipping the last bit of the proof). (The easiest way to see this is to hold $b \models q$ fixed and look at $\{a : (a, b) \in Q\}$. These are type-definable because forking and μ are ideals.)

The key fact we will use repeatedly is that if

$$a \mathop{\downarrow}_M b, \quad a \equiv_M a', \quad b \equiv_M b', \quad \text{and } \left(a' \mathop{\downarrow}_M b' \text{ OR } b' \mathop{\downarrow}_M a'\right)$$

THEN

$$qa^{-1} \cap qb^{-1} \text{ is wide} \iff q(a')^{-1} \cap q(b')^{-1} \text{ is wide} .$$

Using this, we prove a series of claims.

Claim 1 $q^{-1}q \subseteq QQ$

Proof. Write an element of $q^{-1}q$ as $b^{-1}a$ with $b, a \in q$. By (F), we can find $c \in q$ with $a \mathop{\downarrow}_M c$ and $c \mathop{\downarrow}_M a$. Moving c over Ma, we may assume $c \mathop{\downarrow}_M ab$. Then

$$b^{-1}a = b^{-1}c \cdot c^{-1}a \in QQ \text{ because } c \mathop{\downarrow}_M b \text{ and } a \mathop{\downarrow}_M c.$$

$\square$

Claim 2 For all $a, b \in q$, if $b \downarrow_M a$, then $qa^{-1} \cap qb^{-1}$ is wide.

Proof. By the technical fact, we only need to prove this for one pair of independent realizations of q. Take a realization $a_1, a_2, \cdots$ of $q^{\otimes\omega}|M$. Then $qa_i^{-1} \cap qa_j^{-1}$ is wide, or else the S1 property would be violated. But $a_2 \downarrow_M a_1$. $\square$

Claim $2\frac{1}{2}$ If $c_1 \in q^{-1}q$ and $c_2 \in Q'$ and $c_2 \downarrow_M c_1$, then $qc_1^{-1} \cap qc_2^{-1}$ is wide.

Proof. Let a realize q. We can find b_1 and b_2 realizing q such that

$$a^{-1}b_i \equiv_M c_i$$

and $\mathrm{tp}(b_2/M, a)$ is wide. Extending this to a wide type over M, a, b_1 and moving b_2, we may assume $\mathrm{tp}(b_2/M, a, b_1)$ is wide. Then

$$b_2 \underset{M}{\downarrow} b_1$$

so by Claim 1,

$$qb_2^{-1} \cap qb_1^{-1} \text{ is wide}$$

so by translation invariance

$$q(a^{-1}b_2)^{-1} \cap q(a^{-1}b_1^{-1}) \text{ is wide.}$$

Meanwhile, by translation invariance,

$$\mathrm{tp}(a^{-1}b_2/M, a, b_1) \text{ is wide, so } a^{-1}b_2 \underset{M}{\downarrow} a^{-1}b_1$$

Now $a^{-1}b_i \equiv_M c_i$ and so by the technical fact, it follows that

$$qc_1^{-1} \cap qc_2^{-1} \text{ is wide.}$$

$\square$

Claim 3 (the worst) If $c \in q^{-1}q$ and $d \in Q$ and $d \downarrow_M c$, then $qc^{-1} \cap qd^{-1}$ is wide.

Proof. Write d as $a^{-1}b$ with $b \downarrow_M a$. Now we do a series of moves:

- Let $(a_1, b_1) \equiv_M (a, b)$ be such that $a_1 \downarrow_M c$. Possible since $a, b \downarrow_M M$.
- Let $b_2 \equiv_{M,a_1} b_1$ be such that $b_2 \downarrow_M c, a_1$. Possible since $b_1 \downarrow_M a_1$.
- Let b_3 be such that $b_3 \equiv_M b_2$ and $\mathrm{tp}(b_3/M, c, a_1)$ is wide. This is possible because $\mathrm{tp}(b_2/M) = q$ is wide.

We claim that

$$\begin{aligned} qc^{-1} \cap qb^{-1}a \text{ is wide } &\iff qc^{-1} \cap qb_2^{-1}a_1 \text{ is wide} \\ &\iff qc^{-1}a_1^{-1} \cap qb_2^{-1} \text{ is wide} \\ &\iff qc^{-1}a_1^{-1} \cap qb_3^{-1} \text{ is wide} \\ &\iff qc^{-1} \cap qb_3^{-1}a_1 \text{ is wide } \iff \top \end{aligned}$$

1. The first holds by the technical fact: we have $(b_2, a_1) \equiv_M (b, a)$, and $b^{-1}a \mathop{\smile\!\!\!\!\!\!|}_M c$ (by assumption), and

$$(b_2, a_1) \mathop{\smile\!\!\!\!\!\!|}_M c \text{ because } a_1 \mathop{\smile\!\!\!\!\!\!|}_M c \text{ and } b_2 \mathop{\smile\!\!\!\!\!\!|}_M c, a_1$$

2. The second holds by translation invariance of μ.
3. The third holds by the technical fact, seeing as $b_2 \equiv_M b_3$, and

$$b_3 \mathop{\smile\!\!\!\!\!\!|}_M (a_1, c) \text{ because } \mathrm{tp}(b_3/M, a_1, c) \text{ is wide}$$

$$b_2 \mathop{\smile\!\!\!\!\!\!|}_M (a_1, c) \text{ by choice of } b_2.$$

4. The fourth holds by translation invariance of μ.
5. The fifth holds by Claim $2\frac{1}{2}$. Note that $c \in q^{-1}q$. The product $a_1^{-1}b_3 \in Q'$ because $\mathrm{tp}(b_3/Ma_1)$ is wide (by choice of b_3). Finally, why is $a_1^{-1}b_3 \mathop{\smile\!\!\!\!\!\!|}_M c$? Well,

$$\mathrm{tp}(b_3/M, a_1, c) \text{ is wide, so } \mathrm{tp}(a_1^{-1}b_3/M, a_1, c) \text{ is wide, so doesn't fork..}$$

□

Claim 4−ϵ If $a \in q^{-1}q$ and $b \in Q$ and $a \mathop{\smile\!\!\!\!\!\!|}_M b$, then $qa^{-1} \cap qb^{-1}$ is wide.

Proof. This is Claim 3 plus the symmetry part of the technical fact. □

Claim 4+ϵ If $a \in q^{-1}q$ and $b \in Q$ and $a \mathop{\smile\!\!\!\!\!\!|}_M b$, then $ab \in q^{-1}q$.

Proof. Since $a^{-1} \mathop{\smile\!\!\!\!\!\!|}_M b$, by the previous claim, $qa \cap qb^{-1}$ is wide. So it's non-empty. If $c \in qa \cap qb^{-1}$, then $ca^{-1} \in q$ and $cb \in q$, so

$$ab = (ca^{-1})^{-1}(cb) \in q^{-1}q.$$

□

Claim 5 Let $a \in q^{-1}q$, and $b_1, \ldots, b_n$ be such that

$$\mathrm{tp}(a/M, b_1, \ldots, b_n) \text{ is wide.}$$

Then $ab_1 \cdots b_n \in q^{-1}q$.

Proof. By translation invariance of wideness,

$$\mathrm{tp}(ab_1 \cdots b_i/M, b_1, \ldots, b_n) \text{ is wide}$$

hence

$$ab_1 \cdots b_i \underset{M}{\downarrow} b_{i+1}.$$

So by Claim 4+ϵ,

$$ab_1 \cdots b_i \in q^{-1}q \implies ab_1 \cdots b_{i+1} \in q^{-1}q.$$

So by induction

$$a \in q^{-1}q \implies ab_1 \cdots b_n \in q^{-1}q.$$

□

Claim 6 $Q^n \subseteq (q^{-1}q)^2$.

Proof. Let $b_1, \ldots, b_n \in Q$. Since q is wide, so is $q^{-1}q$. Let $a \in q^{-1}q$ with $\mathrm{tp}(a/M(b_1, \ldots, b_n))$ wide. Then by Claim 5, $ab_1 \cdots b_n \in q^{-1}q$. So

$$b_1 \cdots b_n = a^{-1} \cdot ab_1 \cdots b_n \in q^{-1}qq^{-1}q.$$

□

Now from Claim 1 and Claim 6, $q^{-1}q$ and Q generate the same group S, and $S = (q^{-1}q)^2 = \bigcup_n Q^n$.

Because $q^{-1}q$ is wide, S is wide, so S has bounded index in $\tilde{G}$. We claim that S is $\tilde{G}^{00}_M$. If not, then there is some M-definable proper subgroup T of S, of bounded index in S. If r is a global M-invariant extension of M, then realizations of r all live in the same coset of T. This coset is a type-definable set, invariant under $\mathrm{Aut}(\mathbb{M}/M)$, so it's type-definable over M. Therefore, realizations of q live in this coset. Hence realizations of $q^{-1}q$ all live in T, so $S \subseteq T$, a contradiction.

So S is $\tilde{G}^{00}_M$, and S is normal.

Let d realize q. We claim that $qq^{-1}q = dS$, so $qq^{-1}q$ is a coset of S. First of all, if $ab^{-1}c$ is some element of $qq^{-1}q$, then

$$d^{-1}ab^{-1}c \in q^{-1}qq^{-1}q = S,$$

so

$$ab^{-1}c \in dS.$$

Thus $qq^{-1}q \subseteq dS$.

To see $dS \subseteq qq^{-1}q$, since $S = \bigcup_n Q^n$, we need to show that if $b_1, \ldots, b_n \in Q$, then $db_1 \cdots b_n \in qq^{-1}q$. Let a be a realization of q such that $\mathrm{tp}(a/M(d, b_1, \ldots, b_n))$ is wide. Then $\mathrm{tp}(a^{-1}d/M(d, b_1, \ldots, b_n))$ is wide, so by claim 5,

$$a^{-1}db_1 \cdots b_n \in q^{-1}q.$$

Then

$$db_1 \cdots b_n \in aq^{-1}q \subseteq qq^{-1}q.$$

So $qq^{-1}q$ is a coset.

We omit the proof of the last claim (about $S \setminus q^{-1}q$ being small), because it isn't needed for Corollary 1.2 = Hrushovski's Corollary 3.6. $\square$