Interpretable sets in o-minimal structures

Will Johnson

March 27, 2015
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Let $G$ be an interpretable group in an o-minimal structure $M$. Then $G$ is $M$-definably isomorphic to a definable group.
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- Yes, if they expand RCF.
- *Usually*, if they expand DOAG.
Interpretable groups in o-minimal theories

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But don’t o-minimal theories eliminate imaginaries?

- **Yes**, if they expand RCF.
- **Usually**, if they expand DOAG.
- **No**, in general.
The affine line

Consider \((\mathbb{R}, <, \sim)\), where

\[(x, y) \sim (a, b) \iff x - y = a - b\]
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\[x \mapsto x + 1\]

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**Remark**

After naming any constant, \(\mathbb{R}^2 / \sim\) becomes definably isomorphic to the home sort.
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Is this really a property of groups?
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**Conjecture**

If $X$ is an interpretable set in an o-minimal structure $M$, then there is an $M$-definable bijection to a definable set.
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Conjecture

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Unfortunately, this is false...
Consider $M = (\mathbb{R}, <, \sim)$ where the relation

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and

$$\cot(x - z) - \cot(y - z) = \cot(x' - z) - \cot(y' - z)$$
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Morally, $M$ is the universal cover of the real projective line.
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- $\text{Aut}(M)$ acts transitively on $M$
- For any $a \in \mathbb{R}$, $\text{dcl}(a) = a + \mathbb{Z} \cdot \pi$. 
Automorphisms of $M$

**Lemma**

- $\text{Aut}(M/\text{dcl}(0))$ is isomorphic to the group $A$ of affine transformations $x \mapsto ax + b$ with $a > 0$. 

- The non-singleton orbits of $\text{Aut}(M/\text{dcl}(0))$ are exactly the open intervals $(n\pi, (n+1)\pi)$. Each orbit is $A$-isomorphic to the affine line via $\cot(\cdot)$. 

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- Each orbit is $A$-isomorphic to the affine line via $\cot(-)$.
We can identify the quotient of $\sim_0$ with $\mathbb{R}$, via

$$(x, y) \mapsto \cot(x) - \cot(y)$$

Under this identification, an affine transformation $x \mapsto ax + b$ acts by multiplication by $a$. Any $\sim_0$-equivalence class is fixed by translations, but most aren’t fixed by scalings. No tuple from the home sort has this property.

Corollary

Most $\sim_0$-equivalence classes can’t be coded by reals, so $M$ doesn’t eliminate imaginaries.
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**Corollary**

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Fact

We can lay two copies of $M$ “end to end,” getting a structure $M_1 \cup M_2$. Then:
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- \( \text{Aut}(M_1 \cup M_2) \cong \text{Aut}(M_1) \times \text{Aut}(M_2) \).
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*We can lay two copies of $M$ “end to end,” getting a structure $M_1 \cup M_2$. Then:*

- $M_1 \preceq M_1 \cup M_2 \succeq M_2$
- $\text{Aut}(M_1 \cup M_2) \cong \text{Aut}(M_1) \times \text{Aut}(M_2)$.

If all quotients could be eliminated by naming parameters, the structure $M_1 \cup M_2$ would have elimination of imaginaries after naming all elements of $M_2$. But then

$$\text{Aut}(M_1 \cup M_2/M_2) = \text{Aut}(M_1)$$

and we can still run the automorphisms argument in $M_1$. 

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Tracing through the proof, $X$ is actually the quotient of

$$\{(x, y, z) : x < y < x + \pi, \ x < z < x + \pi\}$$

by the equivalence relation

$$(x, y, z) \equiv (x', y', z') \iff (x = x' \text{ and } (y, z) \sim_x (y', z'))$$.
What can be said about interpretable sets?

Invariants of definable sets can be extended:
- Dimension theory (Peterzil)
- Euler characteristic (Kamenkovich and Peterzil)

Interpretable sets can be definably topologized.
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- Interpretable sets can be definably topologized.
Fix $M$ a dense o-minimal structure.

**Theorem**

Let $Y \subset M^n$ be definable, and $E$ be a definable equivalence relation on $Y$. Then there is $Y' \subset Y$ definable, such that

- The quotient topology on $Y'/E$ is definable, Hausdorff, regular, and “locally Euclidean.”

For any $Y'' \subset Y'$, the quotient topology on $Y''/E$ is the subspace topology of that on $Y'/E$.

All these properties remain true in elementary extensions of $M$.

By recursively handling $(Y \setminus Y')/E$, one can topologize $Y/E$ as an “interpretable manifold” with finitely many connected components.
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References

- Sofya Kamenkovich and Ya’acov Peterzil. Euler characteristic of imaginaries in o-minimal structures, 2014.