Interpretable sets in o-minimal structures

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- *Yes*, if they expand RCF.
- *Usually*, if they expand DOAG.
- *No*, in general.

The affine line

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After naming any constant, $\mathbb{R}^2/\sim$ becomes definably isomorphic to the home sort.

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If X is an interpretable set in an o-minimal structure M, then there is an M-definable bijection to a definable set.

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Unfortunately, this is false. . .

My counterexample

Consider $M = (\mathbb{R}, <, \sim)$ where the relation

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Morally, M is the universal cover of the real projective line.

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- $\mathrm{Aut}(M)$ acts transitively on M
- For any $a \in \mathbb{R}$, $\mathrm{dcl}(a) = a + \mathbb{Z} \cdot \pi$.

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Lemma

- $\mathrm{Aut}(M/\operatorname{dcl}(0))$ *is isomorphic to the group* $\mathcal{A}$ *of affine transformations*

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- *The non-singleton orbits of* $\mathrm{Aut}(M/\mathrm{dcl}(0))$ *are exactly the open intervals* $(n\pi, (n+1)\pi)$.
- *Each orbit is* $\mathcal{A}$*-isomorphic to the affine line via* $\cot(-)$.

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Corollary

Most $\sim_0$-equivalence classes can't be coded by reals, so M doesn't eliminate imaginaries.

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If all quotients could be eliminated by naming parameters, the structure $M_1 \cup M_2$ would have elimination of imaginaries after naming all elements of M_2. But then

$$\mathrm{Aut}(M_1 \cup M_2 / M_2) = \mathrm{Aut}(M_1)$$

and we can still run the automorphisms argument in M_1.

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Proposition (J.)

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Tracing through the proof, X is actually the quotient of

$$\{(x, y, z) : x < y < x + \pi,\ x < z < x + \pi\}$$

by the equivalence relation

$$(x, y, z) \approx (x', y', z') \iff \left(x = x' \text{ and } (y, z) \sim_x (y', z')\right).$$

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- Invariants of definable sets can be extended:
    - Dimension theory (Peterzil)
    - Euler characteristic (Kamenkovich and Peterzil)
- Interpretable sets can be definably topologized.

Topologizing interpretable sets

Fix M a dense o-minimal structure.

Theorem

Let $Y \subset M^n$ be definable, and E be a definable equivalence relation on Y. Then there is $Y' \subset Y$ definable, such that

- *The quotient topology on Y'/E is definable, Hausdorff, regular, and "locally Euclidean."*

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By recursively handling $(Y \setminus Y')/E$, one can topologize Y/E as an "interpretable manifold" with finitely many connected components.

References

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