Topologizing interpretable sets in O-minimal Structures

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1 Definitions and Conventions

If \( X \) is a subset of a topological space, \( \overline{X} \) will mean the closure of \( X \), and \( \partial X \) will mean the frontier of \( X \), \( \overline{X} \setminus X \). The interior of \( X \) will be denoted \( X^{\text{int}} \). The boundary \( \overline{X} \setminus X^{\text{int}} \) will be denoted \( bd(X) \).

1.1 Definable Topologies

Let \( M \) be a structure in some language. Assume \( M \) has elimination of imaginaries. Let \( X \) be a definable set. Definable will mean “definable with parameters.” By a definable topology, we mean a definable family of subsets \( \{ B_y \subset X \}_{y \in Y} \) which form the basis for some topology on \( X \). The fact that these form a basis for a topology amounts to the claim that if \( y_1, y_2 \) have \( B_{y_1} \cap B_{y_2} \neq \emptyset \), then for every \( x \in B_{y_1} \cap B_{y_2} \) there is a \( y_3 \in Y \) such that \( x \in B_{y_3} \subset B_{y_1} \cap B_{y_2} \). This is a first-order condition, so a definable topology on \( X \) remains a definable topology in elementary extensions of \( M \). But note that if \( M \preceq M' \), the topology on \( X(M) \) need not agree with the subspace topology on \( X(M') \) as a subset of \( X(M') \).

If \( D \) is a definable subset of \( X \), and \( X \) has a definable topology, then the subspace topology on \( D \) is also definable. If \( X \) and \( Y \) are two sets with definable topologies, then the product topology on \( X \times Y \) is definable.

If \( X \) and \( Y \) are two sets with definable topologies, and \( f : X \to Y \) is a definable function, then we can express whether or not \( f \) is continuous using some first-order statement. So the continuity of \( f \) is invariant under elementary extensions, and definable in families.

We say that \( X \) is definably connected if there is no definable clopen set \( D \) with \( \emptyset \subsetneq D \subsetneq X \). The definable connectedness of \( X \) is invariant under elementary extensions, and type-definable in families. If \( f : X \to Y \) is a continuous definable function, and \( X \) is definably connected, then so is \( Y \).

A continuous map \( f : X \to Y \) between abstract topological spaces is an open map if \( f(U) \) is open for every open subset \( U \subset X \). If \( B \) is a basis of opens on \( X \), it suffices to check \( U \in B \). If \( f : X \to Y \) is a continuous definable map between two definable topological spaces, then we can express that \( f \) is an open map via a first-order statement. So elementary extensions preserve whether or not \( f \) is open, and this is definable in families.
1.2 The Quotient Topology

If \( X \) is an abstract topological space, and \( E \) is an equivalence relation on \( X \), then there is a natural quotient topology on \( X/E \). Letting \( \pi : X \to X/E \) be the natural projection, a subset \( U \subseteq X/E \) is open in this quotient topology if and only if \( \pi^{-1}(U) \) is open. Note that \( \pi \) is then continuous.

A continuous surjection \( f : X \to Y \) of abstract topological spaces is identifying if \( Y \) has the quotient topology of \( X/\ker f \), where \( \ker f \) is the equivalence relation \( \{(x,y) : f(x) = f(y)\} \). In other words, a subset \( U \subseteq Y \) is open if and only if \( f^{-1}(U) \) is open.

If \( X \) is a definable topological space and \( E \) is a definable equivalence relation, the quotient topology on \( X/E \) need not be definable, as far as I know. If \( X \) and \( Y \) have definable topologies, and \( f : X \to Y \) is a definable continuous function, there is not any general way of expressing that \( f \) is identifying.

Remark 1.1. If \( f : X \to Y \) is a continuous open surjection of abstract topological spaces, then \( f \) is identifying. Indeed, if \( f^{-1}(U) \) is open for some subset \( U \subseteq Y \), then \( f(f^{-1}(U)) = U \) is also open. Moreover, if \( X_0 \) is an open subset of \( X \), then the restriction of \( f \) to \( X_0 \) is a continuous open surjection to \( f(X_0) \), where \( f(X_0) \) has the subspace topology from \( Y \). In particular, the quotient topology on \( f(X_0) \) as a quotient of \( X_0 \) agrees with the subspace topology on \( f(X) \), as a subspace of \( f(X) \).

In a definable setting, we can express that \( f \) is a continuous open surjection.

Definition 1.2. An equivalence relation \( E \) on a set \( X \) is an open equivalence relation if the natural quotient map \( \pi : X \to X/E \) (with the quotient topology on \( X/E \)) is an open map. Equivalently, for every open set \( U \subseteq X \), the set

\[
\pi^{-1}(\pi(U)) = \{x \in X : xEu \text{ for some } u \in U\}
\]

is open. If \( \mathcal{B} \) is a basis of open sets on \( X \), it suffices to check the \( U \in \mathcal{B} \).

Remark 1.3.

(a) If \( E \) is an open equivalence relation on \( X \), and \( U \subseteq X \) is open, then \( E \upharpoonright U = E \cap (U \times U) \) is an open equivalence relation on \( E \), by Remark 1.1.

(b) If \( X \) is a definable topological space and \( E \) is a definable equivalence relation, then we can write a first-order sentence that expresses that \( E \) is an open equivalence relation on \( X \). (We only need to check that \( \pi^{-1}(\pi(U)) \) is open for \( U \) in a basis of opens on \( X \).) In particular, this property is definable in families and invariant under elementary extensions.

(c) If \( X \) is a definable topological space and \( E \) is an open definable equivalence relation on \( X \), then the quotient topology on \( X/E \) is definable. In fact, if \( \pi : X \to X/E \) denotes the natural surjection, and \( \mathcal{B} \) is a definable basis of opens for \( X \), then \( \{\pi(U) : U \in \mathcal{B}\} \) is a definable basis of opens for \( X/E \).
1.3 Separation Axioms

Recall that a topological space $X$ is Hausdorff if for every two distinct points $x, x' \in X$, we can find two disjoint open sets $U, U'$ with $x \in U$, $x' \in U'$. Given a basis, we can require that $U$ and $U'$ be basic opens. Recall that $X$ is $T_0$ if for every two distinct points $x, x' \in X$, we can find an open set $U$ which contains exactly one of $\{x, x'\}$. Again, we can require that $U$ be a basic open.

Recall that $X$ is regular if for every $x \in X$ and every closed set $C \subset X$ with $x \notin C$, we can find disjoint open sets $U$ and $V$ with $x \in U$ and $C \subset V$. Equivalently, whenever $D$ is open and $x \in D$, we can find a smaller open neighborhood $U \ni x$ with $\overline{U} \subset D$. In this second definition, we may assume that $D$ and $U$ are basic opens.

Because it suffices to check basic opens, if $X$ is a definable topological space, then we can therefore express by a first-order statement that the topology is Hausdorff, $T_0$, or regular. Such properties are therefore preserved in elementary extensions and definable in families.

Note that Hausdorff implies $T_0$. Also, regular plus $T_0$ implies Hausdorff. Indeed, suppose $X$ is regular and $T_0$. If $x$ and $x'$ are two distinct points in $X$, then by $T_0$ we can find a closed set $C$ containing exactly one of $x$ and $x'$. Now use regularity to separate $C$ from the unique point of $\{x, x'\} \setminus C$.

If $X$ is Hausdorff or $T_0$ or regular, and $Y$ is a subset of $X$, then the subspace topology on $Y$ has the same property.

1.4 Definable Compactness?

If $X$ is a set, a filtered collection of subsets of $X$ is a collection $\mathcal{F}$ of subsets of $X$ with the property that for every $D_1, D_2 \in \mathcal{F}$, there is some $D_3 \in \mathcal{F}$ with $D_3 \subset D_1 \cap D_2$. For example, if $X$ is a topological space, $\mathcal{B}$ is a basis, and $x \in X$, then the set of basic opens neighborhoods of $x$ is a filtered collection of non-empty subsets of $X$.

If $X$ is a definable set and $\mathcal{F}$ is a definable family of subsets of $X$, we can express by a first-order statement that $\mathcal{F}$ is a filtered collection.

If $X$ is a definable topological space, let’s say that $X$ is definably compact if every definable filtered collection of closed non-empty subsets of $X$ has non-empty intersection. This property is preserved in elementary extensions, and type-definable in families. If $D$ is a closed subset of a definably compact topological space, then $D$ with the subspace topology is definably compact as well.

Lemma 1.4. Let $X$ and $Y$ be definable topologies, and suppose $X$ is definably compact. If $D \subset X \times Y$ is definable and closed, then $\pi(D)$ is closed as a subset of $Y$, where $\pi : X \times Y \to Y$ is the projection.

Proof. Suppose that $y \in \overline{\pi(D)}$. Let $\mathcal{B}$ be a the (definable) collection of basic open sets containing $y$. For each $V \in \mathcal{B}$, let $C_V$ be the set of $x \in X$ for which there is some open neighborhood $U \ni x$ with $(U \times V) \cap D = \emptyset$. The family of the $C_V$’s is definable. Let $X_V = X \setminus C_V$. Each $C_V$ is open, so each $X_V$ is closed. If some $X_V$ is empty, then $C_V = X$. In particular, for every $x \in X$ we have $(\{x\} \times V) \cap D = \emptyset$, so $(X \times V) \cap D = \emptyset$ and $V$ is
disjoint from \( \pi(D) \). But then \( y \in V \) so \( y \notin \pi(D) \), a contradiction. Therefore each \( X_V \) is non-empty.

Note that if \( V' \subset V \), then \( C_V \subset C_{V'} \), so \( X_{V'} \subset X_V \). Therefore, \( \{X_V\}_{V \in B} \) is a filtered collection of non-empty closed subsets of \( X \). So there is some \( x \in \bigcap_{V \in B} X_V \). Now for every basic open neighborhood \( U \times V \ni (x, y) \), if \( (U \times V) \cap D = \emptyset \), then \( x \in C_V \), a contradiction. So every basic open neighborhood of \((x, y)\) intersects \( D \). As \( D \) is closed, \((x, y) \in D \). Thus \( x \in \pi(D) \).

**Lemma 1.5.** If \( X \) and \( Y \) are definably compact, then so is \( X \times Y \).

**Proof.** Let \( \mathcal{F} \) be a definable filtered collection of non-empty closed subsets of \( X \times Y \). Let \( \pi : X \times Y \rightarrow Y \) be the projection. By Lemma [1.4], \( \pi(F) \) is closed for every \( F \in \mathcal{F} \). In particular, \( \{\pi(F) \mid F \in \mathcal{F} \} \) is a definable filtered collection of non-empty closed subsets of \( X \times Y \). So there is some point \( y \) with \( y \in \pi(F) \) for every \( F \in \mathcal{F} \). Now for each \( F \in \mathcal{F} \), let \( G_F = \{x \in X : (x, y) \in F \} \). Then \( G_F \) is non-empty and closed for every \( F \in \mathcal{F} \), and the collection \( \mathcal{G} := \{G_F \}_{F \in \mathcal{F}} \) is a definable filtered collection of non-empty closed subsets. So there is some \( x \) in \( \bigcap_{F \in \mathcal{F}} G_F \). Then \((x, y) \in \bigcap \mathcal{F} \).

**Lemma 1.6.** If \( f : X \rightarrow Y \) is a continuous definable surjection of definable topological spaces, and \( X \) is definably compact, then so is \( Y \).

**Proof.** Let \( \mathcal{F} \) be a definable filtered collection of closed non-empty subsets of \( Y \). Let \( f^* \mathcal{F} \) be the collection

\[
\{f^{-1}(F) : F \in \mathcal{F}\}.
\]

Then \( f^* \mathcal{F} \) is a definable filtered collection of closed non-empty subsets of \( X \). So there is some \( x \in \bigcap_{F \in \mathcal{F}} f^{-1}(F) \). Equivalently, \( f(x) \in \bigcap \mathcal{F} \).

**Lemma 1.7.** Let \( X \) be a Hausdorff definable topological space, and let \( D \) be a definable subset of \( X \). Suppose that \( D \) with the subspace topology is definably compact. If \( x \notin D \), then there is an open neighborhood \( U \) of \( x \) with \( \overline{U} \) disjoint from \( D \).

**Proof.** Let \( \mathcal{B} \) be a definable basis for the topology on \( X \). Let \( \mathcal{F} \) be the collection of sets \( \{ \overline{U} \cap D : x \in U \in \mathcal{B} \} \). This is a filtered definable collection of closed subsets of \( D \).

Note that if \( y \in D \), then \( y \neq x \) and so by Hausdorffness we can find \( x \in U \) and \( y \in V \) with \( U \) and \( V \) basic opens, and \( U \cap V = \emptyset \). This means that \( y \notin \overline{U} \). So every point \( y \in D \) is not in some \( \overline{U} \), meaning that \( \bigcap \mathcal{F} = \emptyset \). By definable compactness of \( D \), some element of \( \mathcal{F} \) must be empty. Therefore there is some open \( U \ni x \) with \( \overline{U} \cap D = \emptyset \).

**Corollary 1.8.** If \( X \) is a Hausdorff definable topological space, and \( D \) is a subset of \( X \) which is definably compact (with the subspace topology), then \( D \) is closed.

**Corollary 1.9.** If \( X \) is a Hausdorff definable topological space, and \( X \) is definably compact, then \( X \) is regular.

**Proof.** If \( C \) is a closed subset of \( X \) and \( x \in X \setminus C \), then \( C \) is compact (as we noted above), so by Lemma [1.7] there is an open neighborhood \( U \ni x \) with \( \overline{U} \) disjoint from \( C \).
1.5 The o-minimal Setting

Now restrict to the setting of $M^{eq}$, where $M = (M, <, \ldots)$ is an o-minimal structure. All o-minimal structures will be dense, i.e., expand DLO. A definable topological space $X$ in $M^{eq}$ will be said to be locally Euclidean if every $x \in X$ has a definable neighborhood $U$ which is definably homeomorphic to an open subset of $M^n$. The dimension $n$ might vary with $x$. The property of being locally Euclidean is preserved downwards but perhaps not upwards in elementary extensions.

Lemma 1.10. Any closed interval $[a, b] \subset M$ is definably compact (in the sense defined above).

Proof. Let $\mathcal{F}$ be a definable filtered collection of non-empty closed subsets of $[a, b]$. For $F \in \mathcal{F}$, let $x_F$ be $\inf F$. Note $x_F \in F$, by closedness of $F$. The set $D = \{x_F : F \in \mathcal{F}\}$ is a definable subset of $[a, b]$; let $y$ be $\sup D$. I claim that $y \in \bigcap \mathcal{F}$. If not, then $y \notin F_0$ for some $F_0 \in \mathcal{F}$. As $F_0$ is closed, there is some $z < y$ such that $[z, y] \cap F_0$ is empty. Now as $y = \sup D$, there must be some $x_{F_1} \in [z, y]$. Take $F_2 \subseteq F_1 \cap F_0$. Because $F_2 \subset F_1$, we have $x_{F_2} = \inf F_2 \geq \inf F_1 = x_{F_1} \geq z$. So $z \leq x_{F_2} \leq y$, and $x_{F_2} \in F_2 \subset F_0$. Therefore, $F_0$ does intersect $[z, y]$, at least at the point $x_{F_2}$. So we have a contradiction. □

Corollary 1.11. Closed and bounded subsets of $M^n$ are definably compact (in the sense defined above).

Proof. Use Lemma 1.5 □

The converse is also true: if a subset $X$ is definably compact, then it is closed and bounded. It is closed by Corollary 1.8. If it is unbounded, then $\pi(X) \subset M^1$ is unbounded, for some coordinate projection $\pi : M^n \to M^1$. But then $\pi(X)$ is definably compact by Lemma 1.6. However, an unbounded subset of $M^1$ is not definably compact because of the filtered collection of closed non-empty subsets obtained by intersecting with $[a, \infty)$ or $(-\infty, a]$ for $a \in M^1$. So the notion of definably compact that we are using agrees with the usual notion in o-minimal structures.

Remark 1.12. If $X \subset M^n$ is definable, then the frontier $\partial X := \overline{X} \setminus X$ always has lower dimension than $X$. Also, if $X \subset Y \subset M^n$, then the relative boundary $\partial_Y(X)$ of $X$ as a subset of the topological space $Y$, always has $\dim \partial_Y(X) < \dim Y$.

Lemma 1.13. Let $X$ be a definable topological space in $M^{eq}$. Suppose that $X$ is Hausdorff and locally Euclidean. Then $M$ is regular.

Proof. Let $C$ be a closed subset of $X$ and $x \in X \setminus C$. Take some open neighborhood $U$ of $x$ which is homeomorphic to an open subset of $M^n$. Shrinking $U$, we may assume that $U \cap C = \emptyset$. Suppose that $U$ is homeomorphic to an open subset $V \subset M^n$ via a definable homeomorphism $f$. As $f(x)$ is in the interior of $V$, we can find a closed box $B$ with $x \in B^\text{int}$ and $B \subset V$. Now $f^{-1}(B)$ is definably compact, so it is closed as a subset of $X$, by Corollary 1.8. Also, $f^{-1}(B^\text{int})$ is an open neighborhood of $x$. Its closure is contained in $f^{-1}(B)$ which is contained in $U$. In particular, $f^{-1}(B)$ is an open neighborhood of $x$ whose closure is disjoint from $C$. □
Definition 1.14. A definable manifold is a Hausdorff definable topological space $X$ which is locally Euclidean in all elementary extensions of the model.

This definition forces some uniformity in the manifold charts. It suffices to check the local Euclideanity in some saturated elementary extension of the original model. Moreover, because the property of being locally Euclidean uniformly must be witnessed by a definable family of charts, the property of being locally Euclidean is expressible as a small disjunction of first-order statements, and is consequently ind-definable in families.

Note by Lemma 1.13 that definable manifolds are always regular.

2 Quotients in o-minimal structures

Recall that if $M$ is o-minimal, then $M^{eq}$ is a supperrosy structure of finite definable rank. We will use $\dim X$ to refer to the rank of a definable set. We will use $R(a/b)$ to denote the rank of $a$ over $b$. We will use $\perp$ to denote thorn-forking independence, not usual forking independence.

2.1 The Theorem

Fix some o-minimal structure $M$.

Theorem 2.1. Let $Y \subset M^n$ be a definable set. Let $E$ be a definable equivalence relation on $Y$. Then there is some definable open subset $Y'$ of $Y$ such that $\dim(Y \setminus Y') < \dim Y$, and such that the natural quotient topology on $Y'/E$ is definable, Hausdorff, regular, and locally Euclidean, and the map $Y' \to Y'/E$ is an open map. Moreover, we may arrange that these topological properties remain true in all elementary extensions.

Corollary 2.2. Let $X$ be a definable set in $M^{eq}$. Then $X$ can be put in definable bijection with a finite disjoint union of definably connected interpretable manifolds.

Proof. Write $X$ as $Y/E$, with $Y \subset M^n$ for some $n$. Proceed by induction on $\dim Y$. The base case where $Y = \emptyset$, is trivial. Assume $Y$ is non-empty. By the Theorem, we can find $Y'$ such that $Y'/E$ is a definable manifold. The map $Y' \to Y'/E$ is continuous, and $Y'$ has finitely many definably connected components (by cell-decomposition), so $Y'/E$ also has finitely many definably connected components. Each of these is a definably connected definable manifold. Letting $\pi : Y \to X$ be the natural quotient map, let $X' = \pi(Y')$. Then we have just put $X'$ in definable bijection with a disjoint union of definably connected definable manifolds. Also, $X \setminus X'$ can be written as a quotient of $\pi^{-1}(X \setminus X')$. But

$$\pi^{-1}(X \setminus X') \subset Y \setminus Y'$$

so $\dim(\pi^{-1}(X \setminus X')) < \dim Y$. By induction, $X \setminus X'$ can be put in definable bijection with a finite disjoint union of definably connected interpretable manifolds. Writing $X$ as $X' \bigsqcup (X \setminus X')$, we are done. \qed
In proving Theorem 2.1, we may replace $M$ with a saturated elementary extension—a monster model. The topological properties other than local Euclideanity are all expressible by first-order statements (because of the assumption that $Y' \to Y'/E$ is open), and uniform local Euclideanity is a small disjunction of first-order statements, so if we can find a satisfactory $Y'$ in the monster, we can also find one in the original model $M$.

Henceforth, assume that $M$ is a monster model. Work in $M^{eq}$. Hold $Y$ fixed. Let $\pi : Y \to X$ be the projection. So the equivalence class of $y$ is $\pi^{-1}(\pi(y))$, for $y \in Y$. Denote this by $E(y)$, for simplicity.

If $D$ is a definable set in $M^{eq}$, $[D]$ will denote the code of $D$ (as an element of $M^{eq}$). We will frequently use the following fact:

**Remark 2.3.** If $U$ is an open subset of $M^n$, and $p$ is a point in $U$, and $S$ is some small subset of $M^{eq}$, then we can find an open neighborhood $V$ with $p \in V \subset U$ and with $Sp[U] \downarrow [V]$.

Indeed, we can take $V$ to be an open box with generically chosen corners.

More generally, if $Y$ is a definable set defined over some set $T$, and $p \in Y$ is a point, and $U$ is an open subset of $Y$ containing $p$, and $S$ is a small subset of $M^{eq}$, then we can find a neighborhood $V$ of $p$ in $Y$ with $p \in V \subset U$ and with $[V] \downarrow_T Sp[U]$. Specifically, we can take a random box, sufficiently small, and intersect it with $Y$.

The proof of Theorem 2.1 will proceed in several steps. At each step, we will replace $Y$ with an open subset $Y'$ such that $\dim Y \setminus Y' < \dim Y$, in such a way that certain properties will be true of $Y'$. These properties will be preserved in each subsequent step.

### 2.2 Step 1: Pure-dimensional Equivalence Classes

Recall that if $Z \subset M^n$ is a definable set, then for $z \in Z$, we can discuss the local dimension $\dim_z Z$ of $Z$ at $z$. This is $\dim U \cap Z$ for $U$ a sufficiently small open around $z$. The value $\dim_z Z$ is definable in families (as $Z$ and $z$ vary in a family).

Moreover, if $z$ is a generic point in $Z$, then $\dim_z Z = \dim Z$. Indeed, suppose $Z$ is defined over $S$, and $R(z/S) = \dim Z$. Take some $U$ containing $z$ such that $\dim U \cap Z = \dim_z Z$. By Remark 2.3 we can find $V$ independent from $z$ over $S$, with $z \in V \subset U$. Then $\dim V \cap Z = \dim_z Z$. By the independence, $R(z/S) = R(z/[V]S)$. But $z$ is in the $[V]S$-definable set $V \cap Z$, so $R(z/[V]S) \leq \dim V \cap Z = \dim_z Z$. In particular,

$$\dim_z Z \leq \dim Z = R(z/S) = R(z/[V]S) \leq \dim V \cap Z = \dim_z Z,$$

so $\dim_z Z = \dim Z$.

Say that a set $Z$ is pure-dimensional if $\dim_z Z = \dim Z$ for every $z \in Z$, or equivalently, $z \mapsto \dim_z Z$ is a constant function on $Z$. Equivalently, every non-empty open subset of $Z$ has the same dimension as $Z$. Note that any open subset of a pure-dimensional set is pure-dimensional.
Lemma 2.4. In the context of the theorem... there is an open subset \( Y' \) of \( Y \) with \( \dim Y \setminus Y' < \dim Y \), such that every equivalence class of \( E \setminus Y' \) is pure-dimensional.

Proof. Let \( Z \) be the set of \( y \in Y \) such that \( \dim_y E(y) = \dim E(y) \). Then \( Z \) is a definable subset of \( Y \). Any generic point \( y \) in \( Y \) is in \( Z \). Indeed, let \( S \) be the set over which everything is defined, and suppose \( y \in Y \) with \( R(y/S) = \dim Y \). Let \( x = \pi(y) \), so \( \pi^{-1}(x) \) is \( E(y) \), the equivalence class of \( y \). Then \( R(y/Sx) = \dim E(y) \); if not then taking \( y' \in E(y) \) with \( R(y'/Sx) = \dim E(y) \), we have \( R(y'/Sx) > R(y/Sx) \), and then

\[
R(y'/S) = R(y'x/S) = R(y'/xS) + R(x/S) > R(y/xS) + R(x/S) = R(y/S) = \dim Y,
\]

which is absurd. Now since \( y \) is a generic point of \( E(y) \), we have \( \dim_y E(y) = \dim E(y) \). So \( y \in Z \).

Since every generic point of \( Y \) is in \( Z \), it follows that \( \dim Y \setminus Z < \dim Y \). Let \( Y' \) be \( Y \setminus (Y \setminus Z) \). Then

\[
\dim Y \setminus Y' \leq \dim Y \setminus Z < \dim Y.
\]

And \( Y' \) is an open subset of \( Y \). Let \( E' \) be the restriction of \( E \) to \( Y' \). So if \( y \in Y' \), then \( E'(y) = E(y) \cap Y' \). Since \( Y' \) is open,

\[
\dim E'(y) \geq \dim_y E'(y) = \dim_y E(y) = \dim E(y) \geq \dim E'(y)
\]

where the second equality holds because \( y \in Z \). Thus \( \dim_y E'(y) = \dim E'(y) \) for every point \( y \in Y' \). So the equivalence classes of \( E' \) are pure-dimensional.

Consequently, replacing \( Y \) with \( Y' \) and \( X \) with \( \pi(Y') \), we may assume that every equivalence class of \( E \) is pure-dimensional. In subsequent reductions, we will replace \( Y \) with smaller open sets. Because open subsets of pure-dimensional sets are still pure-dimensional, we will not lose the pure-dimensionality property.

2.3 Step 2: Open quotients

Assume that every equivalence class of \( E \) is pure-dimensional.

Let \( S \) be a set over which \( Y \) and \( E \) are defined.

Lemma 2.5. Suppose \( y \in Y \) is generic, i.e., \( R(y/S) = \dim Y \). Let \( B \) be an open subset of \( Y \), and suppose \( \pi(y) \in \pi(B) \). Then there is an open neighborhood \( U \) of \( y \) in \( B \) with \( \pi(U) \subset \pi(B) \). We can take \( U \) to be a basic open neighborhood (i.e., an intersection of an open box with \( Y \)).

Proof. Since \( \pi(y) \in \pi(B) \), there is some \( y' \in B \) with \( \pi(y) = \pi(y') \). By Remark 2.3 take some open neighborhood \( V \) of \( y' \) such that \( V \subset B \) and

\[
[V] \downarrow [B]y.
\]
Now $\pi(y) = \pi(y') \in \pi(V) \subset \pi(B)$. Also, $\pi^{-1}(\pi(V))$ is $[V]S$-definable. If $Q$ is $\pi^{-1}(\pi(V)) \subset Y$, then the boundary $\text{bd}_Y(Q)$ of $Q$ as a subset of $Y$ has dimension less than $\dim Y$, by Remark 1.12. In particular, $R(y/[V]S) = R(y/S) > \dim \text{bd}_Y(Q)$, so $y \notin \text{bd}_Y(Q)$. As $y \in Q$, we see that some open neighborhood $U$ of $y$ in $Y$ is contained in $Q$. Thus $y \in U$ and $\pi(U) \subset \pi(V) \subset \pi(B)$. We can take $U$ to be basic by shrinking it.

Let $Z$ be the set of all $y \in Y$ with the following property: if $B$ is a basic open set in $Y$ and $\pi(y) \in \pi(B)$, then there is a basic open $U$ containing $y$, with $\pi(U) \subset \pi(B)$. By the Lemma, every generic point of $Y$ is in $Z$, so $\dim Y \setminus Z < \dim Y$. Let $W = Y \setminus Z$. Let $R \subset X$ be those $x$ for which $\dim \pi^{-1}(x) \cap W = \dim \pi^{-1}(x)$. Let $W' = W \cup \pi^{-1}(R)$. Let $Z' = Y \setminus W'$.

Now for each $x \in X$, one of the following holds:

- $x \in R$. Then $\pi^{-1}(x) \subset W'$, and $x \notin \pi(Z')$. Also, $\pi^{-1}(x) \cap W' = \pi^{-1}(x)$ has the same dimension as $\pi^{-1}(x) \cap W$, by definition of $R$. In this case, $x \notin \pi(Z')$.

- $x \notin R$. Then $\pi^{-1}(x)$ is disjoint from $\pi^{-1}(R)$, so $\pi^{-1}(x) \cap W$ equals $\pi^{-1}(x) \cap W$ and has lower dimension than $\pi^{-1}(x)$.

Either way, $\pi^{-1}(x) \cap W'$ and $\pi^{-1}(x) \cap W$ have the same dimension. Since this holds for every $x$, $\dim W' = \dim W$. Also, we see from this dichotomy that

$$
\text{If } x \in \pi(Z'), \text{ then } \dim \pi^{-1}(x) \cap W' < \dim \pi^{-1}(x).
$$

(1)

Claim 2.6. The restriction of $E$ to $Z'$ is an open equivalence relation.

Proof. Let $X'$ be $\pi(Z')$ and let $\tau : Z' \rightarrow X'$ be the restriction of $\pi$ to $Z'$. We need to show that for $B$ a basic open subset of $Y$, $\tau^{-1}(\pi(B \cap Z'))$ is an open subset of $Z'$. Suppose $y \in \tau^{-1}(\pi(B \cap Z'))$. Because $y \in Z' \subset Z$, there is some open neighborhood $U \supseteq y$ of $y$ in $Y$ such that $\pi(U) \subset \pi(B)$. It suffices to show that $U \cap Z'$ is in $\tau^{-1}(\pi(B \cap Z'))$, i.e., that $\tau(U \cap Z') = \pi(U \cap Z')$ is a subset of $\tau(B \cap Z') = \pi(B \cap Z')$.

$$
\pi(U \cap Z') \subseteq \pi(B \cap Z').
$$

Clearly

$$
\pi(U \cap Z') \subseteq \pi(B) \cap \pi(Z'),
$$

so it suffices to show that $\pi(B) \cap \pi(Z') \subseteq \pi(B \cap Z')$. Suppose $x \in \pi(B) \cap \pi(Z')$. Because $B$ is open, $B \cap \pi^{-1}(x)$ is an open subset of $\pi^{-1}(x)$. By the previous section, $\pi^{-1}(x)$ is pure-dimensional. So $\dim B \cap \pi^{-1}(x) = \dim \pi^{-1}(x)$. Since $x \in \pi(Z')$, by (1), we have that

$$
\dim \pi^{-1}(x) \cap W' < \dim \pi^{-1}(x) = \dim B \cap \pi^{-1}(x).
$$

So some point $z \in B \cap \pi^{-1}(x)$ is not in $W'$, i.e., it is in $Z'$. Then $z \in B \cap Z'$ and $\pi(z) = x$, so $x \in \pi(B \cap Z')$. As $x$ was arbitrary, $\pi(B) \cap \pi(Z') \subseteq \pi(B \cap Z')$.

$\square$
Now let \( Y' \) be \( Y \setminus W' \). So
\[
\dim Y \setminus Y' \leq \dim W' \leq \dim W'' \leq \dim W < \dim Y.
\]
Also, \( Y' \) is an open subset of \( Y \), and \( Y' \cap Z' = Y' \) is an open subset of \( Z' \). By Remark 1.3(a) applied to \( E \upharpoonright Z' \) and \( E \upharpoonright Y' \), the restriction of \( E \) to \( Y' \) is an open equivalence relation on \( Y' \).

Therefore, replacing \( Y \) with \( Y' \) and \( X \) with \( \pi(Y') \), we may assume that \( E \) is an open equivalence relation. Also, \( Y' \) is an open subset of \( Y \), and \( Y' \cap Z' = Y' \) is an open subset of \( Z' \). By Remark 1.3(a) applied to \( E \upharpoonright Z' \) and \( E \upharpoonright Y' \), the restriction of \( E \) to \( Y' \) is an open equivalence relation on \( Y' \).

Therefore, replacing \( Y \) with \( Y' \) and \( X \) with \( \pi(Y') \), we may assume that \( E \) is an open equivalence relation. Note that the equivalence classes of \( E \) on \( Y' \) are open subsets of the equivalence classes on \( Y \), so they remain pure-dimensional.

So we may assume that the equivalence classes of \( E \) are pure-dimensional, and that \( E \) is an open equivalence relation. By Remark 1.3(b), the topology on \( Y/E = X \) is now definable. Moreover, if we subsequently replace \( Y \) with an open subset \( Y' \), then \( \pi(Y') \) will have the subspace topology from \( X \), by Remark 1.3(c).

### 2.4 Step 3: Separation axioms

Let \( F = \bigcup_{x \in X} \partial \pi^{-1}(x) \).

**Claim 2.7.** \( \dim F < \dim Y \).

**Proof.** Let \( S \) be a set over which \( Y, E, F \) are defined. Take \( z \in F \) with \( R(z/S) = \dim F \). Then \( z \in \partial \pi^{-1}(x) \) for some \( x \in X \). Let \( y \in \pi^{-1}(x) \) have \( R(y/xS) = \dim \pi^{-1}(x) \). Now
\[
\dim F = R(z/S) \leq R(zx/S) = R(z/xS) + R(x/S) \leq \dim \partial \pi^{-1}(x) + R(x/S)
\]
Because the frontier of a set always has lower dimension than the set itself,
\[
\dim \partial \pi^{-1}(x) + R(x/S) < \dim \pi^{-1}(x) + R(x/S) = R(y/xS) + R(x/S) = R(xy/S).
\]
But \( x = \pi(y) \in \text{dcl}(Sy) \), so
\[
R(xy/S) = R(y/S) \leq \dim Y.
\]
Putting everything together, \( \dim F < \dim Y \). \( \square \)

Consequently, \( \overline{F} < \dim Y \). Let \( Y' \) be \( Y \setminus F \). For any \( y \in Y' \), \( E(y) \cap Y' \) is a closed subset of \( Y' \): if not, then there is \( z \in Y' \cap \overline{E(y) \setminus E(y)} = Y' \cap \partial E(y) \subset Y' \cap F = \emptyset \). Replacing \( Y \) by \( Y' \) and \( X \) by \( \pi(Y') \), we may therefore assume that the equivalence classes are closed. This preserves the properties obtained above, that the equivalence relation is open and that the equivalence classes are pure-dimensional.

Therefore, we may assume that the equivalence classes are closed (as subsets of \( Y \)), on top of the assumptions that the equivalence relation is open and the equivalence classes are pure dimensional. In terms of the quotient topology on \( X \), we have arranged that singletons in \( X \) are closed.\(^1\)

\(^1\)In other words, the topology on \( X \) is now \( T_1 \).
The next step is to make the topology on $X$ be Hausdorff. Say that $x, x'$ in $X$ can be separated by neighborhoods if there exist open neighborhoods $V \ni x$, $V' \ni x'$ with $V \cap V' = \emptyset$. Let $H \subset X$ consist of those $x$ which can be separated by neighborhoods from every $x' \neq x$. Then $H$ is a definable set.

**Claim 2.8.** Let $S$ be a set over which $Y, E, H$ are defined. If $y \in Y$ has $R(y/S) = \dim Y$, then $\pi(y) \in H$.

**Proof.** Let $x = \pi(y)$. Suppose $x' \in X$ is not equal to $x$. Write $x'$ as $\pi(y')$ for some $y' \in Y$. Then $y' \notin E(y)$. We arranged that $E(y)$ is closed (as a subset of $Y$). Therefore there is a basic open neighborhood $U' \ni y'$ such that $U' \cap E(y) = \emptyset$. By Remark 2.3 we can shrink $U'$ a bit, and arrange that

$$[U'] \downarrow yy'.$$

Then $R(y/[U']S) = R(y/S) = \dim Y$. Let $Q = \pi^{-1}(\pi(U'))$; this is $[U']S$-definable. As $U' \cap E(y) = \emptyset$, $y \notin Q$. By Remark 1.12, $\dim bd_Y(Q) < \dim Y = R(y/[U']S)$, so $y \notin bd_Y(Q)$. Therefore some neighborhood of $y$ is not in $Q$. Let $U$ be this neighborhood. Then $U \cap \pi^{-1}(\pi(U')) = \emptyset$, so $\pi(U) \cap \pi(U') = \emptyset$. But $\pi(U)$ is an open neighborhood of $\pi(y) = x$, and $\pi(U')$ is an open neighborhood of $\pi(y') = x'$. So we have separated $x$ and $x'$. As $x'$ was arbitrary, $x \in H$. \qed

It follows that $\dim Y \setminus \pi^{-1}(H) < \dim Y$. Let $Y'$ be $Y \setminus (Y \setminus \pi^{-1}(H))$. Then $Y'$ is an open subset of $Y$, and $\dim Y \setminus Y' < \dim Y$. Also, $X' := \pi(Y')$ is a subset of $H$, and the quotient topology on $X'$ as a quotient of $Y'$ is the subspace topology from $X$. Since each point of $X'$ can be separated by neighborhoods from any other point in $X$, it follows that $X'$ is Hausdorff.

Replacing $Y$ with $Y'$ and $X$ with $X'$, we may therefore assume that the quotient topology is Hausdorff, on top of the assumptions that the equivalence classes are pure-dimensional and the equivalence relation is open. If we subsequently replace $Y$ with a smaller open set, all these properties will be preserved.

**2.5 Step 4: Local Euclideanity**

**Lemma 2.9.** If $g$ is a definable function $X \to M^m$, and $g \circ \pi : Y \to X \to M^m$ is continuous at some $y \in Y$, then $g$ is continuous at $\pi(y)$.

**Proof.** Given an open neighborhood $U$ around $g(\pi(y))$, there is an open neighborhood $V$ around $y$ such that $g(\pi(V)) \subseteq U$, by definition of continuity. Then $V' = \pi(V)$ is an open neighborhood of $\pi(y)$, because $\pi$ is open. So $V'$ is an open neighborhood of $\pi(y)$ and $g(V') \subseteq U$. As $U$ was arbitrary, $g$ is continuous at $\pi(y)$. \qed

**Lemma 2.10.** If $Y, E$ are definable over some set $T$ and if $f : X \to M^m$ is a $T$-definable function and if $x \in X$ is the image under $\pi$ of a generic point $y \in Y$ (over $T$), then $f$ is continuous at $x$. In fact, it is continuous on a neighborhood of $x$. 

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Proof. By cell decomposition, \( f \circ \pi \) is continuous at \( y \). By the previous lemma, \( f \) is continuous at \( x \). Now let \( f' : X \to M^1 \) be the characteristic function of where \( f \) is continuous. Applying this argument to \( f' \) in place of \( f \), we see that \( f' \) is locally constant around \( x \), and so \( f \) is continuous on a neighborhood of \( x \).

If \( A \) is a definable (i.e., interpretable) topological space, definable over a set \( T \), say that a point \( \alpha \in A \) is **nice** if every \( T \)-definable function \( f : A \to M^n \) is continuous at \( \alpha \). By the argument in the proof of the previous lemma, this also implies that every \( T \)-definable function \( f : A \to M^n \) is continuous on a neighborhood of \( \alpha \), and that any \( T \)-definable subset \( D \) of \( A \) has \( x \notin \text{bd}(D) \).

- If \( Y \) and \( E \) are \( T \)-definable, and \( y \in Y \) has \( R(y/T) = \dim Y \), then \( \pi(y) \in X \) is a nice point of \( X \), by Lemma 2.10
- If \( Z \) is some \( T \)-definable subset of \( M^n \), and \( z \in Z \) has \( \dim Z = R(z/T) \), then \( z \) is a nice point of \( Z \). This follows by cell decomposition (\( z \) cannot be in the closure of any cell other than the top-dimensional cell which it belongs to).

**Lemma 2.11.** Let \( A \) and \( B \) be two definable topological spaces, definable over a set \( T \). Suppose that \( f : A \to B \) is a continuous \( T \)-definable function. Suppose that \( A \subset M^m \) for some \( m \). Suppose that \( \alpha \) is a nice point of \( A \) and \( f(\alpha) \) is a nice point of \( B \). Suppose that \( \alpha \) and \( f(\alpha) \) are inter-definable over \( T \). Then some definable neighborhood of \( \alpha \) in \( A \) is definably homeomorphic via \( f \) to some definable neighborhood of \( f(\alpha) \) in \( B \).

**Proof.** Let \( \beta = f(\alpha) \), and write \( \alpha = g(\beta) \) for some \( T \)-definable function \( g \). Let \( \Gamma \) be the set of pairs \((a, b) \in A \times B \) such that \( b = f(a) \), \( a = g(b) \), and \( g \) is continuous at \( b \). Note that \((\alpha, \beta) \in \Gamma \). Indeed, since \( A \subset M^m \), the \( T \)-definable function \( g : B \to A \subset M^m \) must be continuous at \( \beta \), because \( \beta \) is nice. Now let \( A' \) be the projection of \( \Gamma \) to \( A \), and \( B' \) be the projection of \( \Gamma \) to \( B \). So \( f \mathrel{\upharpoonright} A' \) is a continuous bijection from \( A' \) to \( B' \), and the inverse is the continuous bijection \( g \mathrel{\upharpoonright} B' \). In particular, \( f \) induces a homeomorphism from \( A' \) to \( B' \). As \( \alpha \) is nice and \( A' \) is \( T \)-definable, \( \alpha \) is not in the boundary of \( A' \). So we can find some definable open neighborhood \( U \) of \( \alpha \) such that \( \alpha \in U \subset A' \). Similarly, we can find some definable open neighborhood \( V \) of \( \beta \) such that \( \beta \in V \subset B' \). Then \( f \) induces a homeomorphism from \( U \cap g(V) \) to \( f(U) \cap V \). But \( g(V) \) is a relatively open subset of \( A' \), so \( U \cap g(V) \) is a relatively open subset of \( U \cap A' = U \), hence open as a subset of \( A \). Similarly, \( f(U) \cap V \) is an open neighborhood of \( \beta \) in \( B \). Therefore \( \alpha \) and \( \beta \) have definably homeomorphic definable open neighborhoods, with the homeomorphism induced by \( f \).

**Lemma 2.12.** If \( y \in Y \) is generic (i.e., \( R(y/S) = \dim Y \)), then \( X \) is locally Euclidean around \( \pi(y) \).

**Proof.** Let \( x = \pi(y) \). Let \( k = R(y/Sx) \). Some \( k \) of the coordinates of \( y \) form a basis for the tuple \( \vec{y} \) in the \( \text{dcl}_{Sx}(-) \) pregeometry. Permuting the coordinates, we may assume that \( y_1, \ldots, y_k \) form a basis for \( \vec{y} \). Then \( y \in \text{dcl}(Sx y_1, \ldots, y_k) \). As

\[
k \geq R(y_1, \ldots, y_k/S) \geq R(y_1, \ldots, y_k/Sx) = k,
\]
we have \( y_1 \ldots y_k \downarrow_S x \). Let \( T \) be \( S \cup \{ y_1, \ldots, y_k \} \). Then \( y \) and \( x \) are interdefinable over \( T \).

Let \( Y' \) be the set of \( y' \in Y \) whose first \( k \) coordinates agree with those of \( y \). Note that \( Y' \) is \( T \)-definable. Note that \( \pi \restriction Y' \) is a \( T \)-definable continuous function from \( Y' \) to \( X \), and that \( y \) and \( \pi(y) = x \) are interdefinable over \( T \). We can apply Lemma 2.1 with \( A = Y' \), \( B = X \), \( f = \pi \), and \( \alpha = y \) assuming we prove the following:

- \( x \) is a nice point of \( X \) (with respect to \( T \)).
- \( y \) is a nice point of \( Y' \) (with respect to \( T \)).

Take \( y' \in \pi^{-1}(x) = E(y) \) with \( R(y'/y) = \dim E(y) \). As \( E(y) \) is \( Sx \)-definable, \( R(y'/Sx) = \dim E(y) \) and \( y' \downarrow_{Sx} y \). By monotonicity, \( y' \downarrow_{Sx} T \). As \( x \downarrow_{S} T \), transitivity yields \( xy' \downarrow_{S} T \). In particular, \( R(y'/T) = R(y'/S) \). Now

\[
R(y'/S) = R(y'/Sx) + R(x/S) = \dim E(y) + R(x/S) \geq R(y/Sx) + R(x/S) = R(y/S) = \dim Y,
\]

so \( R(y'/T) \geq \dim Y \). Therefore, \( y' \) is a generic point in \( Y \) as far as \( T \) is concerned. It follows by Lemma 2.10 that \( \pi(y') = x \) is nice, with respect to \( T \).

Meanwhile, \( y \) is a nice point of \( Y' \) because \( R(y/T) = \dim Y' \). Indeed, take \( y'' \in Y' \) with \( R(y''/T) = \dim Y' \). Then the first \( k \) coordinates of \( y'' \) are \( y_1, \ldots, y_k \), so \( T \in \text{dcl}(Sy'') \). Thus

\[
R(y''/S) = R(y''/T) + R(T/S) = \dim Y' + R(T/S) \geq R(y/T) + R(T/S) = R(y/S) = \dim Y.
\]

Since \( y'' \in Y \) and \( Y \) is \( S \)-definable, equality must hold. Then \( R(y/T) = \dim Y' \). It follows that \( y \) is a nice point of \( Y' \).

So Lemma 2.11 applies. In particular, \( \pi \) induces a homeomorphism from some open neighborhood of \( y \) in \( Y' \) to some open neighborhood of \( y \) in \( X \). It remains to show that some open neighborhood of \( y \) in \( Y' \) is Euclidean. But this is clear, using the cell-decomposition of the \( T \)-definable set \( Y' \), and the fact that \( R(y/T) = \dim Y' \). (This ensures that \( y \) is in a top-dimensional cell, and is not in the closure of any lower-dimensional cell. Consequently, \( Y' \) looks like the interior of a cell, around the point \( y \).)

Now let \( Q \) be the set of points \( y \in Y \) such that \( X \) is locally Euclidean around \( \pi(y) \). Then \( Q \) is ind-definable (the complement of a type-definable set). We have just seen that \( Q \) contains the type-definable set \( \{ y \in Y : R(y/S) = \dim Y \} \). By compactness, there must be some definable set \( D \subset Y \) such that

\[
\{ y \in Y : R(y/S) = \dim Y \} \subset D \subset Q
\]

The first inclusion implies that \( \dim Y \setminus D < \dim Y \). Letting \( Y'' = Y \setminus Y \setminus D \), we get that \( Y'' \) is an open subset of \( Y \), \( Y \setminus Y'' \) has lower dimension than \( \dim Y \), and \( Y'' \subset Q \), so that the topology on \( \pi(Y'') \) is locally Euclidean. As usual, we replace \( Y \) with \( Y' \) and \( X \) with \( \pi(Y') \).

At this point we have gotten the topology to be definable, Hausdorff, and locally Euclidean. By Lemma 1.13, we also get regularity. This completes the proof of Theorem 2.1.