Generically stable types are stably dominated in C-minimal expansions of ACVF

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1 Introduction

Fix a C-minimal expansion T of ACVF, with monster model $\mathbb{U}$, home sort K, value group Γ, and residue field k.

If $A \subset \mathbb{U}^{eq}$, write $k(A)$ for $k \cap \mathrm{dcl}(A)$.

Remark 1.1. $k \cap \mathrm{acl}(A) = k \cap \mathrm{acl}(k(A))$.

Proof. Suppose $\alpha \in k$ is algebraic over A. Let S be the finite set of conjugates of α over A. Then S has a code in a power of k, so S has a code in $k(A)$. Therefore α is algebraic over $k(A)$. $\square$

Let $p(x)$ be a generically stable type over C_0, thought of as a C_0-definable type over $\mathbb{U}$. The type $p(x)$ might live in an imaginary sort.

We are going to prove that there is a small set $C \supseteq C_0$ and a C-definable map f into a power of k such that p is "dominated" over C by its pushforward along f. That is, for every $D \supseteq C$ and every a, the following will be equivalent:

- $a \models p|D$
- $a \models p|C$ and $f(a) \models f_*p|D$.

From the general "descent" property for stably dominated types, this should be enough to ensure that the original type over C_0 was stably dominated.

2 Proof

Lemma 2.1. *Let $C = \mathrm{acl}(C)$ be a set of parameters. Suppose $\mathrm{tp}(a/C)$ is generically stable, for some $a \in \mathbb{U}^{eq}$, and suppose $b \in \mathrm{acl}(Ca)$. Then $\mathrm{tp}(b/C)$ is generically stable.*

Proof. Let p be the canonical global extension of $\mathrm{tp}(a/C)$, and let $M \supseteq C$ be a small model in which p is finitely satisfiable. On general grounds, $\mathrm{tp}(ab/C)$ is definable; let q be its canonical global extension. It suffices to show that $q(x, y)$ is finitely satisfiable in M, since this ensures that q is generically stable, hence so is its pushforward along the projection to the second coordinate.

Let d be an element of $\mathbb{U}$, and suppose $\phi(x, y, d) \in q(x, y)$. Let $\psi(x; y)$ be a C-formula such that $\psi(a; b)$ holds and $\psi(a; \mathbb{U})$ is finite for every a. Such a formula exists because $b \in \mathrm{acl}(Ca)$. Let $a'b' \models q|Md$. Then $a'b' \equiv_M ab$, so $\psi(a'; b')$ holds. Also, $\phi(a', b', d)$ holds. The formula

$$\exists y : \phi(x, y, d) \wedge \psi(x; y)$$

is in $p(x)$, because a' satisfies it and $a' \models p|Cd$. Because p is finitely satisfiable in M, there is some $a'' \in M$ such that

$$\exists y : \phi(a'', y, d) \wedge \psi(a''; y)$$

Choose b'' such that $\phi(a'', b'', d) \wedge \psi(a''; b'')$ holds. Then $b'' \in \mathrm{acl}(Ca'') \subset M$. So the pair (a'', b'') is in M, and satisfies $\phi(x, y, d)$. □

Lemma 2.2. *Suppose C is a small set of parameters, and B is a ball in K^1, (possibly a singleton). Suppose $\mathrm{tp}(\ulcorner B \urcorner/C)$ is generically stable. Then either B is C-definable, or there exists $A \supset A' \supseteq B$, where A is a C-definable closed ball of some radius, A' is an open ball of the same radius, and A' is not defined over $\mathrm{acl}(C)$.*

Proof. Let $B = B_1, B_2, \ldots$ be a Morley sequence for the type of $\ulcorner B \urcorner$ over C. Assume B is not C-definable. Then the type is not constant, so the B_i's are distinct. Since the type is generically stable, this sequence is totally indiscernible. Consequently, $B_i \cap B_j = \emptyset$ for $i \neq j$. Let A_{ij} be the smallest ball containing both B_i and B_j, for $i \neq j$. Then A_{ij} is a closed ball. The total indiscernibility of the sequence implies that $A := A_{ij}$ does not depend on i, j. As $A_{1,2}, A_{3,4}, A_{4,5}, \ldots$ is a Morley sequence of a C-definable type (a pushforward of $\mathrm{tp}(\ulcorner B \urcorner/C)^{\otimes 2}$), it follows that A is C-definable.

Let A'_i be the open subball of A of the same radius, containing B_i. Then the sequence $A'_1, A'_2, \ldots$ is a Morley sequence over C. As A is the smallest ball containing B_i and B_j, we must have $A'_i \cap A'_j = \emptyset$ for $i \neq j$. So the elements of the sequence $A'_1, A'_2, \ldots$ are pairwise distinct. As the sequence is C-indiscernible, it follows that the elements are not algebraic over C. In particular, $A' := A'_1 \supseteq B$ is not algebraic over C. □

The next lemma is included to convince myself that I'm not cheating…

Lemma 2.3. *Suppose C is a small set of parameters, and a and b are from $\mathbb{U}^{eq}$ such that $\mathrm{stp}(a/C)$ is generically stable. Then $a \downarrow_C b \iff b \downarrow_C a$. (In particular, this includes the case where a is in a C-definable set.)*

Proof. On general grounds[1], we may replace C with $\mathrm{acl}(C)$, so we may assume $C = \mathrm{acl}(C)$.

[1] A sequence is C-indiscernible if and only if it is $\mathrm{acl}(C)$-indiscernible. So a formula forks over C if and only if it forks over $\mathrm{acl}(C)$. If $\mathrm{tp}(a/b\,\mathrm{acl}(C))$ doesn't fork over $\mathrm{acl}(C)$, then the smaller type $\mathrm{tp}(a/bC)$

Let $p(x)$ be the unique global non-forking extension of $\mathrm{tp}(a/C) = \mathrm{stp}(a/C)$. By Corollary 2.14 in *On NIP and Invariant Measures*, there is some C-invariant type $q(y)$ extending $\mathrm{tp}(b/C)$.

Suppose that $a \mathop{\smile\!\!\!\!\!\!\downarrow}_C b$. Then $\mathrm{tp}(a/Cb)$ doesn't fork over $\mathrm{tp}(a/C)$, so it must be $p|Cb$. Then $a \models p|Cb$ and $b \models q|C$, or equivalently, $(a, b) \models p \otimes q|C$. By one of the characterizations of generic stability, $(b, a) \models q \otimes p|C$. So $\mathrm{tp}(b/Ca) = q|Ca$. Since q doesn't fork over C, $b \mathop{\smile\!\!\!\!\!\!\downarrow}_C a$.

Conversely, suppose that $b \mathop{\smile\!\!\!\!\!\!\downarrow}_C a$. Then by the characterization of forking in NIP theories (Proposition 2.1(i) in HP), $\mathrm{tp}(b/Ca)$ has some global extension $r(y)$ which is Lascar C-invariant. By Corollary 2.14 in HP, $r(y)$ is C-invariant. Then $b \models r|Ca$ and $a \models p|C$, so $(b, a) \models r \otimes p|C$. As before, this implies that $(a, b) \models p \otimes r|C$, so $a \models p|Cb$. As p is C-invariant, $a \mathop{\smile\!\!\!\!\!\!\downarrow}_C b$. □

Now fix a generically stable type $p(x)$, defined over some base set of parameters C_0. The variable x might live in an imaginary sort.

Lemma 2.4. *For $C \supseteq C_0$, let $r(C)$ denote the supremum of $RM(\alpha/C)$, where α is a tuple in $k(Ca)$ and a realizes $p|C$. (By Remark 1.1, we could even let α range over $k \cap \mathrm{acl}(Ca)$, and $r(C)$ would not change.)*

(a) *There is an integer n such that $r(C) \leq n$ for every $C \supseteq C_0$.*

(b) *If $C' \supset C \supseteq C_0$, then $r(C') \geq r(C)$.*

Consequently, there is some $C \supseteq C_0$ such that $r(C') = r(C)$ for every $C' \supset C$.

Proof. **(a)** C-minimal theories are dp-minimal, so the home sort has dp-rank 1. By additivity of dp-rank in NIP theories, every interpretable set in T has finite dp-rank. Let n be the dp-rank of the sort where the variable x lives. Suppose $C \supseteq C_0$, a realizes $p|C$, and α is a tuple in $k(Ca)$. Suppose for the sake of contradiction that $RM(\alpha/C) \geq n + 1$. As k is a strongly minimal set, we can replace α with some subtuple, and assume that α has length $n + 1$, and that it realizes the generic type of k^{n+1}, over C. Write α as $f(a)$ for some C-definable function f. Then the range of f has dp-rank at most n. But the generic type of k^{n+1} over C has dp-rank (at least) n, a contradiction.

(b) Suppose a realizes $p|C$ and $\alpha \in k(Ca)$ has $RM(\alpha/C) = m$. Moving C' over C, we may assume that a realizes $p|C'$. As p is C_0-definable, hence C-invariant, $a \mathop{\smile\!\!\!\!\!\!\downarrow}_C C'$. So $\alpha \mathop{\smile\!\!\!\!\!\!\downarrow}_C C'$. Consequently, $RM(\alpha/C) = RM(\alpha/C')$. And $\alpha \in k(C'a)$.

□

Fix some C as in the conclusion of the lemma. Let $m = r(C)$. Fix some C-definable function f into k^m such that f_*p is the generic type of k^m.

For B a non-degenerate (infinite) closed ball, let $\mathrm{res}\, B$ denote the interpretable set of open subballs of the same radius.

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doesn't fork over C. Conversely, suppose $\mathrm{tp}(a/bC)$ doesn't fork over C, or equivalently, over $\mathrm{acl}(C)$. Then by extension, there is some a' realizing $\mathrm{tp}(a/bC)$ such that $\mathrm{tp}(a'/b\,\mathrm{acl}(C))$ doesn't fork over $\mathrm{acl}(C)$. If σ is an automorphism over bC which sends a' back to a, then $\mathrm{tp}(a/b\sigma(\mathrm{acl}(C)))$ doesn't fork over $\mathrm{acl}(C)$. But as a set, $\sigma(\mathrm{acl}(C)) = \mathrm{acl}(C)$. So $\mathrm{tp}(a/b\,\mathrm{acl}(C))$ doesn't fork over $\mathrm{acl}(C)$.

Lemma 2.5. *Suppose $C' \supseteq C$. Suppose B is a C'-definable closed ball. Suppose $a \models p|C'$ and that $\alpha \in \operatorname{res} B$ is algebraic over $C'a$. Then α is algebraic over $C'f(a)$.*

Proof. Let e and d realize (independently) the generic type of B over $C'a$. Then $ed \mathop{\smile\!\!\!\!|}_{C'} a$, hence $ed \mathop{\smile\!\!\!\!|}_{C'} \alpha f(a)$. By base monotonicity on the right (which holds for forking in arbitrary theories), $ed \mathop{\smile\!\!\!\!|}_{C'f(a)} \alpha$.

Over $C'ed$, $\operatorname{res} B$ is in definable bijection with k, via the map sending the class of $x \in B$ to $\operatorname{res}((x-e)/(d-e))$, for example. So α is interdefinable over $C'ed$ with some $\alpha' \in k$. If $\alpha' \notin \operatorname{acl}(f(a)C'ed)$, then $\alpha' f(a)$ realizes the generic type of k^{m+1} over $C'ed$, so $r(C'ed) = m + 1 > m = r(C)$, a contradiction.[2] Therefore $\alpha' \in \operatorname{acl}(f(a)C'ed)$, and hence $\alpha \in \operatorname{acl}(f(a)C'ed)$. Since $ed \mathop{\smile\!\!\!\!|}_{C'f(a)} \alpha$, it follows that $\alpha \mathop{\smile\!\!\!\!|}_{C'f(a)} \alpha$. This can only happen if $\alpha \in \operatorname{acl}(C'f(a))$. □

Lemma 2.6. *Suppose $C' \supseteq C$. Suppose $a \models p|C'$. Suppose that b is a* singleton in the home sort. *Suppose that the type of $f(a)$ over $C'b$ is the generic type of k^m. Then $a \models p|C'b$.*

Proof. As $\operatorname{tp}(a/C')$ is stationary, it implies $\operatorname{stp}(a/C')$. So $a \models p|\operatorname{acl}(C')$. Similarly, the type of $f(a)$ over $\operatorname{acl}(C'b)$ is still generic in k^m. Replacing C' with $\operatorname{acl}(C')$, we may assume that $C' = \operatorname{acl}(C')$.

Let $\phi(x; y)$ be a C'-formula, and suppose $\phi(x; b) \in p(x)$. We will show that $\phi(a; b)$ holds. Let D be the definable set $\phi(a; \mathbb{U})$. This can be written as a boolean combination of $\operatorname{acl}(aC')$-definable balls $B_1, \ldots, B_n$. By Lemma 2.1, $\operatorname{tp}(\ulcorner B_i \urcorner / C')$ is generically stable for each i.

Claim 2.7. *For each i, either B_i is C'-definable or $b \notin B_i$.*

Proof. Suppose B_i is not C'-definable. By Lemma 2.2, we have the following setup: there is some C'-definable closed ball A containing B_i, and some open ball A' of the same radius, with $A \supset A' \supseteq B_i$, *and (the code for) A' is not algebraic over C'*. Now $\ulcorner A' \urcorner$ is an element $\alpha \in \operatorname{res} A'$, and α is definable from $\ulcorner A \urcorner$ and $\ulcorner B_i \urcorner$. As $\ulcorner A \urcorner$ is C'-definable and $\ulcorner B_i \urcorner$ is algebraic over a and C', it follows that $\alpha \in \operatorname{acl}(C'a)$. By Lemma 2.5, $\alpha \in \operatorname{acl}(C'f(a))$.

Since $f(a)$ realizes the generic type of k^m over $C'b$, we have $f(a) \mathop{\smile\!\!\!\!|}_{C'} b$. Consequently $\alpha \mathop{\smile\!\!\!\!|}_{C'} b$. If $b \in B_i$, then the code α for A' is algebraic over $C'b$, so we would have $\alpha \mathop{\smile\!\!\!\!|}_{C'} \alpha$. This contradicts the fact that A' is not algebraic over C'. □

Let $a^1 = a$ and $B_i^1 = B_i$. Choose $a^2, a^3, \ldots$ and B_i^j such that

$$\langle a^j \ulcorner B_1^j \urcorner \ulcorner B_2^j \urcorner \cdots \rangle_{j=2,3,\ldots}$$

is a Morley sequence over $ba \ulcorner B_1 \urcorner \cdots$ for the type

$$\operatorname{tp}(a \ulcorner B_1 \urcorner \ulcorner B_2 \urcorner \cdots / C')$$

which is generically stable by Lemma 2.1. Then

$$\langle a^j \ulcorner B_1^j \urcorner \ulcorner B_2^j \urcorner \cdots \rangle_{j=1,2,\ldots}$$

[2] This is using Remark 1.1.

is a Morley sequence for this type, over C'. Also, $a^2 \models p|C'b$, so $\phi(a^2; b)$ holds if and only if $\phi(x; b) \in p(x)$. Therefore, it suffices to show for each i that

$$b \in B_i^2 \iff b \in B_i^1.$$

Note that $B_i^1, B_i^2, \ldots$ is a Morley sequence over C', and $B_i^2, B_i^3, \ldots$ is a Morley sequence over $C'b$. If $B_i = B_i^1$ is C'-definable, this sequence is constant, so $b \in B_i^1 \iff b \in B_i^2$. Otherwise, by total indiscernibility, the B_i^j are pairwise disjoint (for fixed i). So $b \notin B_i^j$ for all $j > 1$. But by the claim, $b \notin B_i^1$ either. So we are done. $\square$

Theorem 2.8. *Suppose that $C' \supseteq C$ and $a \models p|C$ and $f(a)$ realizes the generic type of k^m over C'. Then $a \models p|C'$. So the (arbitrary generically stable type p) is stably dominated, in some sense of the words.*

Proof. Take some set C'' of real elements such that $C' \subset \operatorname{dcl}(C'')$. Moving C'' over C', we may assume that $f(a)$ realizes the generic type of k^m over C''. Replacing C' with C'', we may assume that C' is made of real elements.

Let $b_1, \ldots, b_n$ be a tuple from C', and suppose $\phi(x; b)$ is in $p(x)$. It suffices to show that $\phi(a; b)$ holds. It suffices to show that $a \models p|Cb_1 b_2 \cdots b_n$.

We prove by induction on i that $a \models p|Cb_1 \cdots b_i$. The base case where $i = 0$ is given. Suppose that $a \models p|Cb_1 \cdots b_{i-1}$. By Lemma 2.6, we need only show that $\operatorname{tp}(f(a)/Cb_1 \cdots b_i)$ is the generic type of k^m. This is clear, though, since $\operatorname{tp}(f(a)/C')$ was generic in k^m, and $Cb_1 \cdots b_i \subset C'$. $\square$