The fundamental group and the first cohomology group

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1 Introduction

Let T be an almost strongly minimal theory eliminating imaginaries, with monster model $\mathbb{M}$. For S a small set of parameters, let $\mathrm{Gal}(S)$ denote the profinite group $\mathrm{Aut}(\mathrm{acl}(S)/\mathrm{dcl}(S))$. Let M be a small model of T, inside the monster model $\mathbb{M}$. Let G be a connected M-definable group.

1.1 The fundamental group

By a *finite group cover* of G, we will mean an M-definable connected group H with a surjective M-definable map $H \to G$, whose kernel K is finite. The kernel K must lie within the center of H, because the action of H on K by conjugation yields a necessarily-trivial homomorphism from the connected group H to the finite group of automorphisms of K.

If $\pi_i : H_i \to G$ are finite group covers of G for $i = 1, 2$, a morphism of finite group covers from H_1 to H_2 will be an M-definable group homomorphism $f : H_1 \to H_2$ making the obvious diagram commute. This makes finite group covers into a category.

Remark 1.1.

1. *If f is a morphism of finite group covers, the map $f : H_1 \to H_2$ must be a surjection.*
2. *The category of finite group covers is a poset: if f, g are two morphisms from $H_1 \xrightarrow{\pi_1} G$ to $H_2 \xrightarrow{\pi_2} G$, then $f = g$.*
3. *The resulting poset is (co)directed, in the sense that if $H_i \xrightarrow{\pi_i} G$ for $i = 1, 2$ are two finite group covers, then there is a third $\pi_3 : H_3 \to G$ and morphisms from $H_3 \to G$ to $H_i \to G$ for $i = 1, 2$.*

Proof.

1. Since $H_1 \to G$ factors through $H_1 \to H_2$, the kernel of $H_1 \to H_2$ is a subgroup of the finite kernel of $H_1 \to G$. Since $H_1 \to H_2$ has finite kernel, its image has the same rank as G, H_1, and H_2. Consequently, it has finite index in H_2. As H_2 is connected, the image must be all of H_2.

2. Let K be the kernel of $H_2 \to G$. For every $x \in H_1$, let $\tau(x)$ be $f(x) \cdot g(x)^{-1}$. As $f(x)$ and $g(x)$ have the same image in G, $\tau(x) \in K$. So $\tau(x) \in Z(H_2)$. Now for every x, we have $f(x) = \tau(x) \cdot g(x)$. Since f and g are homomorphisms, for any x, y we have

$$\tau(xy)g(x)g(y) = \tau(xy)g(xy) = f(xy) = f(x)f(y) = \tau(x)g(x)\tau(y)g(y).$$

   As $\tau(y)$ is in the center of H_2, it follows that $\tau(xy) = \tau(x)\tau(y)$, so τ is a homomorphism from the connected group H_1 to the finite group K. Therefore τ is trivial, and $f = g$.

3. Given $H_1 \to G$ and $H_2 \to G$, let $H_3 = (H_1 \times_G H_2)^o$. Then we have a commutative diagram

$$\begin{array}{ccc} H_3 & \longrightarrow & H_1 \\ \downarrow & & \downarrow \\ H_2 & \longrightarrow & G \end{array}$$

   Now $H_1 \times_G H_2$ surjects onto G and has kernel equal to the product of the kernels of $H_1 \to G$ and $H_2 \to G$. Therefore $H_1 \times_G H_2$ has the same rank as G, H_1, and H_2. Its connected component, H_3, has this same rank. As H_3 is a subgroup of $H_1 \times_G H_2$, the kernel of $H_3 \to G$ is still finite, so the image has the same rank as G, ans therefore is all of G. So H_3 does indeed surject onto G, and H_3 is a finite group cover.

□

Say that $H' \to G$ *dominates* $H \to G$ if there is a morphism $H' \to H$ (over G).

As a cofiltered category, we can take the inverse limit of $\ker H \to G$ as $H \to G$ ranges over the finite group covers of G. Let $\pi_1(G)$ denote the resluting profinite group.

1.2 The first cohomology group

Let $g_1, g_2, \ldots, g_n$ be a Morley sequence of generics over Mt, of length n, for $n > 1$. For non-empty $I \subset n$, let

$$\mathrm{Gal}_I = \mathrm{Gal}(\{g_i : i \in I\} \cup \mathrm{acl}_M(\{g_i \cdot g_j^{-1} : i, j \in I\}))$$

There are natural restriction maps $\mathrm{Gal}_I \to \mathrm{Gal}_J$ for $I \supset J$; these are always surjections.[1]

[1] In general, if $A \supset C$ are small sets of parameters, $\mathrm{Gal}(A) \to \mathrm{Gal}(C)$ is surjective iff $\mathrm{tp}(A/C)$ is stationary. If $\mathrm{tp}(A/C)$ is stationary and $A \downarrow_C B$, then $\mathrm{tp}(AB/CB)$ is also stationary. In particular, if A and B are independent over $C = \mathrm{acl}(C)$, then $\mathrm{Gal}(ABC) \to \mathrm{Gal}(BC)$ is surjective. Here, apply this with

$$C = \mathrm{acl}_M(g_i \cdot g_j^{-1} : i, j \in J)$$

$$A = \mathrm{acl}_M(g_i \cdot g_j^{-1} : i, j \in I)$$

$$B = g_{j_0}$$

for some $j_0 \in J$. The independence holds because $\mathrm{tp}(g_{j_0}/A)$ is generic.

If A is an abelian group of coefficients, we have been defining $H^1(G, A)$ to be the first cohomology group of the total complex of the double complex

$$\begin{tikzcd}
 & \vdots & \vdots & \\
0 \arrow[r] & \prod_i C^2(\mathrm{Gal}_i, A) \arrow[r] \arrow[u] & \prod_{i<j} C^2(\mathrm{Gal}_i, A) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \prod_i C^1(\mathrm{Gal}_i, A) \arrow[r] \arrow[u] & \prod_{i<j} C^1(\mathrm{Gal}_i, A) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \prod_i C^0(\mathrm{Gal}_i, A) \arrow[r] \arrow[u] & \prod_{i<j} C^0(\mathrm{Gal}_i, A) \arrow[r] \arrow[u] & \cdots
\end{tikzcd} \tag{1}$$

where $C^j(\Pi, A)$ is continuous cochains of the profinite group Π, with (non-twisted) coefficients in A.

Theorem 1.2. *$H^1(G, A)$ is isomorphic to the continuous homomorphisms from $\pi_1(G)$ to A.*

2 Preliminary reductions

Since our coefficients are untwisted, the maps from the zeroth row of (1) to the first row all vanish.[2] Moreover, the cohomology groups of the bottom row are $A, 0, 0, \cdots$, because

$$0 \to A \to \prod_i A \to \prod_{i<j} A \to \prod_{i<j<k} A \to \cdots$$

is exact.

So, to calculate $H^1(G, A)$, we can drop the bottom row of (1) and take the zeroth cohomology group of the total complex of the result. In other words, $H^1(G, A)$ is just the kernel of

$$\prod_i C^1(\mathrm{Gal}_i, A) \to \prod_i C^2(\mathrm{Gal}_i, A) \oplus \prod_{i<j} C^1(\mathrm{Gal}_{ij}, A).$$

Now $\ker C^1(\mathrm{Gal}_I, A) \to C^2(\mathrm{Gal}_I, A)$ is just the set of continuous homomorphisms from Gal_I to A.[3] So, $H^1(G, A)$ is precisely the kernel of

$$\prod_i \mathrm{Hom}_{cts}(\mathrm{Gal}_i, A) \to \prod_{i<j} \mathrm{Hom}_{cts}(\mathrm{Gal}_{ij}, A). \tag{2}$$

[2] $C^0(\Pi, A)$ is just A, and the map $C^0(\Pi, A) \to C^1(\Pi, A)$ sends $a \in A$ to the function $g \mapsto g \cdot a - a$. Since the action on A is trivial, this vanishes.

[3] Recall that $C^1(\Pi, A)$ consists of continuous functions from Π to A. If f is such a function, its image δf in $C^2(\Pi, A)$ is the function $(g, h) \mapsto g \cdot f(h) - f(g \cdot h) + f(g)$. If the action of Π on A is trivial, as it is in our case, then $\delta f = 0$ if and only if f is a homomorphism, clearly.

Let $\mathcal{G}$ denote $\mathrm{Gal}_{1,\ldots,n}$, and let K_I denote the kernel of $\mathcal{G} \to \mathrm{Gal}_I$. Since all the restriction maps are surjective, Gal_I is $\mathcal{G}/K_I$. A continuous homomorphism from Gal_I to A is equivalent to a continuous homomorphism $\mathcal{G} \to A$ whose kernel contains K.

Consequently, an element of the kernel of (2) consists of an n-tuple $(f_1, \ldots, f_n)$ of continuous homomorphisms $\mathcal{G} \to A$ such that $\ker f_i$ contains K_i, and for every $i < j$, f_i and f_j induce the same map from $\mathcal{G}/K_{ij}$ to A. This last condition is equivalent to $f_i = f_j$. So *really*, an element of $H^1(G, A)$ is just a continuous homomorphism $f : \mathcal{G} \to A$ whose kernel contains every K_i, or equivalently, contains $K_1K_2 \cdots K_n$ (which is indeed a normal, topologically closed subgroup).

So it remains to identify $\pi_1(G)$ with $\mathcal{G}/(K_1K_2 \cdots K_n)$. Call this group Π.

For simplicity, we will assume henceforth that $n = 2$. It can be shown that Π doesn't depend on the choice of the g_i, or on n, as long as $n > 1$. At any rate, I showed somewhere else that $H^1(G, A)$ can be calculated with $n = 2$ and doesn't depend on the g_i.

So, $\Pi = \mathcal{G}/K_1K_2$.

2.1 Some categories

2.2 Covers from group extensions

Definition 2.1. *If H is a finite group cover of G, an H-*cover *of G is an M-definable set S with an M-definable transitive action of H on S, and an M-definable map $\pi_S : S \to G$ which is a map of H-sets, viewing G as an H-set via the left action.*

A morphism of H-covers $S \to S'$ is an M-definable map $f : S \to S'$ of H-sets, such that $\pi_{S'} \circ f = \pi_S$. This makes H-covers into a category.

If H' dominates H, then any H-cover can be made canonically into an H'-cover, via the usual way of viewing H-sets as H'-sets under the map $H' \to H$. In fact, H-covers embed into H'-covers via a full and faithful functor, i.e., an embedding of categories.

Definition 2.2. *The category of* finite set covers *of G is the colimit*[4] *of the categories of H-covers, as H ranges over the finite group covers of G.*

Remark 2.3. *This category can also be described more concretely as the category whose objects are finite group covers $\pi_H : H \to G$ of G, but whose morphisms $H \to H'$ are those of the form*

$$h \mapsto \phi(h) \cdot k$$

where ϕ is a morphism of finite group covers, and k is an element of $\ker H' \to G$. That is, we allow homomorphisms and translations by elements of the fiber over 1.

(We may be subtly using the fact that $\pi_1(G)$ is abelian here. I'll try to avoid using this remark in the future.)

[4] Or rather, homotopy colimit, if that makes a difference. As long as the operation of taking this kind of directed colimit respects equivalences, it shouldn't matter.

If $1 \to K \to H \to G$ is a finite group cover of G, there is a functor from the category of H-covers of G to the category of finite sets with a transitive action of K, namely the functor which sends $\pi_S : S \to G$ to $\pi_S^{-1}(1)$.

Exercise 2.4. *This is an equivalence of categories.*

Taking the colimit over H, it follows that there is an equivalence of catgories between finite set covers of G and finite sets with a transitive continuous $\pi_1(G)$ action, and the functor between these categories is given by taking the fiber over the identity.

2.3 Covers?

Recall that $\mathrm{Gal}_{12} = \mathcal{G}$, $\mathrm{Gal}_i = \mathcal{G}/K_i$, and $\Pi = \mathcal{G}/K_1K_2$.

If S is any set of parameters, the category of finite sets with continuous $\mathrm{Gal}(S)$-action is equivalent to the category of finite S-definable sets.[5] Under this equivalence, sets with a transitive $\mathrm{Gal}(S)$-action correspond to finite S-definable sets which isolate algebraic types over S. That is, a finite S-definable set X corresponds to a finite set with transitive $\mathrm{Gal}(S)$-action if and only if all elements of X have the same type over S.

The following categories are equivalent:

- The category of finite sets with continuous Π-action.
- The category of triples (S_1, S_2, f) where S_i is a finite set with Gal_i-action, for $i = 1, 2$, and f is a Gal_{ij}-equivariant bijection S_1 to S_2.
- The category of triples (X_1, X_2, f), where X_i is a finite g_iM-definable set, and f is a $g_1 \operatorname{acl}_M(g_2 \cdot g_1^{-1})$-definable bijection between X_1 and X_2.

Under this equivalence, a triple (X_1, X_2, f) corresponds to a transitive Π-set if one of the following equivalent conditions holds:

- Every element of X_1 has the same type over g_1M.
- Every element of X_2 has the same type over g_2M.

Let $\mathcal{C}$ be the category of triples (X_1, X_2, f) satisfying these equivalent conditions. The map $F_i : (X_1, X_2, f) \mapsto X_i$ is a functor from $\mathcal{C}$ to finite sets, for $i = 1, 2$, and clearly f yields a natural isomorphism between F_1 and F_2. The following diagram of categories commutes, up to natural isomorphism:

$$\begin{tikzcd} FiniteTransitive\Pi Sets \arrow[r] \arrow[dr] & \mathcal{C} \arrow[d, "F_1 \text{ or } F_2"] \\ & FinSet \end{tikzcd}$$

[5] There is an obvious forgetful functor from finite S-definable sets to finite sets with a $\mathrm{Gal}(S)$-action. This is clearly faithful, easily full, and essentially surjective by elimination of imaginaries.

The category of triples (X_1, X_2, f) can be thought of as "covers" defined by glueing data, for the following reason, which won't be used in what follows, since it doesn't seem to line up with the kind of covers of G we were considering earlier. If not interested, skip to §2.4.

We can define a category of "finite covers" of the type-definable set

$$G_1 := \{x \in G : g_1 \cdot x \text{ is generic over } M\}$$

A cover should be a type-definable set Σ over M, and a relatively definable map $\pi : \Sigma \to G_1$ with finite fibers, such that $x \mapsto g_1 \cdot \pi(x)$ is M-definable. A map between such covers should be an M-definable map making the obvious diagram commute.

We can similarly define a notion of "finite covers" of G_2. For

$$G_{12} = \{x \in G : g_1 \cdot x \text{ is generic over } \mathrm{acl}_M(g_2 \cdot g_1^{-1})\}$$

we need to replace M with $\mathrm{acl}_M(g_2 \cdot g_1^{-1})$. There are functors from finite covers of G_i to finite covers of G_{12}. The category of "covers of G" should be[6] the (homotopy or category-theoretic) pullback of

$$\begin{tikzcd} & Covers(G_1) \arrow[d] \\ Covers(G_2) \arrow[r] & Covers(G_{12}) \end{tikzcd}$$

That is, a "cover of G" should be a cover S_i of G_i for $i = 1, 2$, and an isomorphism between the induced covers of G_{12} coming from S_1 and S_2.

Now, it turns out that the category of covers of G_i is equivalent to the category of finite sets defined over $g_i M$, via the functor which takes fibers over the identity $1 \in G$. In fact, one checks that the category of finite covers of G_I is equivalent to the category of Gal_I-sets in a compatible way. So the category of "covers of G" is nothing but the category of finite Π-sets.

2.4 Some Grothendieck Galois Theory

Fact 2.5. *Suppose Π_1 and Π_2 are two profinite groups. Let $\mathcal{C}_\Pi$ denote the category of finite non-empty sets with continuous transitive Π-action, for Π any profinite group. Suppose we have a functor $F : \mathcal{C}_{\Pi_1} \to \mathcal{C}_{\Pi_2}$ such that the following diagram commutes, up to isomorphism of functors:*

$$\begin{tikzcd} \mathcal{C}_{\Pi_1} \arrow[r, "F"] \arrow[dr] & \mathcal{C}_{\Pi_2} \arrow[d] \\ & FinSet \end{tikzcd}$$

where the functors to $FinSet$ are the forgetful functors to finite sets. Then, *F is isomorphic to a functor coming from a surjective homomorphism $\Pi_2 \to \Pi_1$. The functor F is full and faithful, and is essentially surjective if and only if $\Pi_2 \to \Pi_1$ is an isomorphism.*

[6] If G was the union of G_1 and G_2, and G_{12} was their intersection, this would certainly make sense. Neither of these statements is true, but the set of points where they fail has high enough codimension that it shouldn't matter.

This is essentially a part of Grothendieck's Galois theory, if I recall correctly. At any rate, it's not all that hard to see directly. A functor F compatible with the forgetful functors as above amounts to a functorial way of assigning a Π_2-action to Π_1-sets. To show that this must arise from a homomorphism $\Pi_2 \to \Pi_1$, it suffices to show that for every finite quotient Q of Π_1, and every $p \in \Pi_2$, there is some $q \in Q$ such that for every Q-set X, the assigned Π_2-action on X has p act by multiplication by q. (Because then the map $p \mapsto q$ must be a homomorphism $\Pi_2 \to Q$, and as Q varies these must assemble into a homomorphism $\Pi_2 \to \Pi_1$.) This in turn follows by looking at how Π_2 acts on Q with the left regular action: if $p \in \Pi_2$ maps $1 \in Q$ to $q \in Q$, then for any Q-set X, and any $x \in X$, there is a unique morphism of Q-sets from Q to X sending 1 to x. It necessarily sends q to $q \cdot x$. Since it must also be a morphism of Π_2-sets, it sends $p \cdot 1 = q$ to $p \cdot x$. So $p \cdot x = q \cdot x$.

So any such functor must come from a homomorphism $\Pi_2 \to \Pi_1$. The homomorphism must be surjective, or else it would not send connected/transitive Π_1-sets to connected/transitive Π_2-sets. Once we know that $\Pi_2 \to \Pi_1$ is surjective, the functor F is automatically full and faithful, and it is easy to see that it is an equivalence if and only if $\Pi_2 \to \Pi_1$ is an isomorphism. (Otherwise, there are Π_2-sets for which the action does not factor through Π_1.)

2.5 Another reduction

Returning to our model-theoretic setting, we want to produce an isomorphism between the profinite groups Π and $\pi_1(G)$. By §2.4, it suffices to produce a functor from (finite, continuous, transitive) $\pi_1(G)$-sets to Π-sets, that is essentially surjective, and compatible with the forgetful functors.

Above, we have identified the category of suitable $\pi_1(G)$-sets with the category of finite set covers of G. The forgetful functor is identified with the fiber functor at 1.

Likewise, we identified the category of suitable Π-sets with the category of triples (X_1, X_2, f), with X_i a finite g_iM-definable set, and f a $g_1 \operatorname{acl}_M(g_2 \cdot g_1^{-1})$-definable bijection $X_1 \to X_2$, such that each element of X_i has the same type over g_iM, for either/both values of i. Under this equivalence, the forgetful functor to finite sets corresponds to the functor sending (X_1, X_2, f) to X_1.

Let α be some arbitrary element of the inverse limit of $\pi_H^{-1}(g_2 \cdot g_1^{-1})$ as H ranges over finite group covers of G. So α is a system $\langle \alpha_H \rangle$ of elements $\alpha_H \in H$ all of which map to $g_2 \cdot g_1^{-1}$ in G. Each α_H is in $\operatorname{acl}_M(g_2 \cdot g_1^{-1})$, of course.

(If you like, you can think of α as an element of the pro-definable group that is the inverse limit of the finite group covers of G. This pro-definable group acts on the finite set covers of G.)

Using α, we get a functor F_α from the category of finite set covers of G to the category of triples (X_1, X_2, f) as above. Specifically, given a finite set cover $\pi_S : S \to G$, where S is acted on transitively by H a finite group cover of G, we set $X_i = \pi_S^{-1}(g_i)$, and let f be induced by the action of $\alpha_H \in H$ on S (so f comes from "multiplication by α"). This does not depend on the choice of H, and is functorial.

To apply Fact 2.5, we just need to show that the functor

$$S \overset{F_\alpha}{\mapsto} (X_1, X_2, f) \mapsto X_1$$

from finite set covers of G, to finite sets, is isomorphic to the fiber-at-1 functor. An isomorphism can be exhibited by choosing some element β in the inverse limit of $\pi_H^{-1}(g_1)$, as $\pi_H : H \to G$ ranges over finite group covers of G. Then "multiplication by β" gives a functorial map from $\pi_S^{-1}(1)$ to $\pi_S^{-1}(g_1) = X_1$ as S ranges through finite set covers of G, in the same way that f came from "multiplication by α."

In light of all the discussion above, to prove Theorem 1.2, it remains to prove that F_α is essentially surjective.

That is, we need to prove the following statement:

Let X_i be a finite non-empty $g_i M$-definable set for $i = 1, 2$. Suppose every element of X_2 has the same type over $g_2 M$. Suppose f is a $g_1 \operatorname{acl}_M(g_2 \cdot g_1^{-1})$-definable bijection $X_1 \to X_2$. Then there is a finite set cover $\pi_S : S \to G$ and M-definable bijections between $\pi_S^{-1}(g_i)$ and X_i, such that the following diagram commutes:

$$\begin{array}{ccc} \pi_S^{-1}(g_1) & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ \pi_S^{-1}(g_2) & \longrightarrow & X_2 \end{array}$$

where the vertical arrow $\pi_S^{-1}(g_1) \to \pi_S^{-1}(g_2)$ comes from "multiplication by α."

3 Generically given groups

The proof of this basically boils down to a statement about generically given groups and generically given covers of groups. We collect this fact in the following theorem.

Theorem 3.1. *Suppose G is a connected group of finite Morley rank, defined over a model M. Suppose that g_1, g_2, g_3 are independent and generic over M. Suppose that a_i for $1 \le i \le 3$ and d_{ij} for $1 \le i < j \le 3$ are tuples, satisfying the following conditions:*

- *For each i, $a_i \in \operatorname{acl}_M(g_i)$ and $g_i \in \operatorname{dcl}_M(a_i)$.*
- *For each $i < j$, $d_{ij} \in \operatorname{acl}_M(g_j \cdot g_i^{-1})$, and $g_j \cdot g_i^{-1} \in \operatorname{dcl}_M(d_{ij})$.*
- *The type of (a_i, g_i) over M does not depend on i.*
- *The type of (a_i, a_j, d_{ij}) over M does not depend on $i < j$.*
- *a_1 and a_2 are interdefinable over $d_{12}M$.*
- *d_{12} is interdefinable over M with the canonical base of $\operatorname{tp}(a_1 a_2 / d_{12} M)$, which is stationary because $a_2 \in \operatorname{dcl}_M(a_1 d_{12})$ and $\operatorname{tp}(a_1/d_{12}M)$ is a non-forking extension of the stationary type $\operatorname{tp}(a_1/M)$.*

Then there is a finite group cover H and an H-cover S of G (both defined over M), and elements $s_1, s_2, s_3 \in S$ and $h_{ij} \in H$ such that

- $g_j \cdot g_i^{-1} = \pi_H(h_{ij})$
- $g_i = \pi_S(s_i)$
- *h_{ij} is interdefinable with d_{ij} over M, and $\operatorname{tp}(h_{ij}d_{ij}/M)$ doesn't depend on i, j.*
- *s_i is interdefinable with a_i over M, and $\operatorname{tp}(s_i a_i/M)$ doesn't depend on i.*
- *$h_{13} = h_{23} \cdot h_{12}$ and $s_j = h_{ij}s_i$ for $i < j$.*

Proof (sketch). Let p be the type of any of the a_i over M. Let q be the type of any of the d_{ij} over M. The fact that each a_i is interalgebraic with g_i, and that d_{ij} is interalgebraic with $g_j \cdot g_i^{-1}$, easily has as a consequence that $a_i \mathop{\smile\!\!\!\!\!\!\,|}_M d_{ij}$ and $a_j \mathop{\smile\!\!\!\!\!\!\,|}_M d_{ij}$. If $a_j = f(a_i, d_{ij})$, then $f(-, d_{ij})$ induces a function from realizations of p to realizations of p, and the assumption on canonical bases makes d_{ij} be a code for this function. We get the group H by closing the set of germs of such functions under composition. That is, an element of H is the germ at p of a function of the form $f(f(-, d), d')$ for d and d' two realizations of q. In this group, d_{13} ends up being the product of d_{12} and d_{23}.

One gets S by looking at germs of functions from q to p. Each realization of p yields such a function $f(a, -)$, and the closure of this collection under the action of H yields the set S.

The maps $H \to G$ and $S \to G$ come from extending homomorphically the maps on generics, which in turn come from the M-definable function which sends d_{ij} to $g_j \cdot g_i^{-1}$. This function does not depend on i, j, because the assumptions imply that the type of $(a_i, a_j, g_i, g_j, d_{ij})$ over M does not depend on i, j. $\square$

4 Applying generically given groups

Now suppose we are given a g_1-definable set $\phi(\mathbb{M}, g_1)$ and a g_0-definable set $\psi(\mathbb{M}, g_0)$, and

$$f : \phi(\mathbb{M}, g_1) \to \psi(\mathbb{M}, g_0)$$

defined over $g_0 \operatorname{acl}_M(g_1 \cdot g_0^{-1})$, such that all elements of $\phi(\mathbb{M}, g_1)$ have the same type over Mg_1. Write f as f_{g_0,c_1}, for some $c_1 \in \operatorname{acl}_M(g_1 \cdot g_0^{-1})$. Enlarging c_1, we may assume that $g_1 \cdot g_0^{-1} \in \operatorname{dcl}_M(c_1)$. Now c_1 and $g_1 \cdot g_0^{-1}$ are interalgebraic over M, and

$$g_1 \cdot g_0^{-1} \mathop{\smile\!\!\!\!\!\!\,|}_M g_0, \text{ so } c_1 \mathop{\smile\!\!\!\!\!\!\,|}_M g_0.$$

In particular, $\operatorname{tp}(c_1/Mg_0)$ is stationary. Let c_2, c_3 be independent realizations of this type, over Mg_0. Let g_2, g_3 be chosen so that

$$c_1 g_1 \equiv_{Mg_0} c_2 g_2 \equiv_{Mg_0} \equiv c_3 g_3 \equiv_{Mg_0} .$$

As $g_1 \cdot g_0^{-1} \in \mathrm{dcl}_M(c_1)$, it follows that $g_1 \in \mathrm{dcl}_M(c_1, g_0)$. Therefore, the independence of c_1, c_2, c_3 over Mg_0 implies the independence of $c_1 g_1, c_2 g_2, c_3 g_3$ over Mg_0. Also, $c_1 g_1, c_2 g_2$, and $c_3 g_3$ have the same type over Mg_0. In particular, $c_1 g_1, c_2 g_2, c_3 g_3$ and g_1, g_2, g_3 are Morley sequences over Mg_0. As g_1 and g_0 are independent generics, it follows that g_0, g_1, g_2, g_3 is a Morley sequence of generics, over M.

Moreover, for $i = 1, 2, 3$, we have a bijection

$$f_{g_0, c_i} : \phi(\mathbb{M}, g_i) \to \psi(\mathbb{M}, g_0),$$

and $c_i \in \mathrm{acl}_M(g_i \cdot g_0^{-1})$.

Consider the composition

$$h_{12} := f_{g_0,c_2}^{-1} \circ f_{g_0,c_1} : \phi(\mathbb{M}, g_1) \to \phi(\mathbb{M}, g_2).$$

The code for this bijection (viewed as a finite set of ordered pairs) is in $\mathrm{acl}_M(g_1, g_2)$, and also in

$$\begin{aligned} &\mathrm{dcl}_M(g_0, \mathrm{acl}_M(g_1 \cdot g_0^{-1}), \mathrm{acl}_M(g_2 \cdot g_0^{-1})) \\ &= \mathrm{dcl}_M(g_1, \mathrm{acl}_M(g_1 \cdot g_0^{-1}), \mathrm{acl}_M(g_2 \cdot g_0^{-1})) \end{aligned}$$

Using the finite satisfiability of $\mathrm{tp}(g_1 \cdot g_0^{-1} / \mathrm{acl}_M(g_1, g_2))$ in M, it follows that $\ulcorner h_{12} \urcorner$ is actually in

$$\mathrm{dcl}_M(g_1 \, \mathrm{acl}_M(g_2 \cdot g_1^{-1})).$$

(Indeed, if $r(x)$ is $\mathrm{tp}(g_1 \cdot g_0^{-1} / \mathrm{acl}_M(g_1, g_2))$, then some finite subtype of $r(x)$ witnesses the fact that

$$\ulcorner h_{12} \urcorner \in \mathrm{dcl}_M(g_1, \mathrm{acl}_M(x), \mathrm{acl}_M(g_2 \cdot g_1^{-1} \cdot x)).$$

If we now choose $x \in M$ with this property, then

$$\ulcorner h_{12} \urcorner \in \mathrm{dcl}_M(g_1, M, \mathrm{acl}_M(g_2 \cdot g_1^{-1}))$$

as claimed.)

So we can write h_{12} as $h_{g_1, e_{12}}$ for some e_{12} in $\mathrm{acl}_M(g_2 \cdot g_1^{-1})$. Enlarging e_{12}, we may assume that $g_2 \cdot g_1^{-1} \in \mathrm{dcl}_M(e_{12})$.

On the set of realizations of $\mathrm{tp}(e_{12}/M, g_2 \cdot g_1^{-1})$, consider the equivalence relation

$$e \sim e' \iff h_{g_1,e} = h_{g_1,e'}.$$

This equivalence relation is definable over $M \cup \{g_2 \cdot g_1^{-1}\}$ because of $g_1 \mathop{\smile\!\!\!\!\!\!\mid}_M g_2 \cdot g_1^{-1}$, and definability of types in stable theories. Let d_{12} be the equivalence class of e_{12}. Then:

- $d_{12} \in \mathrm{acl}_M(g_2 \cdot g_1^{-1})$
- $g_2 \cdot g_1^{-1}$ is still in $\mathrm{dcl}_M(d_{12})$
- h_{12} is defined over $Mg_1 d_{12}$, so we can write it as $h_{g_1, d_{12}}$.

- If d' is a conjugate of d_{12} over $\mathrm{dcl}_M(g_2 \cdot g_1^{-1})$, then $h_{g_1,d'} \neq h_{g_1,d_{12}}$.

As c_1g_1, c_2g_2, c_3g_3 was a Morley sequence over Mg_0, it follows that $\mathrm{tp}(c_ic_jg_ig_j/Mg_0)$ does not vary for $1 \leq i < j \leq 3$. Therefore we can find d_{23} and d_{13} such that $\mathrm{tp}(c_ic_jg_ig_jd_{ij}/Mg_0)$ does not vary for $1 \leq i < j \leq 3$.

By choice of d_{12}, it follows that

$$\begin{tikzcd} \phi(\mathbb{M}, g_1) \arrow[r, "f_{g_0,c_1}"] \arrow[d, "h_{g_1,d_{12}}"] & \psi(\mathbb{M}, g_0) \\ \phi(\mathbb{M}, g_2) \arrow[ur, "f_{g_0,c_2}"] & \end{tikzcd}$$

commutes. Therefore,

$$\begin{tikzcd} \phi(\mathbb{M}, g_i) \arrow[r, "f_{g_0,c_i}"] \arrow[d, "h_{g_i,d_{ij}}"] & \psi(\mathbb{M}, g_0) \\ \phi(\mathbb{M}, g_j) \arrow[ur, "f_{g_0,c_j}"] & \end{tikzcd}$$

commutes for $1 \leq i < j \leq 3$. Because all the maps are bijections, it follows that

$$\begin{tikzcd} \phi(\mathbb{M}, g_1) \arrow[d, "h_{g_1,d_{12}}"] \arrow[dr, "h_{g_1,d_{13}}"] & \\ \phi(\mathbb{M}, g_2) \arrow[r, "h_{g_2,d_{23}}"] & \phi(\mathbb{M}, g_3) \end{tikzcd}$$

commutes, i.e., $h_{g_2,d_{23}} \circ h_{g_1,d_{12}} = h_{g_1,d_{13}}$.

Now choose some $b_1 \in \phi(\mathbb{M}, g_1)$, and let

$$b_2 = h_{g_1,d_{12}}(b_1) \in \phi(\mathbb{M}, g_2)$$
$$b_3 = h_{g_2,d_{23}}(b_2) = h_{g_1,d_{13}}(b_1)$$

Let $a_i = b_ig_i$ (concatenation) for each i. We claim that Theorem 3.1 applies to our chosen a_i, g_i, d_{ij}.

- As we noted above, g_1, g_2, g_3 are a Morley sequence over Mg_0. Since g_1 is generic over Mg_0, it follows that g_0, g_1, g_2, g_3 is an independent sequence of generics over M. So g_1, g_2, g_3 are independent and generic over M.

- For each i, $g_i \in \mathrm{dcl}_M(a_i)$ because $a_i = b_ig_i$. On the other hand, b_i is in the finite Mg_i-definable set $\phi(\mathbb{M}, g_i)$, so $a_i \in \mathrm{acl}_M(g_i)$.

- It was noted above that $d_{ij} \in \mathrm{acl}_M(g_j \cdot g_i^{-1})$ and $g_j \cdot g_i^{-1} \in \mathrm{dcl}_M(d_{ij})$.

- For $i < j$, $g_i \equiv_M g_j$ of course. If σ is an automorphism over M sending g_i to g_j, then $\sigma(b_i) \in \phi(\mathbb{M}, g_j)$. All elements of $\phi(\mathbb{M}, g_j)$ have the same type over Mg_j (because we assumed this for $j = 1$, and symmetry), so

  $$\sigma(b_i) \equiv_{Mg_j} b_j, \text{ or equivalently, } \sigma(b_i)\sigma(g_i) \equiv_M b_jg_j.$$

  Thus $b_ig_i \equiv_M b_jg_j$. So $\mathrm{tp}(a_ig_i/M)$ does not depend on M.

- By choice of d_{13} and d_{23}, $\mathrm{tp}(g_i g_j d_{ij}/M)$ does not depend on i, j. So for $i < j$ and $i' < j'$, we can find σ over M, sending $g_i g_j d_{ij}$ to $g_{i'} g_{j'} d_{i'j'}$. Now $\sigma(b_i)$ is in $\phi(\mathbb{M}, g_{i'})$. All elements of $\phi(\mathbb{M}, g_{i'})$ have the same type over $M g_{i'}$, so

  $$\sigma(b_i) \equiv_{M g_{i'}} b_{i'}$$

  Equivalently

  $$\sigma(b_i)\sigma(g_i) \equiv_M b_{i'} g_{i'}.$$

  Now $\mathrm{tp}(b_{i'} g_{i'}/M)$ is stationary and

  $$\sigma(b_i) b_{i'} g_{i'} \mathop{\downarrow}_M d_{i'j'}$$

  (because the left side is interalgebraic with $g_{i'}$ and the right side is interalgebraic with $g_{i'}^{-1} \cdot g_{j'}$, which is independent from $g_{i'}$.) By stationarity, it follows that

  $$\sigma(b_i)\sigma(g_i) \equiv_{M d_{i'j'}} b_{i'} g_{i'},$$

  or equivalently

  $$\sigma(b_i, g_i, d_{ij}) \equiv_M b_{i'} g_{i'} d_{i'j'}.$$

  Thus

  $$b_i g_i d_{ij} \equiv_M b_{i'} g_{i'} d_{i'j'}.$$

  Now $g_i \cdot g_j^{-1} \in \mathrm{dcl}_M(d_{ij})$ via a function not depending on i, j, so

  $$b_i g_i g_j d_{ij} \equiv_M b_{i'} g_{i'} g_{j'} d_{i'j'}.$$

  Likewise, $b_j = h_{g_i, d_{ij}}(b_i)$ is definable from g_i and b_i and d_{ij} in a way not depending on i, j, so

  $$b_i g_i b_j g_j d_{ij} \equiv_M b_{i'} g_{i'} b_{j'} g_{j'} d_{i'j'}$$

  As i, j and i', j' were arbitrary, it follows that $\mathrm{tp}(a_i, a_j, d_{ij}/M)$ does not depend on i, j.

- a_1 and a_2 are interdefinable over $d_{12}M$: given $a_1 d_{12} M$, we can define g_1 and $g_2 \cdot g_1^{-1}$, and hence $h_{g_1, d_{12}}(b_1) = b_2$ and g_2. Given $a_2 d_{12} M$, we can define g_2 and $g_2 \cdot g_1^{-1}$, hence g_1 and then $h_{g_1, d_{12}}^{-1}(b_2) = b_1$.

It remains to check that d_{12} is interdefinable over M with a canonical base for the stationary type $\mathrm{tp}(a_1 a_2/d_{12}M)$. If C is this canonical base, then certainly $C \subseteq \mathrm{dcl}_M(d_{12}M)$, so it remains to show that C is no smaller, i.e., that $d_{12} \in \mathrm{dcl}_M(C)$. First note that

$$g_2 \cdot g_1^{-1} \in \mathrm{dcl}(a_1, a_2) \cap \mathrm{dcl}_M(d_{12}M),$$

so certainly $g_2 \cdot g_1^{-1}$ is definable from C. Therefore d_{12} is algebraic over C. If d_{12} were not definable, it would have some conjugate d' over C. This would also be a conjugate over $g_2 \cdot g_1^{-1}$, of course, so by choice of d_{12}, we have $h_{g_1, d'} \neq h_{g_1, d_{12}}$. So there is some $\beta \in \phi(\mathbb{M}, g_1)$ such that $h_{g_1, d'}(\beta) \neq h_{g_1, d_{12}}(\beta)$.

Each of d_{12} and d' is interalgebraic over M with $g_2 \cdot g_1^{-1}$, which is independent from g_1. Since β and b_1 are algebraic over g_1, it follows that

$$b_1 \beta g_1 \underset{M}{\perp} d_{12} d'.$$

Now b_1 and β have the same type over Mg_1, hence $b_1 g_1$ and βg_1 have the same type over M. As types over M are stationary, it follows that

$$b_1 g_1 \equiv_{Md_{12}d'} \beta g_1, \text{ or equivalently } b_1 \equiv_{Md_{12}d'g_1} \beta.$$

The fact that $h_{g_1,d_{12}}(\beta) \neq h_{g_1,d'}(\beta)$ therefore implies that $h_{g_1,d_{12}}(b_1) \neq h_{g_1,d'}(b_1)$.

Let N be a mildly saturated and homogeneous model containing Md. Moving N, we may assume that $a_1 a_2 \perp_{Md} N$. Then

$$\operatorname{tp}(a_1 a_2/N) \text{ is a non-forking extension of } \operatorname{tp}(a_1 a_2/Md_{12})$$

and in particular has the same canonical base. Also,

$$d' \in \operatorname{acl}_M(g_2 \cdot g_1^{-1}) = \operatorname{acl}_M(d_{12}) \subset N,$$

so $d' \in N$, and $C \in N$. Because we chose N to be mildly saturated and homogeneous, we can find an automorphism σ of N which fixes C and sends d_{12} to d'. Because σ fixes the canonical base of $\operatorname{tp}(a_1 a_2/N)$, it can be extended to an automorphism of the monster fixing $a_1 a_2$. Now we have a contradiction, because σ fixes b_1, b_2, g_1, g_2 and

$$h_{g_1,d_{12}}(b_1) = b_2$$

$$h_{g_1,d'}(b_1) \neq b_2,$$

so that $d' \not\equiv_{Mb_1b_2g_1g_2} d_{12}$. So the assumption that d_{12} was not definable over C was false.

We have verified that Theorem 3.1 applies, yielding a finite group cover H of G and a finite set cover S, among other things.

From the conclusion of that theorem, we get $s_1 \in \pi_S^{-1}(g_1)$ which is interdefinable over M with a_1. This yields an Mg_1-definable bijection between $\phi(\mathbb{M}, g_1)$, which is the set of conjugates of b_1 over Mg_1, and the set of conjugates of s_1 over Mg_1. We claim that this latter set is exactly $\pi_S^{-1}(g_1)$. First of all, if $s' \equiv_{Mg_1} s_1$, then certainly $\pi_S(s') \equiv_{Mg_1} \pi_S(s_1) = g_1$, so $\pi_S(s') = g_1$. Conversely, suppose $\pi_S(s') = g_1$. Then s' has the same rank over M as g_1, and is therefore generic. Since S has a single generic type (being an orbit of a connected group H), it follows that $s' \equiv_M s_1$. Then $(s', \pi_S(s')) = (s', g_1) \equiv_M (s_1, \pi_S(s_1)) = (s_1, g_1)$, so $s' \equiv_{Mg_1} s'$. So we get an Mg_1-definable bijection μ_{g_1} between $\phi(\mathbb{M}, g_1)$ and $\pi_S^{-1}(g_1)$, sending b_1 to s_1.

Now consider the commutative square of bijections

$$\begin{tikzcd} \pi_S^{-1}(g_0) \arrow[r, dotted] \arrow[d] & \psi(\mathbb{M}, g_0) \\ \pi_S^{-1}(g_1) \arrow[r, "\mu_{g_1}"] & \phi(\mathbb{M}, g_1) \arrow[u, "f_{g_0,c_1}"'] \end{tikzcd}$$

where the vertical arrow from $\pi_S^{-1}(g_0)$ to $\pi_S^{-1}(g_1)$ comes from multiplication by α or α_H. There is a unique bijective dotted arrow making the diagram commute. If ν denotes this bijection, then ν is on one hand definable over $\operatorname{acl}_M(g_0)$ because it is a bijection between two g_0M-definable sets. On the other hand, it is a composition of three maps which are respectively definable over

- g_0 and $\alpha \in \operatorname{acl}_M(g_1 \cdot g_0^{-1})$
- μ_{g_1} which is g_1M-definable, hence definable from g_0 and $g_1 \cdot g_0^{-1}$.
- f_{g_0,c_1} which is defined over g_0 and $c_1 \in \operatorname{acl}_M(g_1 \cdot g_0^{-1})$.

So

$$\ulcorner \nu \urcorner \in \operatorname{acl}_M(g_0) \cap \operatorname{dcl}_M(g_0 \operatorname{acl}_M(g_1 \cdot g_0^{-1}))$$

Using the finite satisfiability of $\operatorname{tp}(g_1 \cdot g_0^{-1} / \operatorname{acl}_M(g_0))$ in M, it follows that $\ulcorner \nu \urcorner$ is in fact in $\operatorname{dcl}_M(g_0)$.

Therefore, in the category of triples (X_0, X_1, f), our original triple (X_0, X_1, f) is isomorphic to $F_\alpha(S)$, completing the proof of Theorem 1.2.