C-minimal expansions of ACVF eliminate exists infinity

Will Johnson

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Let T be a C-minimal expansion of ACVF in a language L. We will show that T^{eq} eliminates $\exists^\infty$. Suppose not. Let M_0 be a model of T^{eq} and $\phi(x;y)$ be a formula such that

$$\{b \in M_0 : \phi(M_0; b) \text{ is finite}\}$$

is not definable. In particular, for every $n \in \mathbb{N}$, there is some b_n such that $\phi(M_0; b)$ is finite but has size at least n.

Let N_0 be the expansion of M_0 obtained by adding a new sort $\mathbb{N} \cup \{\infty\}$ and a function symbol $f_\psi(y)$ taking values in $\mathbb{N} \cup \{\infty\}$ for every predicate $\psi(x;y) \in L$, interpreted in N_0 as

$$f_\psi(b) = |\psi(M_0; b)|$$

In particular, $n \le f_\phi(b_n) < \infty$ for each $n \in \mathbb{N}$.

Let N be a saturated elementary extension of N_0 and let M be the reduct of M to L. By saturation, there is some $b_\omega \in M$ such that $\mathbb{N} < f_\phi(b_\omega) < \infty$ Say that a definable subset $\psi(M; b)$ of M is *pseudofinite* if $f_\psi(b)$ is less than ∞. By the assumption, M has at least one infinite pseudofinite set, namely $\phi(M; b_\omega)$.

Say that a definable set X in M is *wild* if there is some pseudofinite infinite definable family of subsets of X. That is, there is some infinite pseudofinite definable set Y and some definable family $X_b \subset X$ for $b \in Y$ such that $X_b \neq X_{b'}$ for $b \neq b'$ in Y. Say that a definable set X in M is *tame* if X is not wild.

Let K denote the home sort (the valued field sort). Since $\phi(M; b_\omega)$ is in K^{eq}, the infinite pseudofinite set $\phi(M; b_\omega)$ sits inside some 0-definable quotient of K^n, for some n. Consequently, $\phi(M; b_\omega)$ is (the index set of) an infinite pseudofinite definable family of subsets of K^n, and K^n is wild. Therefore, it suffices to show that K^n is tame for every n.

Claim 1.1. *Every pseudofinite subset of Γ (the value group) is finite.*

Proof. This follows from the fact that Γ is o-minimal and densely-ordered. □

Claim 1.2. *If S is a pseudofinite set of balls of the same radius and the same type (open/closed), then S is finite.*

Proof. This follows by C-minimality.

More precisely, write S as $\psi(M;b)$ for some formula $\psi(x;y)$ and parameter b. Let $\chi(z;y)$ be the formula such that $\chi(M;c)$ is the union of the balls in $\psi(M;c)$ for every $c \in M$. C-minimality ensures that there is an integer n such that $\chi(M;c)$ is a boolean combination of at most n balls, for every c.

The following statement holds in M_0

> For every $c \in M_0$, if $\psi(M_0;c)$ is a set of balls of the same radius and the same type, then
>
> $$|\psi(M_0;c)| < \infty \implies |\psi(M_0;c)| \leq n.$$

Indeed, if $\psi(M_0;c)$ is a finite set $\{B_1, \ldots, B_m\}$ of balls of the same radius and same type, then the B_i are pairwise disjoint, so their union $\chi(M_0;c)$ is the disjoint union of $B_1, \ldots, B_m$. This disjoint union cannot be written as a boolean combination of fewer than m balls, so $n \geq m$.

Since N is an elementary extension of N_0, the following holds:

> For every $c \in M$, if $\psi(M;c)$ is a set of balls of the same radius and the same type, then
>
> $$f_\psi(c) < \infty \implies f_\psi(c) \leq n.$$

In particular, $S = \psi(M;b)$ has cardinality at most n. □

Claim 1.3. *If S is a pseudofinite set of balls, then S is finite.*

Proof. Writing S as a union of the closed balls and the open balls, we may assume that all the elements of S are of the same type. Let $R \subset \Gamma$ be the set of radii of balls in S. Since S surjects onto R and S is pseudofinite, so is R. So R is finite, by Claim 1.1. The fibers of $S \twoheadrightarrow R$ are pseudofinite, hence finite by Claim 1.2. So S is finite. □

Claim 1.4. *K^1 is tame. That is, any pseudofinite set of subsets of K^1 is finite.*

Proof. Let D be a definable family of subsets of K^1. Suppose that D is pseudofinite. By C-minimality, each element of D has a canonical minimal swiss-cheese decomposition. Let S be the set of balls involved in the swiss cheese decompositions of elements of D. Then S is definable. It is also pseudofinite, since D is pseudofinite. By Claim 1.3, S is finite. As every element of D is a boolean combination of elements of S, and boolean algebras are locally finite, it follows that D is finite. □

Claim 1.5. *If X is tame, so is any (definable) subset of X. If X and Y are tame, then so is $X \cup Y$.*

Proof. The first statement is clear, since there is less to check. For the second, let $\mathcal{D}$ be a definable family of subsets of $X \cup Y$ which is pseudofinite. Note that $\{D \cap X : D \in \mathcal{D}\}$ is pseudofinite, because $\mathcal{D}$ is pseudofinite, and hence finite, because X is tame. Similarly $\{D \cap Y : D \in \mathcal{D}\}$ is finite. Finally, the map

$$D \in \mathcal{D} \mapsto (D \cap X, D \cap Y)$$

is an injective map from $\mathcal{D}$ to a cartesian product of finite sets. □

Claim 1.6. *Let $\pi : X \to Y$ be a definable map with finite fibers. If Y is tame, then so is X.*

Proof. By saturation, there is a uniform upper bound k on the size of the fibers. We proceed by induction on k. The base case $k = 1$ is trivial. Suppose $k > 1$. Let $\mathcal{D}$ be a pseudofinite definable family of subsets of X. Let

$$\mathcal{E} = \{\pi(D) : D \in \mathcal{D}\}$$

and

$$\mathcal{F} = \{\pi(X \setminus D) : D \in \mathcal{D}\}$$

Then $\mathcal{E}$ and $\mathcal{F}$ are both pseudofinite definable families of subsets of Y. By tameness of Y, they are both finite. It suffices to show that the fibers of $\mathcal{D} \to \mathcal{E} \times \mathcal{F}$ are finite. Replacing $\mathcal{D}$ with such a fiber, we may assume that $\pi(D)$ and $\pi(X \setminus D)$ don't depend on D, as D ranges over $\mathcal{D}$. Let $U = \pi(D)$ and $V = \pi(X \setminus D)$ for any/every $D \in \mathcal{D}$. Let $Y' = U \cap V$ and $X' = \pi^{-1}(Y')$. Then the map $D \mapsto D \cap X'$ is injective on $\mathcal{D}$, because every element D of $\mathcal{D}$ contains $\pi^{-1}(U \setminus V)$ and is disjoint from $\pi^{-1}(V \setminus U)$. So it suffices to show that X' is tame. Let D be some arbitrary element of $\mathcal{D}$. Then $X' \cap D$ and $X' \setminus D$ each intersect every fiber of $X' \to Y'$, by choice of X'. In particular, the two maps

$$X' \cap D \to Y'$$

$$X' \setminus D \to Y'$$

have finite fibers of size less than k. By Claim 1.5, Y' is tame, and by induction, $X' \cap D$ and $X' \setminus D$ are tame. By Claim 1.5, X' is tame. □

Claim 1.7. *Suppose that $\pi : X \to Y$ is a definable surjection with finite fibers. Suppose that Y is tame. Then any pseudofinite definable set of sections of the surjection π is finite.*

Proof. A section is determined by its image. □

Claim 1.8. *Suppose X and Y are tame. Then so is $X \times Y$.*

Proof. Let $\mathcal{D}$ be a pseudofinite definable family of subsets of $X \times Y$. For each $x \in X$, the set $Y_x := \{x\} \times Y \subset X \times Y$ is tame, so the collection

$$E_x := \{D \cap Y_x : D \in \mathcal{D}\}$$

is finite. So $\pi : \bigcup_{x \in X} E_x \to X$ is a definable map of definable sets, with finite fibers. Each element $D \in \mathcal{D}$ induces a section of π, namely, the map σ_D sending a point $x \in X$ to (the code for) $D \cap Y_x$. This gives a definable injection from $\mathcal{D}$ to sections of π. By Claim 1.7 and the fact that X is tame, it follows that $\mathcal{D}$ is finite. □

By Claims 1.8 and 1.4, K^n is tame for every n, so we have a contradiction.