Etale cohomology of almost strongly minimal groups

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1 The conjectural picture

Let T be an almost strongly minimal theory. Work inside a monster model $\mathbb{M}$ of T^{eq}. Let M be a model (a small elementary substructure of $\mathbb{M}$). If G is an M-definable group, and H is an M-definable subgroup, we can consider the homogeneous space G/H with the left-action by G. We will define "etale cohomology groups" $H^n_M(G/H, A)$ for all $n \geq 0$ and all abelian groups A. We would like the following things to hold:

1. If T is ACF_p and A is finite of order prime to p, then $H^n_M(G/H, A)$ should agree with the usual etale cohomology of the variety G/H with coefficients in A. In particular, it shouldn't depend on M.

2. More generally, if G and H are algebraic groups over a definable field K (not necessarily pure), and A has order prime to char K, then $H^n_M(G/H, A)$ should agree with the usual etale cohomology of the variety G/H.

3. For fixed G and H, there should be some non-zero integer N such that for all finite abelian groups A with order prime to N, $H^n_M(G/H, A)$ is finite and does not depend on the choice of M. We call these "suitable" A, and drop the subscript M.

4. If

$$\begin{tikzcd} H_1 \arrow[r, hook] \arrow[d] & G_1 \arrow[d] \\ H_2 \arrow[r, hook] & G_2 \end{tikzcd}$$

   is a commutative diagram of definable homomorphisms of groups (yielding a definable map $G_1/H_1 \to G_2/H_2$), and A is suitable, there should be a map $H^n(G_2/H_2, A) \to H^n(G_1/H_1, A)$.

5. If $G_1/H_1 \to G_2/H_2$ is an isomorphism, the induced map on suitable cohomology groups should be an isomorphism. For example, $H^n(G, A)$ should be the same whether we

regard G as a homogeneous space under the left regular action of G, or the two-sided action of $G \times G$, corresponding to the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & 1 \arrow[r, hook] \arrow[d] & G \arrow[r, two heads] \arrow[d, "\iota_1"] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & G \arrow[r, hook, "\Delta"] & G \times G \arrow[r, two heads, "\mu"] & G \arrow[r] & 1
\end{tikzcd}$$

where Δ is the diagonal map, μ is the map $(g, h) \mapsto g \cdot h^{-1}$, and ι_1 is the inclusion of the first factor. In particular, we should be able to unambiguously talk about $H^n(G, A)$.

6. Ideally, any map of homogeneous spaces which is defined using the group operation and definable homomorphisms should yield a map on the cohomology groups in the other direction. More precisely, if C is the category of pairs (G, H) with H a subgroup of G, and D is the category of definable sets, there is a natural functor $C \to D$ sending (G, H) to G/H. Let C' be the smallest subcategory of D containing the image of $C \to D$, and closed under taking inverses of morphisms which happen to be isomorphisms in D. Then we would like $H^n(-, A)$ to extend to a well-defined contravariant functor from C' to finite abelian groups.

7. $H^n(G/H, A)$ should depend functorially on A, and short exact sequences $0 \to A' \to A \to A'' \to 0$ should yield the usual long exact sequences on cohomology groups.

8. If S and S' are two homogeneous spaces, there should be a Künneth formula relating $H^\bullet(S \times S', A)$ to $H^\bullet(S, A)$ and $H^\bullet(S', A)$.

9. There should be a cup-product on $H^\bullet(S, R)$, for R a suitable ring.

10. There should be a Hopf-algebra structure on $H^\bullet(G, R)$, for G a group and R a ring. (This would probably follow from the previous points: it would essentially just mean that $H^\bullet(G, R)$ has all the algebraic structure obtained by taking the structure morphisms $\mu : G \times G \to G$ and $\epsilon : 1 \to G$ and the inverse map $G \to G$ and applying the contravariant functor $H^\bullet(-, R)$ to them.)

11. If S has n connected components, then $H^k(S, A)$ should be the direct sum of n copies of $H^k(S^0, A)$, where S^0 is one of the connected components.

12. If S is connected, $H^0(S, A)$ should just be A.

13. If G is connected, $H^1(G, A)$ should classify definable extensions of G by the abelian group A.

14. There should be Serre spectral sequences. For example, the cohomology of G/H should be related to the cohomology of G and of H. More generally, if $G_1 < G_2 < G_3$, there should be a Serre spectral sequence relating the cohomology of G_2/G_1 and G_3/G_2 to the cohomology of G_3/G_1. This would require discussing cohomology with twisted coefficients, however.

15. If G is an abelian group, A should be suitable as long as there are no primes p dividing $|A|$ such that $G[p]$ is infinite. Moreover, for G connected, $H^n(G, \mathbb{Z}/p^m)$ should have an explicit description (as an abstract group) in terms of the p-torsion of G. If the p-torsion of G is isomorphic to a k-dimensional $\mathbb{F}_p$-vector space, then $H^n(G, \mathbb{Z}/p^m)$ ought to agree with the usual topological cohomology group $H^n(T^k, \mathbb{Z}/p^m)$, where T^k is a k-torus.

If everything outlined above holds, it is conceivable that these cohomology groups could prove useful for the study of groups of finite Morley rank.

2 Review of homological algebra

We will always work with cochain complexes $C^\bullet$, so the differentials go $C^i \to C^{i+1}$ (rather than $C_{i+1} \to C_i$, as they would in a chain complex). We will only consider nonnegative gradings, so C^i will vanish for $i < 0$.

A map of complexes $C^\bullet \to D^\bullet$ is a quasi-isomorphism if the induced map on homology groups is trivial. An equivalent condition is that the total complex of the following double complex is exact:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \\
D_0 \arrow[r] \arrow[u] & D_1 \arrow[r] \arrow[u] & D_2 \arrow[r] \arrow[u] & \cdots \\
C_0 \arrow[r] \arrow[u] & C_1 \arrow[r] \arrow[u] & C_2 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

(In general, the total complex of this double complex is the mapping cone of $C^\bullet \to D^\bullet$, which is exact if and only if $C^\bullet \to D^\bullet$ is itself a quasi-isomorphism.)

The derived category $D(\mathbb{Z})$ is obtained by taking the category of cochain complexes of $\mathbb{Z}$-modules (abelian groups), and formally inverting all the quasi-isomorphisms. Formally inverting morphisms in a large category is usually dangerous (the hom-sets can become proper classes), but it is well-known that this does not happen in the case of derived categories, because there is a calculus of fractions. The following facts about $D(\mathbb{Z})$ are well-known:

- $D(\mathbb{Z})$ is an additive category, but not an abelian category.
- The functor $H^n(-)$ sending a cochain complex to its nth cohomology group, extends to $D(\mathbb{Z})$ in a canonical way (this is obvious from the definition).
- If $f, g : C^\bullet \to D^\bullet$ are two maps of cochain complexes, and f and g are related by a homotopy (that is, $f^i - g^i = d^{i-1}s^i + s^{i+1}d^i$ for some maps $s^i : C^i \to D^{i-1}$), then f and g have the same image in $D(\mathbb{Z})$.

- If the cohomology of C is N concentrated in degree n, and the cohomology of D is M concentrated in degree m, then $\mathrm{Hom}(C, D)$ in the derived category is exactly $\mathrm{Ext}^{n-m}(N, M)$.
- Consider the functor which takes a cochain complex $C^{0,\bullet} \to C^{1,\bullet} \to \cdot$ of cochain complexes, and outputs the total complex of the associated double complex. This functor respects quasi-isomorphisms, and consequently extends uniquely to a similar functor from cochain complexes in $D(\mathbb{Z})$, to $D(\mathbb{Z})$. In the case of cochain complexes of the form
$$0 \to C^\bullet \to D^\bullet \to 0$$
the output is just the mapping cone of $C^\bullet \to D^\bullet$, part of the triangulated category structure. (The more general case is probably just an iteration of the mapping cone operation.)

We will frequently use the following fact about double complexes, which is an easy exercise in diagram-chasing:

Fact 2.1. *Let*

$$\begin{tikzcd}
\vdots & \vdots & \\
C^{0,1} \arrow[r] \arrow[u] & C^{1,1} \arrow[r] \arrow[u] & \cdots \\
C^{0,0} \arrow[r] \arrow[u] & C^{1,0} \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

be a double complex. If every row is exact, or every column is exact, then the associated total complex is exact. More generally, if every row is exact in the first k places, then the total complex is exact in the first k places.

3 Assumptions

We assume that the ambient theory T is almost strongly minimal, eliminates imaginaries, and that $\mathrm{acl}(\emptyset)$ is a model. These assumptions have the following (well-known) consequences:

- Stability, total transcendentality, finite Morley rank, $\aleph_1$-categoricity, NFCP
- Morley rank agrees with Lascar rank and weight, and is definable.
- Every algebraically closed set is a model.
- If A is a set of parameters and X is an infinite A-definable set, then X has a strongly minimal $\mathrm{acl}(A)$-definable subset.
- In particular, there is an $\mathrm{acl}(0)$-definable strongly minimal set.

- If A is a set of parameters, X is an A-definable set, and $n \leq U(X)$, then there is an A-definable subset of X of rank n.
- If S is any acl(0)-definable strongly minimal set, and e is any element (any imaginary), then e is interalgebraic with a tuple from S.
- If X, S are A-definable, with S strongly minimal, then there is a finite-to-finite A-definable correspondence between X and a power of S. That is, if $n = U(X)$, then there is an A-definable subset $Y \subset X \times S^n$ whose projections onto X and S^n are surjective with finite fibers.
- We get almost skolem functions: if $f : X \to Y$ is a definable surjection (over some set of parameters), then there is a subset $X' \subset X$ definable over the same set of parameters, such that the restriction $X' \to Y$ of f to X' is surjective with finite fibers.
- Every type-definable group is definable. There are no descending chains of definable groups. Zilber's indecomposability theorem holds.

The base assumptions hold in any simple group of finite Morley rank, after naming all the elements of the prime model, and passing to T^{eq}. Moreover, the base assumption remains true when more parameters are named.

4 Special opens

Fix S a definable set with a transitive definable action of a definable group G. **We assume (for now) that G and S are connected.** This ensures that G and S have unique generic types $gen(G)$ and $gen(S)$. We also assume that G, S, and the group action are all 0-definable. We don't assume that the action of G on S is faithful. Let $K \lhd G$ be the kernel of the action, so G/K acts faithfully and transitively on S.

If A is any small set, let S_A denote the set of $s \in S$ such that $\mathrm{tp}(s/A)$ is generic. If $g \in G$, we can look at the translate $g \cdot S_A$ of S_A by g. Call such sets *special opens*. These will play the role of Zariski open subsets of S in our construction.

Lemma 4.1. *Let H be a connected* acl(0)*-definable subgroup of G, containing K. Suppose H is strictly bigger than K. Let s realize the generic type of S over* acl(0)*. Then H does not fix s.*

Proof. Suppose for the sake of contradiction that H fixes s. The subgroup K is exactly $\bigcap_{s \in S} \mathrm{Stab}(s)$. By Baldwin-Saxl, K can be written as a finite intersection $\bigcap_{i=1}^n \mathrm{Stab}(s_i)$ for some $s_1, \ldots, s_n \in S$. Let $g \in G$ be generic over $s_1, \ldots, s_n$. Then $g \cdot s_i$ is generic over $\emptyset$ for every i. So $g \cdot s_i$ and s have the same strong type over $\emptyset$. As H, G, and S are acl($\emptyset$)-definable, H must fix $g \cdot s_i$. This holds for each i. So if $h \in H$, then $h \cdot g \cdot s_i = g \cdot s_i$, or equivalently, $h^g \in \mathrm{Stab}(s_i)$. Since this holds for every i, $h^g \in K$. Since K is a normal subgroup, $h \in K$. As h was arbitrary, $H \leq K$, a contradiction. $\square$

Lemma 4.2. *There exists an n, and strong types $p_1, \ldots, p_n$ over* $\operatorname{acl}(0)$ *in S, such that K has finite index in $\bigcap_{i=1}^n \operatorname{Stab}(p_i)$, and $U(p_i) = U(S) - 1$.*

Proof. Proceeding by induction, it suffices to show that if H is a connected subgroup of G, containing K, then there is some type p in S, over $\operatorname{acl}(0)$, of rank $U(S) - 1$, such that $\operatorname{Stab}(p)$ does not contain H. (Then we can replace H with $(H \cap \operatorname{Stab}(p))^0$ and iterate the construction, shrinking H until it becomes K, and starting with $H = G$.)

Take s_0 generic in S, over $\operatorname{acl}(0)$. Let c be the code for the orbit $\ulcorner H \cdots s_0 \urcorner$. Since H is not K and s_0 is generic in S, $H \cdot s_0$ is not just s_0. Since H is connected, $H \cdot s_0$ is infinite, hence has positive rank. Let $\alpha \in H \cdot s_0$ be such that $U(\alpha/c) = U(H \cdot s_0) - 1$. Let $p = \operatorname{stp}(\alpha/\emptyset)$. We claim that $U(p) = U(S) - 1$ and that $H \not\subseteq \operatorname{Stab}(p)$.

For the first claim, note first that

$$U(s_0/\emptyset) - U(\alpha/\emptyset) = U(s_0/c) - U(\alpha/c)$$

by the Lascar inequalities, and the fact that $c \in \operatorname{dcl}(\alpha) \cap \operatorname{dcl}(c)$. By choice of α,

$$U(\alpha/c) = U(H \cdot s_0) - 1 \geq U(s_0/c) - 1$$

(as $s_0 \in H \cdot s_0$), so

$$U(\alpha/\emptyset) \geq U(s_0/\emptyset) - 1 = U(S) - 1. \tag{1}$$

On the other hand, if $\beta \in H \cdot s_0$ is chosen so that $U(\beta/c) = U(H \cdot s_0)$, then

$$U(\beta/\emptyset) - U(\alpha/\emptyset) = U(\beta/c) - U(\alpha/c) = 1,$$

so

$$U(\alpha/\emptyset) < U(\beta/\emptyset) \leq U(S).$$

Combining with (1), we see that $U(p) = U(\alpha/\emptyset) = U(S) - 1$.

Next, suppose for the sake of contradiction that $H \subseteq \operatorname{Stab}(p)$. Let h be generic in H, over α. The assertion that $h \in \operatorname{Stab}(p)$ implies that $h \cdot \alpha$ and α have the same strong type over h, hence over $\emptyset$. In particular, $U(h \cdot \alpha/\emptyset) = U(\alpha/\emptyset)$. As $h \cdot \alpha$ and α both define c, this implies that

$$U(h \cdot \alpha/c) = U(h \cdot \alpha/\emptyset) - U(c/\emptyset) = U(\alpha/\emptyset) - U(c/\emptyset) = U(\alpha/c).$$

In particular, $U(h \cdot \alpha/c) < U(H \cdot s)$. But this is impossible, since $\operatorname{tp}(h/c, \alpha)$ is generic in H, so that $\operatorname{tp}(h \cdot \alpha/c)$ should be generic in the homogeneous space $H \cdot s$. $\square$

Lemma 4.3. *The following are equivalent:*

- $g' \cdot S_{A'} \subseteq g \cdot S_A$.
- $\operatorname{acl}(A) \subseteq \operatorname{acl}(A')$ *and $g^{-1}g'$ is in $K \cdot (G \cap \operatorname{acl}(A'))$.*

Proof. We may (and do) assume that $g' = 1$. Suppose that $\operatorname{acl}(A) \subseteq \operatorname{acl}(A')$ and $g \in K \cdot (G \cap \operatorname{acl}(A'))$. Changing g by an element of K, we may assume that $g \in G \cap \operatorname{acl}(A')$. Suppose $x \in S_{A'}$. Then $\operatorname{stp}(x/A')$ is generic in S. As $g \in \operatorname{acl}(A')$, we also see that $\operatorname{stp}(g^{-1} \cdot x/A')$ is generic. As $\operatorname{acl}(A) \subseteq \operatorname{acl}(A')$, we see that $\operatorname{stp}(g^{-1} \cdot x/A)$ is generic in S, or equivalently, $x \in g \cdot S_A$. As x was arbitrary in $S_{A'}$, we conclude that $S_{A'} \subseteq g \cdot S_A$.

Conversely, suppose that $S_{A'} \subseteq g \cdot S_A$. By the previous lemma, we can find $p_1, \ldots, p_n$ strong types over $\emptyset$, in S, of rank one less than S, such that $\left(\bigcap_{i=1}^n \operatorname{Stab}(p_i)\right)/K$ is finite. Let $s_1, \ldots, s_n$ realize $p_1 \otimes \cdots \otimes p_n$, independent from A, g, A'. For each i,

$$U(s_i/A) = U(p_i) = U(S) - 1,$$

so $s_i \notin S_A$. Therefore $g \cdot s_i \notin g \cdot S_A$, so $g \cdot s_i \notin S_{A'}$. By definition of $S_{A'}$, it follows that

$$U(g \cdot s_i/A') \leq U(S) - 1 = U(s_i/A'g) = U(g \cdot s_i/A'g).$$

Consequently $g \cdot s_i \mathop{\smile\!\!\!\!\!\!|}_{A'} g$. Let $q_i = \operatorname{stp}(g \cdot s_i/A')$. Then $g \cdot p_i = q_i$, since $s_i \models p_i|A'g$ and $g \cdot s_i \models q_i|A'g$.

Consider the following set

$$Z = \left\{ g \in G : \bigwedge_{i=1}^{n} g \cdot p_i = q_i \right\}$$

This set is type-definable over $\operatorname{acl}(A')$, because p_i and q_i are based in $\operatorname{acl}(A')$. It is also a coset of $\bigcap_{i=1}^n \operatorname{Stab}(p_i)$, hence, a finite union of cosets of K. Since $\operatorname{acl}(A')$ is a model, there is some $g'' \in \operatorname{acl}(A')$ such that g'' and g are in the same coset of K. Equivalently, $g \in K \cdot (G \cap \operatorname{acl}(A'))$.

It remains to show that $\operatorname{acl}(A) \subseteq \operatorname{acl}(A')$. By the forward direction of this lemma, since $g \in K \cdot (G \cap \operatorname{acl}(A'))$, it follows that

$$S_{A'} = g^{-1} S_{A'}.$$

Since $g^{-1} S_{A'} \subseteq S_A$, we see that

$$S_{A'} \subseteq S_A. \tag{2}$$

Fix some strongly minimal set T definable over $\operatorname{acl}(\emptyset)$. For reasons noted earlier, A and A' are interalgebraic with $T \cap \operatorname{acl}(A)$ and $T \cap \operatorname{acl}(A')$, so it suffices to show that $T \cap \operatorname{acl}(A) \subseteq T \cap \operatorname{acl}(A')$. Suppose this failed. Take some $t_0 \in T$ such that $t_0 \in \operatorname{acl}(A) \setminus \operatorname{acl}(A')$.

Let $s_0 \in S$ be generic over $A \cup A' \cup \{t_0\}$. Let $n = U(S)$. For the same reasons noted earlier, s_0 is inter-algebraic over the empty set with some n-tuple $(t_1, \ldots, t_n) \in T^n$. Then $U(t_1, \ldots, t_n/A \cup A' \cup \{t_0\}) = n$, so $t_1, \ldots, t_n$ is generic in T^n over $A \cup A' \cup \{t_0\}$. The fact that $t_0 \notin \operatorname{acl}(A')$ implies that $(t_0, t_1, \ldots, t_n)$ is generic in T^{n+1}, over A'. Consequently, $(t_0, t_2, \ldots, t_n)$ is generic in T^n over A', and so

$$\operatorname{stp}(t_0, t_2, \ldots, t_n/A') = \operatorname{stp}(t_1, t_2, \ldots, t_n/A').$$

Therefore we can find some s_1 such that

$$s_1 t_0 t_2 \cdots t_n \equiv_{\operatorname{acl}(A')} s_0 t_1 t_2 \cdots t_n$$

In particular, this means that s_1 is inter-algebraic over the empty set with the tuple $(t_0, t_2, \ldots, t_n)$, and that s_1 is generic in S, over A'. So $s_1 \in S_{A'}$. By (2), $s_1 \in S_A$. Then

$$U(S) = n = U(s_1/A) = U(t_0, t_2, \ldots, t_n/A) = U(t_2, \ldots, t_n/A) = n - 1,$$

where the penultimate identity holds because $t_0 \in \operatorname{acl}(A)$. So we have a contradiction, and the assumption that $T \cap \operatorname{acl}(A) \not\subseteq T \cap \operatorname{acl}(A')$ was false. □

Definition 4.4. *If A is a small set of parameters,* $\operatorname{Gal}(A)$ *denotes* $\operatorname{Aut}(\operatorname{acl}(A)/\operatorname{dcl}(A))$*. This is naturally a profinite group.*

Remark 4.5. *We get natural restriction maps* $\operatorname{Gal}(B) \to \operatorname{Gal}(A)$ *whenever $B \supseteq A$.*

Definition 4.6. *Let U be a special open of the form $g \cdot S_A$. Let $s \in U$. Define $\pi_1(U, s)$ to be the profinite group* $\operatorname{Gal}(\{g^{-1} \cdot s\} \cup \operatorname{acl}(A))$.

Remark 4.7. *This doesn't depend on the choice of g and A. If $g' \cdot S_{A'} = g \cdot S_A$, then* $\operatorname{acl}(A) = \operatorname{acl}(A')$*, and $g^{-1}g' \in \operatorname{acl}(A)$ (up to a factor from K), so that $g^{-1} \cdot s$ and $(g')^{-1} \cdot s$ are interdefinable over* $\operatorname{acl}(A)$*. That is,*

$$\operatorname{dcl}(\{g^{-1} \cdot s\} \cup \operatorname{acl}(A)) = \operatorname{dcl}(\{(g')^{-1} \cdot s\} \cup \operatorname{acl}(A')),$$

so one gets the same group.

Remark 4.8. *More generally, suppose that $s \in U \subset U'$. Writing U, U' as $g \cdot S_A$ and $g' \cdot S_{A'}$, we know that* $\operatorname{acl}(A) \supseteq \operatorname{acl}(A')$*, and that $g^{-1} \cdot g' \in \operatorname{acl}(A)$, after changing g by a factor from K. Now it is clear that*

$$\operatorname{dcl}(\{g^{-1} \cdot s\} \cup \operatorname{acl}(A)) \supseteq \operatorname{dcl}(\{(g')^{-1} \cdot s\} \cup \operatorname{acl}(A'))$$

so we get a map $\pi_1(U, s) \to \pi_1(U', s)$.

Remark 4.9. *If U is a special open, and s, s' are two points in U, then $\pi_1(U, s)$ and $\pi_1(U, s')$ are non-canonically isomorphic, because $g^{-1} \cdot s$ and $g^{-1} \cdot s'$ have the same type over* $\operatorname{acl}(A)$.

Remark 4.10. *Suppose we name some small set of parameters B, adding new constant symbols to the language. Suppose $U = g \cdot S_A$ is a special open in the new expansion. Then U was a special open $g \cdot S_{A \cup B}$ in the original expansion, and if $s \in U$, the groups $\pi_1(U, s)$ are the same calculated either way. So in some sense, $\pi_1(U, s)$ doesn't depend on the choice of the base model over which we are working.*

Remark 4.11. *Let G' be a subgroup of G, still acting transitively on S. (Maybe also assume that G' is still connected.) Then every special open in S with respect to G' is (obviously) still a special open in S with respect to G, and $\pi_1(U, s)$ is the same calculated in either setting. So $\pi_1(U, s)$ is in some sense immune to increasing or decreasing the size of the group G acting on S (provided that we stay within the realm of groups acting transitively on S).*

Lemma 4.12. *The poset of special opens is a join semilattice. The join of two special opens $g \cdot S_A$ and $g' \cdot S_{A'}$ is $g \cdot S_B$, where $B = \mathrm{acl}(A \cup A' \cup \{c\})$, and c is the code for the equivalence class of $g^{-1} \cdot g'$ mod K. We denote the join of U and U' by $U \wedge U'$. Note that $U \wedge U'$ does not change when we vary the base model (in so far as that is allowed), and is also unchanged when we increase or decrease G (in so far as that is allowed).*

Proof. An easy exercise involving Lemma 4.3. □

Also note that for a fixed special open $U = g \cdot S_A$, the lattice of special opens inside U is isomorphic to the (order-reversed) lattice of algebraically closed sets containing A.

5 Defining the cohomology groups

Let $U_1, U_2, \dots$ be a collection of special opens in S, of length ω. For I a non-empty finite subset of ω, let U_I denote the join $\bigwedge_{i \in I} U_i$. For $I \subset J$, we have an inclusion $U_I \supseteq U_J$.

Choose some $s \in S$ generic over all the parameters involved with the U_i's. Then $s \in U_I$ for every $I \subset_f \omega$. The map $I \mapsto \pi_1(U_I, s)$ yields a contravariant functor from finite non-empty subsets of ω to pro-finite groups.

Fix some abelian group A. For each I, let $C_I^\bullet$ denote the cochain complex arising from the group cohomology of $\pi_1(U_I, s)$ with coefficients in A. So, C_I^n is the abelian group of continuous *functions* from $(\pi_1(U_I, s))^n$ to A, with the discrete topology on A.

Then we have a covariant functor from the poset of finite non-empty subsets of ω to the category of chain complexes. We would like to view this as an ω-dimensional $\omega \times 2 \times 2 \times 2 \times \cdots$ complex, and take the total complex.

More precisely, we take the cohomology of the total complex of the following double complex:

$$0 \to \prod_i C_i^\bullet \to \prod_{i<j} C_{i,j}^\bullet \to \prod_{i<j<k} C_{i,j,k}^\bullet \to \cdots$$

Denote this by $H^\bullet(\{U_i\}, s, A)$, for now.

In the case where $U_i = g_i \cdot S_{A_i}$, and where the sequence $g_1, g_2, \dots, A_1, A_2, \dots$ is independent over $\mathrm{acl}(\emptyset)$, and $g_1, g_2, \dots$ is a Morley sequence in the generic type of G, this will be our desired group cohomology. The independence of the choices will be seen in §7.

6 Heuristic motivation for the definition

We give a vague heuristic justification for this definition in the case where the theory is that of algebraically closed fields extending $\mathbb{C}$.

First we observe that special opens U are "morally" Eilenberg-Maclane spaces, because of the following fact:

Fact 6.1. *Let V be an irreducible variety over $\mathbb{C}$. Then there is a Zariski open $U \subset V$ such that U is a $K(\pi_1(U), 1)$-space, where $\pi_1(U)$ denotes the usual topological homotopy group.*

Proof. Abusing terminology(?), I'll call something an Eilenberg-Maclane space (EM-space) if its analytification is a $K(G, 1)$-space for some G, or equivalently, if its universal cover is contractible. Any curve of negative Euler characteristic is an EM space, because its analytic universal cover is the open unit disk. If $E \to B$ is a fibration (in the sense of homotopy theory) and B and the fiber are both EM spaces, then E is an EM space, because of the homotopy long exact sequence of the fibration.

We proceed by induction on $\dim V$, the case of curves being already handled. In the case of dimension $n > 1$, take some dominant rational map from V to $\mathbb{A}^{n-1}$. Shrinking V to a Zariski open, we may assume that this is a true morphism of varieties. The fibers are generically one-dimensional; shrinking V we may assume that all the non-empty fibers are one-dimensional. Replacing $\mathbb{A}^{n-1}$ with the (definable) set of pairs (p, X), where $p \in \mathbb{A}^{n-1}$ and X is an irreducible component of the fiber over p, we get a map $V \to W$ whose fibers are all irreducible curves. By the "almost skolem functions" mentioned earlier, we can find some closed subset $Z \subset V$ whose intersection with a generic fiber of $V \to W$ is finite, but sufficiently big that the generic fiber of $V \setminus Z \to W$ has negative Euler characteristic. Replacing V with $V \setminus Z$, we may assume that the generic fibers of $V \to W$ have negative Euler characteristic. Shrinking W to a Zariski open (and hence shrinking V as well), we may assume that the map is a fibration, and that every fiber has negative Euler characteristic. Shrinking W even further, by the induction hypothesis, we may assume that W is an EM space. Then $V \to W$ is a fibration with fibers and base being EM spaces, so V is an EM space. $\square$

Now, since U is morally an Eilenberg-Maclane space, the cochain complex of cochains on U should be quasi-isomorphic to the cochain complex from group cohomology of $\pi_1(U)$. If we take coefficients in $\mathbb{Z}/\ell$, we should be able to replace $\pi_1(U)$ with its profinite completion, which is the $\pi_1(U)$ constructed in Section 4. Let $C_U^\bullet$ be the cochain complex of this profinite group, with coefficients in $\mathbb{Z}/\ell$.

Let $0 \to \mathbb{Z}/\ell \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$ be an injective resolution of the constant sheaf $\mathbb{Z}/\ell$ on the variety S. The restriction of $\mathcal{F}^n$ to the "open" U is still an injective sheaf on $\mathcal{F}^n$, so the cochain complex of cochains on U can also be gotten (up to quasi-isomorphism) as

$$0 \to \mathcal{F}^0(U) \to \mathcal{F}^1(U) \to \cdots$$

In other words, we think or pretend that $C_U^\bullet$ is quasi-isomorphic to $\mathcal{F}^\bullet(U)$.

Now suppose that $g_1, g_2, \ldots$ realize a Morley sequence in the generic type of G, and $U_i = g_i \cdot S_\emptyset$. In this case, the recipe in the previous section yields the total complex of the following double complex:

$$0 \to \prod_i C_{U_i}^\bullet \to \prod_{i<j} C_{U_{ij}}^\bullet \to \prod_{i<j<k} C_{U_{ijk}}^\bullet \to \cdots$$

By what we have just said, this should be quasiisomorphic to the total complex of the double complex

$$0 \to \prod_i \mathcal{F}^\bullet(U_i) \to \prod_{i<j} \mathcal{F}^\bullet(U_{ij}) \to \prod_{i<j<k} \mathcal{F}^\bullet(U_{ijk}) \to \cdots$$

We want this to be quasi-isomorphic to the complex $0 \to \mathcal{F}^0(G) \to \mathcal{F}^1(G) \to \cdots$. To show this, it suffices to show that the total complex of the following double complex is exact:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\mathcal{F}^1(S) \arrow[r] \arrow[u] & \prod_i \mathcal{F}^1(U_i) \arrow[r] \arrow[u] & \prod_{i<j} \mathcal{F}^1(U_{ij}) \arrow[r] \arrow[u] & \cdots \\
\mathcal{F}^0(S) \arrow[r] \arrow[u] & \prod_i \mathcal{F}^0(U_i) \arrow[r] \arrow[u] & \prod_{i<j} \mathcal{F}^0(U_{ij}) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

It suffices to show that the rows of this double complex are exact. It suffices to show that if $\mathcal{F}$ is an arbitrary injective sheaf, then the complex

$$0 \to \mathcal{F}(S) \to \prod_i \mathcal{F}(U_i) \to \prod_{i<j} \mathcal{F}(U_{ij}) \to \cdots$$

is exact. By injectivity of $\mathcal{F}$, it suffices to show that the following chain complex of sheaves is exact:

$$\cdots \to \bigoplus_{i<j} \mathbb{Z}_{U_{ij}} \to \bigoplus_i \mathbb{Z}_{U_i} \to \mathbb{Z}_S \to 0 \tag{3}$$

where $\mathbb{Z}_U$ denotes the constant sheaf $\mathbb{Z}$ on U, extended by zero off of U.

This can be checked on stalks. If p is a point, the stalk of $\mathbb{Z}_U$ is $\mathbb{Z}$ if $p \in U$, and 0 otherwise (since we are pretending that U is open). From this, the stalk of (3) at a point p more or less becomes the (reduced) simplicial homology groups of the following abstract simplicial complex:

- There is a non-degenerate 0-simplex (a point) for each i such that $p \in U_i$, that is, for each i such that $(g_i)^{-1} \cdot p$ is generic over $\emptyset$.
- There is an edge between i and j if $p \in U_{ij}$, that is, if $(g_i)^{-1} \cdot p$ is generic over $(g_j)^{-1} g_i$.
- There is a face between i, j, and k if $p \in U_{ijk}$, that is, if $(g_i)^{-1} \cdot p$ is generic over $(g_j)^{-1} g_i$ and $(g_j)^{-1} g_k$.
- $\ldots$

But one can see directly that this abstract simplicial complex is contractible. It suffices to show that if Δ is any finite subcomplex of the above complex, then there is some vertex k such that the cone of Δ to the point k is also in the above abstract simplicial complex. That is, if Σ is a finite subset of ω, we want to find some $k \notin \Sigma$ such that for every $I \subseteq \Sigma$, $p \in U_I \iff p \in U_{Ik}$. We can do this by taking k such that

$$g_k \perp p, \{g_i : i \in I\}$$

Indeed, suppose that $p \in U_I$. Choose some $j \in I$. The statement that $p \in U_I$ means that $g_j^{-1} \cdot p$ is generic over $\{g_j^{-1} \cdot g_i : i \in I\}$. Since g_k is independent from p and g_i for $i \in I$, g_k is generic in G over the same. Generic types are fixed by translation, so $g_j^{-1} \cdot g_k$ is also generic over p and g_i for $i \in I$. Therefore

$$g_j^{-1} \cdot g_k \mathop{\smile\!\!\!\!\!\!|}\, p, \{g_i : i \in I\}$$

This implies

$$g_j^{-1} \cdot g_k \mathop{\smile\!\!\!\!\!\!|}_{\{g_j^{-1} \cdot g_i : i \in I\}} g_j^{-1} \cdot p$$

Since $\operatorname{tp}(g_j^{-1} \cdot p/\{g_j^{-1} \cdot g_i : i \in I\})$ is generic, so is its non-forking extension

$$\operatorname{tp}(g_j^{-1} \cdot p/\{g_j^{-1} \cdot g_i : i \in I \cup \{k\}\})$$

Therefore $p \in U_{Ik}$. Conversely, if $p \in U_{Ik}$ then $p \in U_I$. Therefore the abstract simplicial complex on the vertices $I \cup \{k\}$ is a cone of the abstract simplicial complex on the vertices in I. This ensures that the entire thing is contractible, and so the stalk of (3) at p is exact.

This completes the heuristic justification of why one should get the correct cohomology groups in this way. Later, I may work on making this argument precise.

7 Well-definedness

In §5 we defined $H^\bullet(\{U_i\}, s, A)$ for U_i special opens in a homogeneous space S with a group action G, for s a point in $U_I := \bigwedge_{i \in I} U_i$ for every $I \subset_f \omega$, and for A an abelian group.

In this section, we will show that for properly chosen U_i, the choice of U_i does not matter. We may also show that the choice of s does not matter.

Assume for simplicity that G acts faithfully on S. Fix some abelian group A. Fix some medium-sized model M, and some s_0 generic in S over M. If $U = g \cdot S_A$ is a special open, with $g, A \subset M$, then Π_U will denote $\pi_1(U, s)$, and $C_U^\bullet$ will denote the complex of continuous cohains of Π_U with coefficients in A (with the discrete topology and trivial action of Π_U).

Lemma 7.1. *Suppose that $A_1, \ldots, A_n$ is an independent sequence of small sets over $\emptyset$. For $I \subset \{1, \ldots, n\}$, let A_i denote the algebraic closure of the union of the A_i for $i \in I$. Suppose that $a_1, \ldots, a_m$ are elements of $A_{\{1,\ldots,n\}}$. Suppose that $I \subset \{1, \ldots, n\}$. Suppose that $\mathbb{M} \models \chi(a_1, \ldots, a_m)$ for some formula χ. Then $\mathbb{M} \models \chi(a'_1, \ldots, a'_m)$ where each a'_i is in A_I.* Moreover*, we can arrange that the following things are true for each i:*

- *If $a_i \in A_I$, then $a'_i = a_i$.*
- *If $a_i \in A_J$, then $a_i \in A_{J \cap I}$.*

Proof. Note that the second bullet point subsumes the requirement that $a'_i \in A_I$, by taking $J = \{1, \ldots, n\}$.

For each i, J such that $a_i \in A_J$, let $\phi_{i,J}(x; b_{i,J}; c_{i,J})$ be a formula such that

- $\phi_{i,J}(a_i; b_{i,J}; c_{i,J})$ holds
- for any b, c in the monster, $\phi_{i,J}(\mathbb{M}; b, c)$ is finite
- $b_{i,J} \in A_{J\cap I}$
- $c_{i,J} \in A_{J\setminus I}$

Such a formula exists because $a_i \in \operatorname{acl}(A_{J\cap I} \cup A_{J\setminus I})$.

By the independence assumption,

$$A_I \mathop{\smile\!\!\!\!\!\!|}\, A_{\{1,\ldots,n\}\setminus I}.$$

It follows that

$$\{a_i : a_i \in A_I\} \cup \{b_{i,J} : a_i \in A_J\} \mathop{\smile\!\!\!\!\!\!|}\, \{c_{i,J} : a_i \in A_J\}$$

The type of $\{c_{i,J} : a_i \in A_J\}$ over $\{a_i : a_i \in A_I\} \cup \{b_{i,J} : a_i \in A_J\}$ is finitely satisfiable in $\operatorname{acl}(\emptyset)$, because $\operatorname{acl}(\emptyset)$ is a model. It contains the following formula (witnessed by taking x_i to be a_i):

$$\exists x_1, \ldots, x_n : \chi(x_1, \ldots, x_n) \wedge \left(\bigwedge_{i : a_i \in A_I} x_i = a_i \right) \wedge \left(\bigwedge_{i,J : a_i \in a_J} \phi_{i,J}(x_i; b_{i,J}; y_{i,J}) \right)$$

Consequently, we can find $c'_{i,J}$ in $\operatorname{acl}(\emptyset)$ and a'_i in $\mathbb{M}$ such that the following things hold:

- $\chi(a'_1, \ldots, a'_n)$
- If $a_i \in A_I$, then $a'_i = a_i$.
- If $a_i \in A_J$, then $\phi_{i,J}(a'_i; b_{i,J}; c'_{i,J})$. In particular, a'_i is algebraic over $b_{i,J} \in A_{J\cap I}$ and $c'_{i,J} \in \operatorname{acl}(\emptyset)$. So $a'_i \in A_{J\cap I}$.

□

Lemma 7.2. *Suppose that $A_1, \ldots, A_n$ is an independent sequence of small sets over $\emptyset$. For $I \subset \{1, \ldots, n\}$, let A_i denote the algebraic closure of the union of the A_i for $i \in I$. Suppose that $a_1, \ldots, a_m$ are elements of $A_{\{1,\ldots,n\}}$. Suppose that $I \subset \{1, \ldots, n\}$. Suppose that $\mathbb{M} \models \chi(a_1, \ldots, a_m)$ for some formula χ. Then $\mathbb{M} \models \chi(a'_1, \ldots, a'_m)$ where each a'_i is in A_I.* Moreover, *we can arrange that the following things are true for each i:*

- *If $a_i \in A_I$, then $a'_i = a_i$.*
- *If $J_1, \ldots, J_k$ are such that*

  $$a_i \in \operatorname{dcl}(A_{J_1} \cup \cdots \cup A_{J_k}),$$

  then

  $$a'_i \in \operatorname{dcl}(A_{J_1\cap I} \cup \cdots \cup A_{J_k\cap I}).$$

Proof. For each i and $J_1, \ldots, J_k$ such that $a_i \in \mathrm{dcl}(A_{J_1} \cup \cdots \cup A_{J_k})$, write a_i as $f(b_1, \ldots, b_k)$ where $b_i \in A_{J_i}$. Add the b_i's to the list $(a_1, \ldots, a_n)$, and add replace χ with $\chi \wedge a_i = f(b_1, \ldots, b_k)$. Do this for all the original i and $J_1, \ldots, J_k$. Apply the previous lemma. Now we have a'_i and b'_j such that $a'_i = f(b'_1, \ldots, b'_k)$, where each $b'_j \in A_{J_j \cap I}$. □

Lemma 7.3. *Suppose that $A_1, \ldots, A_n$ is an independent sequence over $\emptyset$. Suppose $A_i = \mathrm{acl}(A_i)$ for each i. For $I \subset \{1, \ldots, n\}$, let*

$$A_I = \mathrm{acl}\left(\bigcup_{i \in I} A_i\right).$$

Let s be generic in S over $\bigcup_{i=1}^n A_i$. Let Π_I denote $\mathrm{Gal}(sA_I)$. Let $m \geq 0$ be an integer. For $I \subset J$ there are natural restriction maps $\Pi_J \to \Pi_I$. Suppose that $I, J_1, \ldots, J_\ell$ are subsets of $\{1, \ldots, n\}$, and let $J = \bigcup_{i=1}^{\ell} J_i$. Let $J_0 = I$. Let R be an abelian group, with the discrete topology. For $0 \leq j \leq \ell$, let g_j be a continuous function from $\Pi_{J_j}^m$ to R. Suppose that when pulled back to $\Pi_{J \cup I}^m$, we have

$$g_0 = \sum_{j=1}^{\ell} g_j \tag{4}$$

Then there exist continuous maps $g'_j : \Pi_{J_j \cap I}^m \to R$ such that when pulled back to Π_I^m,

$$g_0 = \sum_{j=1}^{\ell} g'_j.$$

Proof. For each i, choose an element $c_i \in \mathrm{acl}(sA_{J_i})$ such that $g_i(\sigma_1, \ldots, \sigma_m)$ is determined by $(\sigma_1(c_i), \ldots, \sigma_m(c_i))$. Let Σ_i be an enumeration of the conjugates of c_i over $\mathrm{dcl}(sA_{J_i})$, viewed alternatively as a tuple or a set. Let $\phi_i(x; d_i)$ be a formula such that $\phi_i(\mathbb{M}; d_i) = \Sigma_i$, $d_i \in \mathrm{dcl}(sA_{J_i})$, and $\phi_i(\mathbb{M}; d)$ is finite for every $d \in \mathbb{M}$.

Consider the finite set $\Sigma := \prod_{i=1}^{\ell} \Sigma_i$. Let E be some finite subset of $\mathrm{dcl}(sA_{J \cup I})$ and $\phi(x; y)$ be a formula such that two elements of Σ have the same type over $\mathrm{dcl}(sA_{J \cup I})$ if and only if they have the same ϕ-type over E.

By the previous lemma, applied to $s, A_1, \ldots, A_n$, with $s \in A_I$, we can find Σ'_i, Σ', E', and c'_i such that all the following things hold:

- c'_i is the first element of Σ'_i, and Σ'_i is a tuple of the same length as Σ_i
- The underlying *set* of Σ'_i is $\mathrm{dcl}(sA_{J_i \cap I})$-definable.
- $c'_0 = c_0$ and $\Sigma'_0 = \Sigma_0$ remain unchanged
- Σ' is $\prod_{i=1}^{\ell} \Sigma'_i$
- $t_1, t_2 \in \Sigma$ have the same ϕ-type over E if and only if $t'_1, t'_2 \in \Sigma'$ have the same ϕ-type over E'.

- E' is a finite subset of $\mathrm{dcl}(sA_I)$.

From the last two bullet points (and the choice of E), we see that if $t'_1, t'_2 \in \Sigma'$ have the same type over $\mathrm{dcl}(sA_I)$, then t_1 and t_2 have the same type over $\mathrm{dcl}(sA_{I\cup J})$.

Next we define $g'_i : \Pi^m_{J_i \cap I} \to R$. Given $\sigma'_1, \ldots, \sigma'_m \in \Pi_{J_i\cap I} = \mathrm{Gal}(s\,\mathrm{acl}(A_{J_i\cap I}))$, look at $\sigma'_j(c'_i)$. Since $c'_i \in \Sigma'_i$ and the underlying set of Σ'_i is $\mathrm{dcl}(s\,\mathrm{acl}(A_{J_i\cap I}))$-definable, $\sigma'_j(c'_i)$ is in Σ'_i. So it is q'_j for some $q_j \in \Sigma_i$. Let σ_j be an element of $\mathrm{Gal}(sA_{J_i})$ sending c_i to q_j. Define

$$g'_i(\sigma'_1, \ldots, \sigma'_m) := g_i(\sigma_1, \ldots, \sigma_m).$$

The choice of $\sigma_1, \ldots, \sigma_m$ doesn't matter, because we arranged that

$$g_i(\sigma_1, \ldots, \sigma_m)$$

depends only on $(\sigma_1(c_i), \ldots, \sigma_m(c_i))$, which we have specified as $(q_1, \ldots, q_m)$. Also, g'_i is continuous, because $g'_i(\sigma'_1, \ldots, \sigma'_m)$ depends only on $(\sigma'_1(c'_i), \ldots, \sigma'_m(c'_i))$.

For $i = 0$, note that $A_{J_0\cap I} = A_{J_0}$, $\Sigma'_0 = \Sigma_0$, $c'_0 = c_0$, and $q'_j = q_j$, so that

$$\sigma_j(c_0) = q_j = q'_j = \sigma'_j(c'_0) = \sigma'_j(c_0),$$

so we may as well take $\sigma_j = \sigma'_j$. Thus $g'_0 = g_0$.

It remains to show that

$$g'_0 = \sum_{i=1}^{\ell} g_i$$

holds, where the summands are pulled back to be functions on Π^m_I.

Let $\sigma'_1, \ldots, \sigma'_m$ be elements of $\Pi_I = \mathrm{Gal}(sA_I)$. Let $t = c_0 c_1 \cdots c_\ell$. For each $1 \le j \le m$, let $t'_j = \sigma'_j(t')$. Since t'_j and t' have the same type over $\mathrm{dcl}(sA_I)$, and both are in Σ', it follows that t_j and t have the same type over $\mathrm{dcl}(sA_{I\cup J})$. So there is some automorphism $\sigma_j \in \mathrm{Gal}(sA_{I\cup J})$ sending t to t_j.

Now for every i, σ_j restricts to an automorphism in $\mathrm{Gal}(sA_{J_i\cap I})$. This automorphism sends c_i to $q_{i,j}$, the ith entry of t_j. Then $q'_{i,j}$ is the ith entry of $t'_j = \sigma'_j(t')$, so $q'_{i,j} = \sigma'_j(c'_i)$. By definition of g'_i, it follows that

$$g'_i(\sigma'_1, \ldots, \sigma'_m) = g_i(\sigma_1, \ldots, \sigma_m).$$

Therefore we have

$$g_0 = \sum_{i=1}^{\ell} g_i \implies g'_0 = \sum_{i=1}^{\ell} g'_i.$$

$\square$

Lemma 7.4. *Suppose that $A_1, \ldots, A_n$ is an independent sequence over $\emptyset$. Suppose $A_i = \mathrm{acl}(A_i)$ for each i. For $I \subset \{1, \ldots, n\}$, let*

$$A_I = \mathrm{acl}\left(\bigcup_{i\in I} A_i\right).$$

Let s be generic in S over $\bigcup_{i=1}^n A_i$. Let Π_I denote $\mathrm{Gal}(sA_I)$. Let $m \geq 0$ be an integer. Let R be an abelian group, with the discrete topology. Let C_I^m denote the abelian group of continuous functions from Π_I to R. Then the following sequence is exact, except at the last place:

$$0 \to C_\emptyset^m \to \prod_i C_i^m \to \prod_{i<j} C_{ij}^m \to \cdots \to \prod_i C_{1,\ldots,\hat{i},\ldots,n}^m \to C_{1,\ldots,n}^m \to 0 \tag{5}$$

Proof. We proceed by induction on n. For $n = 1$, we need to show that the sequence

$$0 \to C_\emptyset^m \to C_1^m$$

is exact, i.e,. that $C_\emptyset^m \to C_1^m$ is injective. It suffices to show that the map

$$\mathrm{Gal}(s\,\mathrm{acl}(A_1)) = \Pi_1 \to \Pi_\emptyset = \mathrm{Gal}(s\,\mathrm{acl}(\emptyset))$$

is surjective. Given σ an automorphism of $\mathrm{acl}(s)$ fixing s and $\mathrm{acl}(\emptyset)$ pointwise, we need to extend it to an automorphism of the monster fixing s and $\mathrm{acl}(A_1)$ pointwise. If c and d are tuples enumerating $\mathrm{acl}(s)$ and $\mathrm{acl}(A_1)$, respectively, then $c \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{acl}(\emptyset)} d$. By stationarity of strong types over models, $\sigma(c)$ and c have the same type over $d\,\mathrm{acl}(\emptyset)$. So $\sigma(c)d$ and cd have the same strong type over $\emptyset$. So we can find σ' an automorphism of the monster, fixing $\mathrm{acl}(\emptyset)$ pointwise, acting like σ on $c = \mathrm{acl}(s)$, and fixing $d = \mathrm{acl}(A_1)$ pointwise. Since σ fixes s pointwise, so does σ', so σ' fixes s and $\mathrm{acl}(A_1)$ pointwise, as desired.

For the inductive step, note that the complex in question is the total complex of the double complex

$$\begin{array}{ccccccc}
C_n^m & \longrightarrow & \prod_{i<n} C_{in}^m & \longrightarrow & \prod_{i<j<n} C_{ijn}^m & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
C_\emptyset^m & \longrightarrow & \prod_{i<n} C_i^m & \longrightarrow & \prod_{i<j<n} C_{ij}^m & \longrightarrow & \cdots
\end{array}$$

By induction, each row is exact except possibly in the last place. (To apply induction to the top row, add names for the elements of A_n to the language.) It follows that (5) is exact in the first $n-1$ places. Therefore we only need to check exactness at the penultimate place.

So, for each i, we are given f_i a continuous function from $\Pi_{1,\ldots,\hat{i},\ldots,n}^m$ to R, such that when pulled back to $\Pi_{1,\ldots,n}^m$,

$$f_1 + f_2 + \cdots + f_n = 0.$$

We need to find g_{ij} for $i < j$, a continuous function from $\Pi_{\{1,\ldots,n\}\setminus\{i,j\}}^m$ to R, such that for each i,

$$f_i = \sum_{j<i} g_{ji} - \sum_{j>i} g_{ij}. \tag{6}$$

We show by induction on $0 \leq m \leq n$ that there exists g_{ij} such that (6) holds for $i \leq m$. The base case $m = 0$ is trivial; take all the $g_{ij} = 0$.

Suppose that (6) holds for all $i \leq m$. Replacing f_i with $f_i - \sum_{j<i} g_{ji} - \sum_{j>i} g_{ij}$, we may assume that $f_i = 0$ for $i \leq m$. If $m = n$, then $\sum_i f_i = f_n = 0$, so every f_i vanishes and we can take $g_{ij} = 0$. So assume that $m < n$. By the previous lemma, applied to $I = \{1, \ldots, n\} \setminus m$ and $J_j = \{1, \ldots, n\} \setminus (m + j)$, we can find $f'_{m+1}, \ldots, f'_n$ such that

$$f_m + \sum_{i=m+1}^{n} f'_i = 0,$$

and f'_i comes from $\Pi_{I \cap J_{i-m}} = \Pi_{\{1,\ldots,n\} \setminus \{m,i\}}$. Set $g_{ij} = f'_{ij}$ for $i = m$ and $j > m$, and $g_{ij} = 0$ otherwise. Then

$$f_m = \sum_{j<i} g_{ji} - \sum_{j>i} g_{ij}.$$

Also, for $i < m$, g_{ij} and g_{ji} vanish, so

$$f_i = 0 = \sum_{j<i} g_{ji} - \sum_{j>i} g_{ij}.$$

Thus (6) holds for all $i \leq m$, completing the proof of the inductive step (for the induction on m). By taking $m = n$, we get that (6) holds for all i, completing the inductive step for the induction on n. $\square$

Lemma 7.5. *Let $A_1, A_2, \ldots$ be an independent sequence over $\emptyset$. Suppose each A_i is algebraically closed, and let $A_I = \operatorname{acl}(A_i : i \in I)$, as usual. Let s be generic in S over all the A_I's. Let $\Pi_I = \operatorname{Gal}(sA_I)$. Let R be some abelian group. Let $C_I^\bullet$ be the cochain complex from group cohomology of Π_I with coefficients in R. Then the rows of the following double complex are exact, making the total complex also be exact:*

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C^1_\emptyset \arrow[u] \arrow[r] & \prod_i C^1_i \arrow[u] \arrow[r] & \prod_{i<j} C^1_{ij} \arrow[u] \arrow[r] & \cdots \\
C^0_\emptyset \arrow[u] \arrow[r] & \prod_i C^0_i \arrow[u] \arrow[r] & \prod_{i<j} C^0_{ij} \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

Proof. Fix m, and consider the complex

$$C^m_\emptyset \to \prod_i C^m_i \to \prod_{i<j} C^m_{ij} \to \cdots \tag{7}$$

For each n, (7) happens to be the total complex of the following double complex (pretend there is a column of zeros on both the left and the right sides):

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & {} \\
\prod\limits_{n<j_1<j_2} C^m_{j_1,j_2} \arrow[r] \arrow[u] & \prod\limits_{i_1\le n<j_1<j_2} C^m_{i_1,j_1,j_2} \arrow[r] \arrow[u] & \cdots \arrow[r] \arrow[u] & \cdots \arrow[r] \arrow[u] & \prod\limits_{n<j_1<j_2} C^m_{1,\dots,n,j_1,j_2} \arrow[u] \\
\prod\limits_{n<j_1} C^m_{j_1} \arrow[r] \arrow[u] & \prod\limits_{i_1\le n<j_1} C^m_{i_1,j_1} \arrow[r] \arrow[u] & \prod\limits_{i_1<i_2\le n<j_1} C^m_{i_1,i_2,j_1} \arrow[r] \arrow[u] & \cdots \arrow[r] \arrow[u] & \prod\limits_{n<j_1} C^m_{1,\dots,n,j_1} \arrow[u] \\
C^m_\emptyset \arrow[r] \arrow[u] & \prod\limits_{i_1\le n} C^m_{i_1} \arrow[r] \arrow[u] & \prod\limits_{i_1<i_2\le n} C^m_{i_1,i_2} \arrow[r] \arrow[u] & \cdots \arrow[r] \arrow[u] & C^m_{1,\dots,n} \arrow[u]
\end{tikzcd}
$$

Suppose each row is exact except in the last place. Then the total complex (7) is exact in the first n places. Since n was arbitrary, (7) is exact. So it suffices to show that each row is exact except in the last place. Since a product of exact sequences is exact, it suffices to show for each $J \subset \{n+1, n+2, \dots\}$ that

$$0 \to C^m_J \to \prod_{i\le n} C^m_{i,J} \to \cdots \to C^m_{1,\dots,n,J} \to 0$$

is exact except at the last place. This actually follows from the previous Lemma, after adding new constant symbols for all the elements of $\mathrm{acl}(A_J)$ to our base language. $\square$

Definition 7.6. *A sequence $g_1, A_1, g_2, A_2, \dots$ is* pleasant *if it is independent over* $\mathrm{acl}(\emptyset)$*, and if $g_1, g_2, \dots$ is a Morley sequence in the generic type of G, over $\emptyset$.*

Lemma 7.7. *Let $\langle g_i, A_i\rangle_{i\in I}$ be an infinite pleasant sequence, and suppose J is an infinite and coinfinite subset of I. Then the cohomology groups computed using the special opens $\langle g_i \cdot S_{A_i}\rangle_{i\in I}$ agree with the ones computed using $\langle g_j \cdot S_{A_j}\rangle_{j\in J}$. In fact, the natural map between the associated chain complexes is a quasi-isomorphism.*

Proof. Let $J' = I \setminus J$. For $i \in I$, let U_i be the special open $g_i \cdot S_{A_i}$. For T a finite subset of I, let U_T be the join of U_i for $i \in T$.

Consider the double complex of complexes whose (i, i')th entry is the product of $C^\bullet_{T\cup T'}$, where T ranges over i-element subsets of J, and T' ranges over the i'-element subsets of J'. (Take the $(0, 0)$ entry to be 0). We want the bottom row of this complex to be quasiisomorphic to the entire complex. This will hold as long as the other rows are quasi-isomorphic to zero. Each row other than the bottom one is a product of sequences of the form

$$0 \to C^\bullet_{T'} \to \prod_{j\in J} C^\bullet_{T'j} \to \prod_{j_1<j_2\in J} C^\bullet_{T'j_1j_2} \to \cdots \qquad (8)$$

for T' a finite subset of J', so it suffices to show that each of these is exact (after passing to the total complex).

Write $U_{T'}$ as $g_0 \cdot S_{A_0}$ for some g_0 and A_0. Note that g_0 and A_0 are algebraic over $\langle g_j, A_j \rangle_{j \in J'}$. So

$$g_0, A_0 \perp \langle g_j, A_j \rangle_{j \in J}.$$

Since

$$\langle g_j \rangle_{j \in J} \perp \langle A_j \rangle_{j \in J}$$

it follows that in fact

$$g_0, A_0 \langle A_j \rangle_{j \in J} \perp \langle g_j \rangle_{j \in J}$$

So $\langle g_j \rangle_{j \in J}$ is a Morley sequence in the generic type of G, over g_0, A_0, and A_j for $j \in J$. As the generic type is stabilized by everything, $\langle g_0^{-1} \cdot g_j \rangle_{j \in J}$ also is a Morley sequence in this same type, over g_0, A_0, and A_j for $j \in J$. Thus

$$\langle g_0^{-1} \cdot g_j \rangle_{j \in J} g_0 A_0 \langle A_j \rangle_{j \in J} \equiv_{\emptyset} \langle g_j \rangle_{j \in J} g_0 A_0 \langle A_j \rangle_{j \in J}$$

It follows that the sequence $\langle g_0^{-1} \cdot g_j, A_j \rangle_{j \in J}$ is independent over A_0. Let B_j denote $\operatorname{acl}(g_0^{-1} \cdot g_j, A_j)$.

Now for any $j_1 < \cdots < j_n$ in J,

$$U_{T' j_1 \cdots j_n} = g_0 \cdot S_{A \cup B_{j_1} \cup \cdots \cup B_{j_n}}.$$

Let $s' = g_0 \cdot s$. Then

$$\pi_1(U_{T' j_1 \cdots j_n}, s) = \operatorname{Gal}(s' \operatorname{acl}(A \cup B_{j_1} \cup \cdots \cup B_{j_n})) = \pi_1(S_{A \cup B_{j_1} \cup \cdots \cup B_{j_n}}, s').$$

Since $B_{j_1}, B_{j_2}, \ldots$ is independent over A, (8) is just the sequence that is exact by Lemma 7.5. □

If I_1 and I_2 are two pleasant sequences, we can always find a pleasant sequence I_3 such that the concatenations $I_1 I_3$ and $I_2 I_3$ are pleasant. By four applications of Lemma 7.7, the cohomology groups computed by I_1 and by I_2 are isomorphic. In fact, the isomorphism is canonical:

Theorem 7.8. *For each (infinite) pleasant sequence I, let $H_I^\bullet$ denote the cohomology groups computed using I. There is a unique system of isomorphisms $\eta_{I,I'} : H_I^\bullet \xrightarrow{\sim} H_{I'}^\bullet$ for any two pleasant sequences I, I', such that $\eta_{I'',I'} \circ \eta_{I,I'} = \eta_{I,I''}$ and such that if $I \supset I'$, then $\eta_{I,I'}$ is the natural restriction map from $H_I^\bullet$ to $H_{I'}^\bullet$. (In fact, we could also work with the underlying chain map in the derived category, rather than working with just the cohomology groups.)*

Proof. Uniqueness follows from four applications of Lemma 7.7, as mentioned above. Existence remains to be shown.

The map $I \mapsto H_I^\bullet$ is a contravariant functor from the poset of pleasant sequences to the category of sequences of groups. If I, J are two pleasant sequences such that $I \cap J = \emptyset$ and the union $I \cup J$ is a pleasant sequence, let $\nu_{I,J}$ denote the concatenation

$$H_I^\bullet \xleftarrow{\sim} H_{I \cup J}^\bullet \xrightarrow{\sim} H_J^\bullet.$$

Now suppose I, J, K are pairwise disjoint, and $I \cup J \cup K$ is pleasant. Then $\nu_{J,K} \circ \nu_{I,J} = \nu_{I,K}$, because of the following diagram in which every map is an isomorphism:

$$\begin{tikzcd}
H^\bullet_{I\cup K} \arrow[rr] \arrow[dd] & & H^\bullet_I \arrow[ddll, bend right, "\nu_{I,K}"'] \arrow[dddr, "\nu_{I,J}"] & \\
& H^\bullet_{I\cup J\cup K} \arrow[ul] \arrow[rr] \arrow[dd] & & H^\bullet_{I\cup J} \arrow[ul] \arrow[dd] \\
H^\bullet_K & & & \\
& H^\bullet_{J\cup K} \arrow[ul] \arrow[rr] & & H^\bullet_J \arrow[ulll, "\nu_{J,K}"]
\end{tikzcd}$$

The diagram commutes because of functoriality of $I \mapsto H^\bullet_I$.

For any I, I', let $\eta_{I,I'}$ be the concatenation

$$H^\bullet_I \overset{\nu_{I,J}}{\to} H^\bullet_J \overset{\nu_{J,I'}}{\to} H^\bullet_{I'},$$

where J is some pleasant sequence such that IJ and $I'J$ are pleasant.

We claim that the choice of J does not matter. Indeed, let J' be some other choice. Let K be some pleasant sequence, independent from I, I', J, J'. Then the following diagram commutes:

$$\begin{tikzcd}
& H^\bullet_J \arrow[d, "\nu_{J,K}"] \arrow[dr, "\nu_{J,I'}"] & \\
H^\bullet_I \arrow[ur, "\nu_{I,J}"] \arrow[r, "\nu_{I,K}"] \arrow[dr, "\nu_{I,J'}"'] & H^\bullet_K \arrow[r, "\nu_{K,I'}"] & H^\bullet_{I'} \\
& H^\bullet_{J'} \arrow[u, "\nu_{J',K}"] \arrow[ur, "\nu_{J',I'}"'] &
\end{tikzcd}$$

Now if I, I', I'' are any three pleasant sequences, we can find some pleasant sequence J independent from $I \cup I' \cup I''$, and then

$$\eta_{I,I''} = \nu_{J,I''} \circ \nu_{I,J} = \nu_{J,I''} \circ \nu_{I',J} \circ \nu_{J,I'} \circ \nu_{I,J} = \eta_{I',I''} \circ \eta_{I,I'}.$$

It remains to show that if $I \supset I'$, then $\eta_{I,I'}$ is the original map from the functor $H^\bullet_{\_}$. Take J a pleasant sequence independent from I. Then functoriality of $H^\bullet_{\_}$ gives a commutative diagram of isomorphisms

$$\begin{tikzcd}
H^\bullet_{I'} & H^\bullet_{I'J} \arrow[l] \arrow[r] & H^\bullet_J \\
H^\bullet_I \arrow[u] & H^\bullet_{IJ} \arrow[l] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

The map from $H^\bullet_I$ to $H^\bullet_J$ is $\nu_{I,J}$, and the map from $H^\bullet_J$ to $H^\bullet_{I'}$ is $\nu_{J,I'}$, so the map from $H^\bullet_I$ to $H^\bullet_{I'}$ is indeed $\eta_{I,I'}$. $\square$

Lemma 7.9. *Let $\langle g_i, A_i\rangle_{i\in I}$ and $\langle g_j, A_j\rangle_{j\in J}$ be two sequences, with the g_i and g_j elements of G, with the A_i and A_j small sets, and $I \cap J = \emptyset$. Suppose that $\langle g_i, A_i\rangle_{i\in I}$ is pleasant. Let B_I and B_J be the union of the underlying sets of $\langle g_i, A_i\rangle_{i\in I}$ and $\langle g_j, A_j\rangle_{j\in J}$, respectively. Suppose $B_I \mathop{\smile\!\!\!\!\!\!\;|}\, B_J$. Then the cohomology groups computed using the cover $\langle g_i \cdot S_{A_i}\rangle_{i\in I}$ agree with those computed using $\langle g_i \cdot S_{A_i}\rangle_{i\in I\cup J}$.*

Proof. Same as the proof of Lemma 7.7, with I and J in place of J and J'; we never needed to assume that $\langle g_j, A_j\rangle_{j\in J'}$ was pleasant. □

Lemma 7.10. *Let $\langle g_i, A_i\rangle_{i\in I}$ be an infinite sequence. Let I' be a cofinite subset of I such that $\langle g_i, A_i\rangle_{i\in I'}$ is pleasant. Suppose that each A_i is in the algebraic closure of a finite tuple. Then the cohomology groups computed using $\langle g_i \cdot S_{A_i}\rangle_{i\in I}$ agree with those computed using $\langle g_i \cdot S_{A_i}\rangle_{i\in I'}$.*

Proof. Each A_i has finite weight. From this, it is not hard to find an infinite subset $I'' \subset I'$ such that

$$\langle g_i, A_i\rangle_{i\in I''} \mathop{\smile\!\!\!\!\!\!\;|}\, \langle g_i, A_i\rangle_{i\in I\setminus I''} \tag{9}$$

Specifically, we just take I'' to be $I' \setminus C$, where $C \subset I'$ is a finite subset such that $\langle g_i, A_i\rangle_{i\in C}$ contains the canonical base of the strong type of $\langle g_i, A_i\rangle_{i\in I\setminus I'}$ over $\langle g_i, A_i\rangle_{i\in I'}$. Then (9) is implied by

$$\langle g_i, A_i\rangle_{i\in I\setminus I'} \mathop{\smile\!\!\!\!\!\!\;|}_{\langle g_i, A_i\rangle_{i\in C}} \langle g_i, A_i\rangle_{i\in I''} \text{ and } \langle g_i, A_i\rangle_{i\in C} \mathop{\smile\!\!\!\!\!\!\;|}\, \langle g_i, A_i\rangle_{i\in I''}$$

Now the cohomology groups computed using I'' agree with those using I' (by Lemma 7.7), and also with those using I' (by Lemma 7.9). □

Lemma 7.11. *Let $\langle g_i, A_i\rangle_{i\in I}$ be an infinite sequence, with each A_i of finite rank (i.e., in the algebraic closure of a finite tuple). Suppose that some cofinite subsequence of $\langle g_i, A_i\rangle_{i\in I}$ is pleasant. Suppose I is totally ordered, with a least element i_0. Let U_I and $C_I^\bullet$ denote the usual things. Then the total complex of*

$$0 \to C_{i_0}^\bullet \to \prod_{i_0<i} C_{i_0 i}^\bullet \to \prod_{i_0<i<j} C_{i_0 ij}^\bullet$$

is exact.

Proof. This measures the difference between the cohomology calculated with and without $g_{i_0} \cdot S_{A_{i_0}}$. □

Lemma 7.12. *Let $\langle g_i, A_i\rangle_{i\in I}$ be an infinite sequence. Let I' be an* infinite *subset of I such that $\langle g_i, A_i\rangle_{i\in I'}$ is pleasant. Suppose that each A_i is in the algebraic closure of a finite tuple. Then the cohomology groups computed using $\langle g_i \cdot S_{A_i}\rangle_{i\in I}$ agree with those computed using $\langle g_i \cdot S_{A_i}\rangle_{i\in I'}$.*

Proof. Same proof as Lemma 7.7, but using Lemma 7.11. □

Remark 7.13. *An analog of Theorem 7.8, proven the same way, is true, with "pleasant" replaced with "having an infinite pleasant subsequence, and A_i's of finite weight." In fact, tracing through the proofs, we only need the A_i's to have finite weight for i in the complement of the pleasant subsequence.*

8 The first cohomology group