Orthogonality to the value group is the same as generic stability in C-minimal expansions of ACVF

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1 Introduction

Let T be some C-minimal expansion of ACVF. Let $\mathbb{U}$ be the monster model of T. Let K be the home sort, k be the residue field, and Γ be the value group. The value group Γ of $\mathbb{U}$ is an o-minimal expansion of a divisible ordered abelian group. Let $\Gamma(A)$ denote $\mathrm{dcl}^{eq}(A) \cap \Gamma$ for any subset $A \subset \mathbb{U}^{eq}$.

Remark 1.1. *Let p be a global C-invariant type. The following are equivalent:*

- *For every function f into Γ (defined with parameters from $\mathbb{U}$), the pushforward f_*p is a constant type.*
- *For every $B \supseteq C$, we have $\Gamma(Ba) = \Gamma(B)$ for a realizing $p|B$.*

We say that p is orthogonal to Γ *if these conditions hold. In particular, from the first bullet point, this is a property of p, rather than the pair (p, C).*

Proof. Suppose the first condition holds. Let $B \supseteq C$ and let a be any realization of $p|B$. For $\gamma \in \Gamma(Ba)$, we can write γ as $f(a)$ for some B-definable function. Then $\gamma \models f_*p|B$. Also, p is B-invariant and f is B-definable, so the type f_*p is B-invariant. Since it is constant, it must contain the formula $x = \gamma_0$ for some γ_0, and γ_0 must be B-definable. Therefore the formula $x = \gamma_0$ is in $f_*p|B$, and so $\gamma = \gamma_0 \in \Gamma(B)$. As γ was an arbitrary element of $\Gamma(Ba)$, we conclude that $\Gamma(Ba) = \Gamma(B)$.

Conversely, suppose that the second condition holds. Let f be an $\mathbb{U}$-definable function into Γ. Let B be a set containing C, over which f is defined. Let a realize $p|B$. Then $f(a) \in \Gamma(Ba) = \Gamma(B)$. Since $f(a) \models f_*p|B$, and $f(a)$ is B-definable, the formula $x = f(a)$ must be in $f_*p|B$, so f_*p is a constant type. □

We want to show that a global invariant type p is orthogonal to Γ if and only if it is generically stable. (In particular, this means that types orthogonal to Γ are definable, and stationary.)

One direction is easy: if p is generically stable, and f is a definable function into Γ, then f_*p is a generically stable type in Γ. The Morley sequence of this type is totally indiscernible. But a totally indiscernible sequence in a totally ordered set must be constant. This ensures that f_*p is constant.

The other direction will take more work. We want to do this without discussing stable domination, since I don't know whether stable domination always works in the expansions of ACVF.

2 The hard direction

Lemma 2.1. *If $\langle a_i \rangle_{i \in I}$ is A-indiscernible for some small set A, and $\phi(x; y)$ is a formula over A such that $\phi(\mathbb{U}; a_i)$ is a finite non-empty set for any/every $i \in I$, then there is a sequence $\langle b_i \rangle_{i \in I}$ such that $\langle a_i b_i \rangle_{i \in I}$ is A-indiscernible and $\models \phi(b_i; a_i)$ for every i.*

Proof. For each i, choose some c_i such that $\phi(c_i; a_i)$ holds for every i. Let $\langle b_i' a_i' \rangle_{i \in I}$ be an A-indiscernible sequence of length I extracted from $\langle c_i a_i \rangle_{i \in I}$. Then $a_I \equiv_A a_I'$, and $\models \phi(b_i'; a_i')$ for every i. Let σ be an automorphism over A sending a_I' to a_I, and let b_I be the image of b_I' under σ. Then $\langle b_i a_i \rangle_{i \in I}$ is A-indiscernible, and for every i, $\models \phi(b_i; a_i)$. $\square$

Note that T is shatterproof (NIP), because it is C-minimal. Also, the swiss cheese decomposition still holds.

Lemma 2.2. *Let $\langle S_i \rangle_{i \in I}$ be an indiscernible sequence of subsets of K^1. Suppose that $S_i \subsetneq S_j$ for $i < j$. Let A be any set over which the S_i's are all defined. Then $|\Gamma(A)| \geq |I|$.*

Proof. Suppose not. For each i, let T_i be the finite set of radii of balls occurring in the canonical swiss cheese decomposition of S_i. By the previous lemma, we can choose a tuple t_i enumerating T_i, for each i, in such a way that $\langle t_i \rangle_{i \in I}$ is indiscernible. Since $\bigcup_i T_i \subset \Gamma(A)$, and $|\Gamma(A)| < |I|$, the set of t_i's must have size less than I. Therefore, the sequence $\langle t_i \rangle_{i \in I}$ is constant, and T_i does not depend on i. Write T for T_i.

Let T be $\{\gamma_1, \ldots, \gamma_n\}$. For $1 \leq j \leq n$, let E_j be the equivalence relation on K^1 defined by $xE_jy \iff \operatorname{val}(x - y) > \gamma_j$, and let E_j' be defined similarly using $\geq$ rather than $>$. Then $(K^1, E_1, E_1', E_2, E_2', \ldots, E_n, E_n')$ is a model of the model companion of the theory of a set with $2n$ nested equivalence relations. This theory is stable, hence NSOP. Also, the S_i's are uniformly definable in this model (each is a boolean combination of d equivalence classes, where d does not depend on i), so we get a contradiction (to NSOP). $\square$

Lemma 2.3. *Let p be a global C-invariant type that is orthogonal to Γ. Let $b_1, \ldots, b_n$ realize $p^{\otimes n}|C$. Let $\phi(x; y)$ be a C-formula with x a singleton in the home sort. Let σ be a permutation of $\{1, \ldots, n\}$. Then for every $a \in K^1$, there is $a' \in K^1$ such that for every i,*

$$\models \phi(a; b_i) \iff \phi(a'; b_{\sigma(i)})$$

Proof. We easily reduce to the case where σ is a permutation of two adjacent elements j and $j+1$. Let κ be a cardinal much larger than $|T|$ and $|C|$, and let I be a κ-saturated DLO extending the ordered set $\{1, \ldots, n\}$. Let $\langle b_i \rangle_{i \in I}$ be a Morley sequence in p over C of length I extending the given $b_1, \ldots, b_n$. By orthogonality to Γ, we know that $\Gamma(Cb_I) = \Gamma(C)$. In particular, $\Gamma(Cb_I)$ has cardinality less than κ.

Fix some $a \in K^1$. We want to find $a' \in K^1$ such that

$$\phi(a'; b_i) \iff \phi(a; b_i) \text{ for } i \in \{1, \ldots, j-1, j+2, \ldots, n\}$$

$$\phi(a'; b_j) \iff \phi(a; b_{j+1})$$

$$\phi(a'; b_{j+1}) \iff \phi(a; b_j).$$

If $\phi(a; b_{j+1}) \leftrightarrow \phi(a; b_j)$, then we can just take $a' = a$. So assume otherwise. Then exactly one of $\phi(a; b_j)$ and $\phi(a; b_{j+1})$ holds. Replacing ϕ with $\neg\phi$, we may assume that $\phi(a; b_j)$ holds and $\phi(a; b_{j+1})$ does not hold. Let $\psi(x)$ be the formula

$$\bigwedge_{i \in \{1, \ldots, j-1, j+2, \ldots, n\}} \phi(x; b_i) \leftrightarrow \phi(a; b_i);$$

this is a formula over $C \cup \{b_1, \ldots, b_{j-1}, b_{j+2}, \ldots, b_n\}$, in spite of appearances to the contrary. It suffices to show the consistency of

$$\psi(x) \wedge \phi(x; b_{j+1}) \wedge \neg\phi(x; b_j).$$

Suppose this does not hold. We are given the consistency of

$$\psi(x) \wedge \phi(x; b_j) \wedge \neg\phi(x; b_{j+1}),$$

since a satisfies this.

Let I' be the subset of I between $j-1$ and $j+2$. By κ-saturation of I, the cardinality of I' is at least κ. Moreover,

$$\langle b_i \rangle_{i \in I'}$$

is indiscernible over $B := C \cup \{b_1, \ldots, b_{j-1}, b_{j+2}, \ldots, b_n\}$. Let $\chi(x; y)$ be the B-formula $\psi(x) \wedge \phi(x; y)$. Then

$$\chi(x; b_j) \wedge \neg\chi(x; b_{j+1})$$

is consistent, and

$$\chi(x; b_{j+1}) \wedge \neg\chi(x; b_j)$$

is not. In other words,

$$\chi(K; b_{j+1}) \subsetneq \chi(K; b_j)$$

For $i \in I'$, let S_i be $\chi(K; b_i)$. Then by indiscernibility of $\langle b_i \rangle_{i \in I'}$ over B, it follows that $S_x \supsetneq S_y$ for any $x < y$ in I'. By Lemma 2.2, $|\Gamma(Bb_{I'})| \geq |I'| \geq \kappa$. But this is absurd, since $\Gamma(Bb_{I'}) = \Gamma(Cb_I)$ has size less than κ. So we have a contradiction. $\square$

Lemma 2.4. *Let p be a global C-invariant type that is orthogonal to Γ. Let $\langle b_i \rangle_{i \in I}$ be a Morley sequence for p over C. If $a \in K^1$ and if $\phi(a; y) \in p(y)$ for some C-formula $\phi(x; y)$, then $\phi(a; b_i)$ holds for all but at most n values of i, where $n < \omega$ depends only on $\phi(x; y)$.*

Proof. Let $c_1, c_2, \ldots$ be a Morley sequence for p over $Cb_I a$. Then $\phi(a; c_i)$ holds for every i, and $b_I c_1 c_2 \cdots$ is a Morley sequence for p over C. Replacing b_I with $b_I c_1 c_2 \cdots$, we may assume that $\phi(a; b_i)$ holds for infinitely many i.

Now suppose that $\phi(a; b_i)$ fails for more than n values of i, where n is the alternation number of $\phi(x; y)$, which exists because T is NIP. Then we can find $i_1 < i_2 < \cdots < i_{2n}$ such that $\phi(a; b_{i_j})$ holds for n values of j, and fails for n values of j. By Lemma 2.3, we can find a' such that $\phi(a'; b_{i_j})$ holds for even j and fails for odd j. Since $b_{i_1}, b_{i_2}, \ldots, b_{i_{2n}}$ is the beginning of a C-indiscernible sequence, this contradicts the choice of n. □

Lemma 2.5. *Let p be a global C-invariant type that is orthogonal to Γ. Let κ be a regular cardinal greater than $|C|$ and $|T|$. Let $\langle b_\alpha \rangle_{\alpha < \kappa}$ be a Morley sequence in p over C of length κ. Then for any $a \in K^1$, there is some $\lambda < \kappa$ such that $\langle b_\alpha \rangle_{\lambda \leq \alpha < \kappa}$ is a Morley sequence in p over Ca.*

Proof. Every power of p is orthogonal to Γ: if $B \supseteq C$ and $(a_1, a_2, \ldots, a_n)$ realizes $p^{\otimes n} | B$, then by orthogonality of p to Γ,

$$\Gamma(B) = \Gamma(Ba_1) = \cdots = \Gamma(Ba_1 a_2 \cdots a_n).$$

Of course each power of p is also a global C-invariant type.

Claim 2.6. *For each C-formula $\phi(x; y_1, \ldots, y_n)$, there is a $\lambda_\phi < \kappa$ such that for all*

$$\lambda_\phi \leq \alpha_1 < \cdots < \alpha_n < \kappa$$

we have

$$\phi(a; y_1, \ldots, y_n) \in p^{\otimes n} \iff \models \phi(a; b_{\alpha_1}, \ldots, b_{\alpha_n}).$$

Proof. Suppose no such λ_ϕ existed. Then for each $\lambda < \kappa$ we can find $\lambda < \alpha_1(\lambda) < \cdots < \alpha_n(\lambda) < \kappa$ such that

$$\phi(a; y_1, \ldots, y_n) \in p^{\otimes n} \not\Leftrightarrow \models \phi(a; b_{\alpha_1(\lambda)}, \ldots, b_{\alpha_n(\lambda)}).$$

Inductively build a sequence

$$\alpha_{1,0} < \cdots < \alpha_{n,0} < \alpha_{1,1} < \cdots < \alpha_{n,1} < \cdots$$

by letting $\alpha_{j,0}$ be $\alpha_j(0)$, and letting $\alpha_{j,k+1}$ be $\alpha_j(\alpha_{n,k})$. Let c_k be

$$c_k = (b_{\alpha_{1,k}}, \ldots, b_{\alpha_{n,k}})$$

Then $c_1, c_2, \ldots$ is a Morley sequence for $p^{\otimes n}$ over C. And for every k,

$$\phi(a; \vec{y}) \in p^{\otimes n} \not\Leftrightarrow \models \phi(a; c_k)$$

This contradicts Lemma 2.4 applied to $p^{\otimes n}$. □

Now let λ be the supremum of λ_ϕ for every ϕ. As κ was a regular cardinal bigger than $|C|$ and $|T|$, $\lambda < \kappa$. And now, for any

$$\lambda \leq \alpha_1 < \cdots < \alpha_n < \kappa,$$

and any C-formula $\phi(x; y_1, \ldots, y_n)$, we have

$$\phi(a; y_1, \ldots, y_n) \in p^{\otimes n} \iff \models \phi(a; b_{\alpha_1}, \ldots, b_{\alpha_n})$$

This means that $b_{\alpha_1} \cdots b_{\alpha_n}$ realizes $p^{\otimes n}|Ca$. So $\langle b_\alpha \rangle_{\lambda \leq \alpha < \kappa}$ is a Morley sequence for p over Ca. $\square$

Lemma 2.7. *Let p be a global C-invariant type that is orthogonal to Γ. Let κ be a regular cardinal greater than $|C|$ and $|T|$. Let $\langle b_\alpha \rangle_{\alpha<\kappa}$ be a Morley sequence in p over C of length κ. Then for any $a \in K^{eq}$, there is some $\lambda < \kappa$ such that $\langle b_\alpha \rangle_{\lambda \leq \alpha < \kappa}$ is a Morley sequence in p over Ca.*

Proof. The imaginary element a is in the definable closure of some real tuple. Replacing a with this real tuple, we may assume that $a = (a_1, \ldots, a_n)$, where each $a_i \in K^1$. By Lemma 2.5, there is some $\lambda_1 < \kappa$ such that after discarding the first λ_1 terms of the Morley sequence, the remainder is a Morley sequence over Ca_1. Now applying Lemma 2.5 to the resulting Morley sequence of the Ca_1-invariant type p, we find that there is some $\lambda_2 < \kappa$ such that after discarding the first λ_2 terms of the Morley sequence, the result will be a Morley sequence over Ca_1a_2. Continuing on in this fashion, we get the desired result. $\square$

Theorem 2.8. *Let p be a global C-invariant type that is orthogonal to Γ. Then p is generically stable.*

Proof. Suppose p is not generically stable. Let κ be a regular cardinal, bigger than $|T|$ and $|C|$. Let $\langle b_\alpha \rangle_{\alpha<2\kappa}$ be a Morley sequence of length $\kappa + \kappa$. Since p is not generically stable, C is not totally indiscernible. So there is some formula $\chi(y_1; y_2)$ such that $\chi(b_\alpha; b_\kappa)$ holds for $\alpha > \kappa$, and fails for $\alpha < \kappa$. By Lemma 2.7, there is some $\lambda < \kappa$ such that $\langle b_\alpha \rangle_{\lambda \leq \alpha < \kappa}$ is a Morley sequence for p over Cb_κ. But $\langle b_\alpha \rangle_{\kappa < \alpha \leq 2\kappa}$ is also a Morley sequence for p over Cb_κ, so in particular, b_λ and $b_{\kappa+1}$ should have the same type over Cb_κ. But

$$\phi(b_{\kappa+1}, b_\kappa) \text{ holds and } \phi(b_\lambda; b_\kappa) \text{ does not,}$$

a contradiction. $\square$