Computing the cohomology groups

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1 Intro

Let K be a field, and let T be ACF_K, the theory of algebraically closed fields extending K. Then ACF_K is strongly minimal, eliminates imaginaries, and has the property that every algebraically closed subset of the monster model is an elementary substructure (i.e., a model). Denote the monster by $\mathbb{M}$.

Let G be a connected algebraic group over K. Fix $i \geq 0$, and a finite group M with order prime to $\operatorname{char} K$. We are going to give a purely model-theoretic construction which should recover the ith etale cohomology group with coefficients in M of the variety G, base changed to $\overline{K}$ (or to the monster—these should be the same).

If A is a set of parameters, let $\operatorname{Gal}(A)$ denote the profinite group $\operatorname{Aut}(\operatorname{acl}(A)/\operatorname{dcl}(A))$. Because we eliminate finite imaginaries, the category of sets with continuous action of $\operatorname{Gal}(A)$ is equivalent to the category of finite A-definable sets.[1]

If L is a field, $\operatorname{Gal}(L)$ agrees with the usual Galois group $\operatorname{Aut}(L^{sep}/L)$. If $A \subset B$, there is a natural restriction map $\operatorname{Gal}(B) \to \operatorname{Gal}(A)$.

Let n be sufficiently big. Let $g_1, \ldots, g_n, t$ be a Morley sequence over $\emptyset$ (i.e., over K) in the generic type of G. For each non-empty $I \subseteq \{1, \ldots, n\}$, let

$$L_I = \operatorname{acl}\left(\{g_j \cdot g_i^{-1} : i, j \in I\}\right)$$

$$S_I = \{g \in G : g_i \cdot g \models p|L_I\}$$

where p is the generic type of G and the choice of $i \in I$ doesn't matter.

$$A_I = \operatorname{dcl}\left(\{g_i \cdot t\} \cup L_I\right)$$

where the choice of $i \in I$ doesn't matter.

$$\operatorname{Gal}_I = \operatorname{Gal}(A_I).$$

Note that for $I \subset J$, we have inclusions $L_I \subseteq L_J$, $A_I \subseteq A_J$. Consequently we get restriction maps $\operatorname{Gal}_J \to \operatorname{Gal}_I$. Also, $S_J \subset S_I$, and t is in every S_I.

[1] In particular, Grothendieck's Galois theory applies to the category of finite A-definable sets, and $\operatorname{Gal}(A)$ is the automorphism group of the forgetful functor from finite A-definable sets to finite sets.

For each I, we have a cochain complex

$$0 \to C_I^0 \to C_I^1 \to \cdots$$

coming from group cohomology of Gal_I with coefficients in M and continuous cochains. So C_I^i is the abelian group of continuous functions from $(\mathrm{Gal}_I)^i$ to M. (M was our coefficient group from earlier.) The restriction maps $\mathrm{Gal}_J \to \mathrm{Gal}_I$ induce maps on cochains in the other direction: $C_I^i \to C_J^i$.

Form the total complex of the following double complex

$$0 \to \prod_i C_i^\bullet \to \prod_{i<j} C_{ij}^\bullet \to \prod_{i<j<k} C_{ijk}^\bullet \to \cdots \to C_{\{1,2,\ldots,n\}}^\bullet$$

We are going to (hopefully) show that the ith cohomology group of this complex agrees with the ith etale cohomology group of the variety G, with coefficients in M, assuming n was sufficiently big relative to $\dim G$ and i.

This presumably means that we can (nearly) get the etale cochain complex of G as a homotopy limit of the group-cohomology cochain complexes of the Gal_I.

1.1 Motivation, part 1

One can roughly think of Gal_I as "$\pi_1(S_I)$" (with basepoint t) for the following reason. Say that a *finite cover* of S_I is a type-definable set C and a relatively definable map $\pi : C \to S_I$ such that

- C is type-definable over L_I
- the map $g \mapsto g_i \cdot \pi(g)$ (from C to realizations of the generic type of G over L_I) is relatively L_I-definable. Here, i is some element of I; the choice of i does not matter.
- The map π has finite fibers.

A *morphism* between two finite covers (C_1, π_1) and (C_2, π_2) is an L_I-definable map $f : C_1 \to C_2$ such that $\pi_2 \circ f = \pi_1$. We get a category $\mathcal{C}$ of finite covers of S_I.

If (C, π) is one of these finite covers, the fiber $\pi^{-1}(t)$ is a finite set, and we get a "fiber functor" from $\mathcal{C}$ to finite sets. Grothendieck's Galois theory applies: Gal_I is the automorphism group of this fiber functor and $\mathcal{C}$ is identified with the category of finite sets with continuous Gal_I-action.

Indeed, note that $\pi^{-1}(t)$ is a finite set definable over $\{g_i \cdot t\} \cup L_I$, so we really get a functor from $\mathcal{C}$ to the category of finite sets definable over $\{g_i \cdot t\} \cup L_I$. One checks that this is an equivalence of categories. It remains to recall that the category of $\{g_i \cdot t\} \cup L_I$-definable finite sets is equivalent to the category of finite sets with continuous Gal_I-actions.[2]

From the philosophy of Grothendieck's Galois theory (and from how Grothendieck defines the etale fundamental group), it therefore makes sense to think of Gal_I as the fundamental group of S_I, with basepoint at t.

[2]Everything we are saying here is a variant of the following general fact: Let T be a strongly minimal theory, A be some algebraically closed set of parameters, and p be a stationary type over A. Consider the

1.2 Motivation, part 2

Second of all, it makes sense to think of S_I as an etale $K(\pi_1(S_I), 1)$ (an Eilenberg-Maclane space). Roughly speaking, S_I is an intersection of a number of Zariski-open definable sets. Among these Zariski opens, the ones which are etale Eilenberg-Maclane spaces are cofinal...

Ignoring the translation by g_i, S_I is essentially the set of generics over L_I. So, it is the intersection of all L_I-definable dense opens in G. If U is any L_I-definable dense open in G, we can find a smaller L_I-definable open $V \subset U$ such that V is an Artin local neighborhood.

(Assuming I have the terminology right, an Artin local neighborhood is a hyperbolic curve, or a fibration over such a curve with fibers also being Artin local neighborhoods. From the homotopy long exact sequence of a fibration, it is clear that these are Eilenberg-Maclane spaces, at least in characterstic 0.)

In particular, if we think of S_I as an Eilenberg-Maclane space, then the group cohomology of $\pi_1(S_I) = \mathrm{Gal}_I$ agrees with the geometric cohomology of S_I as a "space." In fact, the cochain complexes are isomorphic in the derived category, morally.[3]

So, the main claim we are making is that we can get the etale cochains of G as a homotopy limit of the "etale cochains" of S_I.[4]

1.3 Motivation, part 3

So, we are roughly asserting that the etale cochains of G can be gotten (up to quasi-isomorphism) as the homotopy limit of the etale cochains of the S_I. Why should this hold?

All schemes will be over the moster $\mathbb{M}$. Recall that M is our group of coefficients. View it as a constant etale sheaf on G. Let $0 \to M \to I^0 \to I^1 \to \cdots$ be an injective resolution

category of type-definable sets over A with relatively A-definable maps. Within this, form the (comma) category of type-definable sets *over* the type-definable set of realizations of p. Within this, take the full subcategory $\mathcal{C}$ of covers with finite fibers. Then, for every $t \models p$, there is a fiber functor from $\mathcal{C}$ to finite tA-definable sets sending a cover (C, π) to the finite set $\pi^{-1}(t)$. This induces an equivalence of categories from $\mathcal{C}$ to finite tA-definable sets, and the category of finite tA-definable sets is in turn equivalent to the category of finite sets with continuous $\mathrm{Gal}(tA)$-action. So we are in the setting of Grothendieck's Galois theory.

[3]If S is a $K(G, 1)$, then the singular cochains of S and the group cohomology cochains of G (with coefficients in $\mathbb{Z}$) are isomorphic in the derived category. If S is a variety which is an etale $K(G, 1)$, where $G = \pi_1^{et}(S)$, and M is a coefficient group which is torsion and prime to the characterstic, then it is presumably also true that the etale cochains of S agree with the continuous cochains of G.

[4]There might be a way to make sense of the etale cochains of the type-definable set S_I. We can try and convert S_I into a scheme by taking the coordinate ring of G (or some affine open containing S) as a variety over $\mathbb{M}$, and then localizing by all the right terms to throw out the translates by g_i of the L_I-definable positive-codimension closed subschemes. Then take Spec of the resulting ring. I don't know whether the resulting object is particularly well-defined, though. If we ignored g_i and started with the coordinate ring over L_I rather than $\mathbb{M}$, this would just yield Spec of the fraction field of G over L_I. This is an etale Eilenberg-Maclane space, I believe, since etale cohomology of fields is the same as group cohomology of their absolute Galois groups. The absolute Galois group of this function field is indeed the same thing as Gal_I. Now, going back to the weird scheme over L_I, it should just be the pullback of this function field along $\operatorname{Spec} \mathbb{M} \to \operatorname{Spec} L_I$. This sort of base change does not usually change etale cohomology groups (with torsion coefficients prime to the characteristic), if I understand correctly.

of M. For each Zariski open $U \subset G$, we have the constant sheaf $\mathbb{Z}_U$ on U, which we can extend to a sheaf on G by extending by zero.

If I understand correctly, extension-by-zero is an exact functor which is the left adjoint to the restriction/pullback functor from etale sheaves on G to etale sheaves on U. It follows that the restriction/pullback functor sends injectives to injectives. In particular,

$$0 \to M|_U \to I^0|_U \to I^1|_U \to \cdots$$

is an injective resolution of the constant sheaf M on U.

Now, the etale cochains of G with coefficients in M are

$$0 \to \Gamma(G, I^0) \to \Gamma(G, I^1) \to \cdots$$

or equivalently,

$$0 \to \mathrm{Hom}(\mathbb{Z}_G, I^0) \to \mathrm{Hom}(\mathbb{Z}_G, I^1) \to \cdots$$

where $\mathbb{Z}_G$ is the constant sheaf $\mathbb{Z}$ on G.

Similarly, the etale cochains of U with coefficients in M will be

$$0 \to \Gamma(U, I^0|_U) \to \Gamma(U, I^1|_U) \to \cdots$$

or equivalently

$$0 \to \mathrm{Hom}(\mathbb{Z}_U, I^0) \to \mathrm{Hom}(\mathbb{Z}_U, I^1) \to \cdots,$$

where the Hom's are in sheaves on G, and $\mathbb{Z}_U$ is the constant sheaf $\mathbb{Z}$ on U, extended by zero to G.

Theorem 1.1. *Suppose we are given a map $I \mapsto U_I$ from non-empty subsets of $\{1, \ldots, n\}$ to Zariski open subsets of G, and suppose that $U_I \supset U_J$ for $I \subset J$. For every point $p \in G$, let $X_{U,p}$ be the abstract simplicial complex with*

- *Vertices the $1 \le i \le n$ such that $p \in U_i$*
- *Faces the $I \subset \{1, \ldots, n\}$ such that $p \in U_I$*

Fix $k > 0$, and suppose that for every $i \le k+1$ and $p \in G$, the ith homology group of $X_{U,p}$ is the same as that of a point, i.e., $\mathbb{Z}$ for $i = 0$, and 0 for $i > 0$. Then the kth etale cohomology group $H^k(G_{et}, M)$ is isomorphic to the kth cohomology group of the totalization of the double complex

$$0 \to \prod_i \mathrm{Hom}(\mathbb{Z}_{U_i}, I^\bullet) \to \prod_{i<j} \mathrm{Hom}(\mathbb{Z}_{U_{ij}}, I^\bullet) \to \cdots \to \mathrm{Hom}(\mathbb{Z}_{U_{1,2,\ldots,n}}, I^\bullet) \to 0 \tag{1}$$

Proof. Consider the double complex

$$0 \to \mathrm{Hom}(\mathbb{Z}_G, I^\bullet) \to \prod_i \mathrm{Hom}(\mathbb{Z}_{U_i}, I^\bullet) \to \prod_{i<j} \mathrm{Hom}(\mathbb{Z}_{U_{ij}}, I^\bullet) \to \cdots \tag{2}$$

This double complex yields a map ψ of chain complexes from its first column $\mathrm{Hom}(\mathbb{Z}_G, I^\bullet)$ to the totalization of the remaining columns, i.e., to the totalization of (1). We claim that ψ induces an isomorphism on cohomology groups up to the kth one. To prove this, it suffices to show that the first k or $k+1$ homology groups of the mapping cone of ψ vanish.[5] But the mapping cone of ψ is just the totalization of (2).

To show that the first k (co)homology groups of the totalization of (2) vanish, it suffices to show that for each *row*, the first k cohomology groups vanish. Now the rth row is obtained by applying the exact contravariant functor $\mathrm{Hom}(-, I^r)$ to the following chain complex of sheaves on G:

$$0 \leftarrow \mathbb{Z}_G \leftarrow \bigoplus_i \mathbb{Z}_{U_i} \leftarrow \bigoplus_{i<j} \mathbb{Z}_{U_{ij}} \leftarrow \cdots . \tag{3}$$

So it suffices to show that (3) is an exact sequence of sheaves, up to the kth degree. This can be checked on stalks, at geometric points. If p is a geometric point, the stalk at p of (3) is just the chain complex which yields reduced simplicial homology of the abstract simplicial complex $X_{U,p}$. By assumption, this reduced homology vanishes.[6] □

For any $p \in G = G(\mathbb{M})$, let X_p be the abstract simplicial complex whose vertices and faces are the non-empty subsets $I \subseteq \{1, \ldots, n\}$ such that $p \in S_I$. This forms an abstract simplicial complex. We will see later (Theorem 2.1) that, if n was chosen sufficiently big relative to i, the first i homology groups of X_p agree with the homology groups of a point, for all p.

Now, pretend that the S_I's are Zariski open sets, rather than type-definable sets. By the theorem, the ith etale cohomology group of G agrees with the ith cohomology group of the homotopy limit of the etale cochain complexes of the S_I's. Since we are pretending that the S_I's are $K(\mathrm{Gal}_I, 1)$'s, and the cochain complex of a $K(G, 1)$ is quasi-isomorphic to the usual group cohomology cochain complex, it should be plausible that we could instead take the homotopy limit of the group cohomology cochain complexes of the Gal_I's.

2 Pointwise Triviality

Theorem 2.1. *Fix $k > 0$. Suppose that n is [sufficiently big]. For each $p \in G$, let X_p be the abstract simplicial complex with*

- *vertices the $i \in \{1, \ldots, n\}$ such that $p \in S_i$*
- *faces the non-empty $I \subseteq \{1, \ldots, n\}$ such that $p \in S_I$.*

[5] We are using the fact that if $K^\bullet$ is the mapping cone of a map of complexes $C^\bullet \to D^\bullet$, then the homology groups of C, D, and K are related by a long exact sequence.

[6] We are sweeping under the rug the fact that the geometric points here not only include the elements of $G(\mathbb{M})$, but also the analogs of scheme-theoretic points, which correspond to elements from elementary extensions of $\mathbb{M}$. The set of p for which $X_{U,p}$ has vanishing reduced homology groups in the right degrees is a definable set, definable over $\mathbb{M}$, because it is a boolean combination of the U_I's. Since every element of $G(\mathbb{M})$ belongs to it, this remains true for elementary extensions of $\mathbb{M}$.

(As $S_I \supset S_J$ for $I \subset J$, this is a well-defined abstract simplicial complex.)

Fix $k \geq 0$. Suppose that $n > k + 2dim(G) + tr.deg(p/\emptyset) - tr.deg(p/g_1, \ldots, g_n)$. Then $H_i(X_p)$ agrees with $H_i(pt)$ for $i \leq k$ and all p.

In particular, if n is sufficiently big (at least $k + 3dim(G)$), then the first $k+1$ reduced homology groups of X_p vanish for all p.

We first prove a slightly simpler variant.

Theorem 2.2. *Let T be a strongly minimal theory; let $R()$ denote rank. Let A be some set of parameters and p be a stationary complete type over A. Let $g_1, \ldots, g_n$ be an independent sequence of tuples (not necessarily realizing p, or the same length), independent over A. Let x realize $p|A$. Let Σ be the abstract simplicial complex on $\{1, \ldots, n\}$ which has I as a face if and only if x realizes the non-forking extension of p to $A \cup \{g_i : i \in I\}$. Equivalently, I is a face if*

$$x \underset{A}{\perp} \{g_i : i \in I\}.$$

Fix kge0. Assuming that $n > k + 2R(x/A) - R(x/Ag_1 \cdots g_n)$, the first $k+1$ homology groups of Σ agree with those of a point.

Proof. We proceed by induction on $k + R(x/A) - R(x/Ag_1 \cdots g_n)$. The base case where this number is negative is trivial, because then pigs fly. For $I \subset \{1, \ldots, n\}$, we let g_I denote $\{g_i : i \in I\}$.

Since $n > R(x/A) = wt(x/A)$, x cannot fork with every g_i. Reordering the g_i, we may assume that

$$g_1 \underset{A}{\perp} x \tag{4}$$

In particular, there is at least one vertex in Σ.

Let Γ be the subcomplex of Σ supported on $\{2, \ldots, n\}$. Let Ψ be the subcomplex of Γ consisting of those I such that $I \cup \{1\}$ is a face of Γ. Note that Σ is the mapping cone of the inclusion $\Psi \hookrightarrow \Gamma$. In particular, we get a long exact sequence of reduced homology groups:

$$\cdots \to \tilde{H}_i(\Psi) \to \tilde{H}_i(\Gamma) \to \tilde{H}_i(\Sigma) \to \tilde{H}_{i-1}(\Psi) \to \cdots$$

To show that $\tilde{H}_i(\Sigma)$ vanishes for $i \leq k$, is suffices to prove one of the following two conditions:

- $\Psi = \Gamma$, in which case *all* reduced homology groups of Σ vanish
- $\tilde{H}_i(\Gamma)$ vanishes for $i \leq k$ and $\tilde{H}_i(\Psi)$ vanishes for $i \leq k-1$.

Note that I is a face of Ψ if and only if $x \perp_A g_I g_1$. But, by (4), this is equivalent to $x \perp_{Ag_1} g_I$. Now $g_2, g_3, \ldots, g_n$ is certainly an independent sequence over Ag_1, and x realizes $p|Ag_1$, and

$$n - 1 > (k-1) + 2R(x/Ag_1) - R(x/Ag_1 \cdots g_n)$$

and

$$(k-1) + R(x/Ag_1) - R(x/Ag_1g_2 \cdots g_n) < k + R(x/A) - R(x/Ag_1g_2 \cdots g_n),$$

all because $R(x/A) = R(x/Ag_1)$, by (4). So the inductive hypothesis applies to Ψ, and $\tilde{H}_i(\Psi)$ vanishes for $i \le k-1$. So we need to show that $\Psi = \Gamma$ or that the first $k+1$ reduced homology groups of Γ vanish.

Case 1: Suppose

$$g_1 \underset{A}{\downarrow} g_2 g_3 \cdots g_n x.$$

Then for any $I \subset \{2, \ldots, n\}$, we have

$$x g_I \underset{A}{\downarrow} g_1, \text{ so } x \underset{Ag_I}{\downarrow} g_1.$$

Therefore

$$x \underset{A}{\downarrow} g_I g_1 \iff x \underset{A}{\downarrow} g_I.$$

It follows that $\Psi = \Gamma$.

Case 2: Suppose that

$$g_1 \underset{A}{\not\downarrow} g_2 g_3 \cdots g_n x.$$

Then

$$R(x/Ag_1 g_2 \cdots g_n) < R(x/Ag_2 g_3 \cdots g_n),$$

so

$$R(x/A) - R(x/Ag_2 \cdots g_n) + k < R(x/A) - R(x/Ag_1 \cdots g_n) + k.$$

Also,

$$n - 1 \ge k + 2R(x/A) - R(x/Ag_1 \cdots g_n) > k + 2R(x/A) - R(x/Ag_2 \cdots g_n)$$

Therefore, we can apply the inductive hypothesis to A and $g_2, \ldots, g_n$, concluding that the first $k+1$ reduced homology groups of Γ vanish. $\square$

Next, we prove Theorem 2.1

Proof. Proceed by induction on $R(p/\emptyset) - R(p/g_1 \cdots g_n)$.

Because $n > dim(G) \ge wt(p/\emptyset)$, the point p does not fork with every g_i. Reordering the g_i, we may assume that $p \downarrow g_1$. Let x be $g_1 \cdot p$. Let $h_2, \ldots, h_n$ denote $g_2 \cdot g_1^{-1}, \cdots g_n \cdot g_1^{-1}$.

As before, let Γ be the subcomplex of X_p supported on $\{2, \ldots, n\}$, and let Ψ be the subcomplex of Γ consisting of I such that $\{1\} \cup I$ is a face of X_p. As before, we have a long exact sequence of reduced homology groups.

Note that the following are equivalent:

- I is a face of ψ
- $p \in S_{I \cup \{1\}}$
- $g_1 \cdot p = x$ is generic over $\{h_i : i \in I\}$.

- $x \downarrow h_I$

Since $g_1 \downarrow x$, the element $g_1 \cdot x$ is generic over x, hence over $\emptyset$. So it realizes a stationary type over $\emptyset$. Also, the sequence $h_2, \ldots, h_n$ is an independent sequence over $\emptyset$. Moreover,

$$n - 1 > k - 1 + 2\dim(G) \geq k - 1 + 2R(x/\emptyset) - R(x/h_2 \cdots h_n).$$

so the first k reduced homology groups of Ψ vanish by Theorem 2.2.

It remains to show that $\Psi = \Gamma$ or that the first $k+1$ reduced homology groups of Γ vanish. As before, we break into two cases:

Case 1: Suppose

$$g_1 \downarrow g_2 g_3 \cdots g_n p$$

We claim that $\Psi = \Gamma$. Indeed, let I be a face of Γ. Let i_0 be an element of I. The fact that I is a face of Γ means that

$$g_{i_0} \cdot p \downarrow \{g_i \cdot g_{i_0}^{-1} : i \in I\}. \tag{5}$$

Now g_1 is generic over pg_I, and thus so is $g_1 \cdot g_{i_0}^{-1}$. In particular,

$$g_1 \cdot g_{i_0}^{-1} \downarrow pg_I.$$

Therefore, (5) implies

$$g_{i_0} \cdot p \downarrow \{g_i \cdot g_{i_0}^{-1} : i \in I\} \cup \{g_1 \cdot g_{i_0}^{-1}\},$$

so $I \cup \{1\}$ is a face of X_p.

Case 2: Suppose

$$g_1 \not\downarrow g_2 g_3 \cdots g_n p$$

Then $R(p/g_2 g_3 \cdots g_n) > R(p/g_1 g_2 \cdots g_n)$. In particular

$$n - 1 \geq k + 2\dim(G) + R(p/\emptyset) - R(p/g_1 \cdots g_n) > k + 2\dim(G) + R(p/\emptyset) - R(p/g_2 \cdots g_n).$$

Also

$$R(p/\emptyset) - R(p/g_2 \cdots g_n) < R(p/\emptyset) - R(p/g_1 \cdots g_n),$$

so by induction, the first $k+1$ reduced homology groups of Γ vanish. $\square$

3 Approximating type-definable sets with opens

Hold k fixed. Assume $n > k + 1 + 3\dim(G)$. We would like to show that the recipe of §1 for computing the kth etale cohomology group of G, with coefficients in M, is correct.

Let $\vec{g}$ denote our fixed sequence of n inependent realizations $g_1, g_2, \ldots, g_n$ of the generic type of G. Let $L = \operatorname{acl}(\vec{g})$. Recall that

$$L_I = \operatorname{acl}(\{g_i \cdot g_j^{-1} : i, j \in I\}),$$

so that $L_I \leq L$ for every I.

Definition 3.1. *A* $\vec{g}$-system *is a map* $I \mapsto U_I$ *from non-empty subsets of* $\{1, \ldots, n\}$ *to non-empty Zariski open subsets of* G *(viewed as a scheme over* $\mathbb{M}$*), subject to the following conditions:*

- *If* $I \subset J$*, then* $U_I \supseteq U_J$*.*
- *For any/every* $i \in I$*, the translate* $g_i \cdot U_I$ *is defined over* $L_I = \operatorname{acl}(g_j \cdot g_i^{-1} : j \in I)$*.*

The set of $\vec{g}$-systems forms a poset, with $U \leq U'$ if $U_I \supseteq U'_I$ for every I. This poset is directed, because $I \mapsto U_I \cap V_I$ is a $\vec{g}$-system if U and V are.

Definition 3.2. *If* U *is a* $\vec{g}$*-system, and* $p \in G(\mathbb{M})$*, let* $X_{U,p}$ *be the abstract simplicial complex from Theorem 1.1. The* bad locus *bad*(U) *of* U *is the set of points* p *for which the first* $k+1$ *reduced homology groups of* $X_{U,p}$ *don't all vanish.*

Being a boolean combination of U's, the set *bad*(U) is always L-constructible.

Recall the simplicial complex X_p from Theorem 2.1, for $p \in G$.

Lemma 3.3. *Let* U *be a* $\vec{g}$*-system. Let* p *be a complete type over* L*. Then there is some* $U' \geq U$ *such that bad*(U') *does not contain the realizations of* p*.*

Proof. Let x realize p. For each $I \subset \{1, \ldots, n\}$, let $g_i \cdot C_I$ be the smallest L_I-definable Zariski closed set containing $g_i \cdot x$, where $i \in I$. One checks that the choice of I does not matter, and also that $C_J \subseteq C_I$ for $I \subseteq J$.

Note that $C_I = G$ exactly when $g_i \cdot x$ is generic over L_I, or equivalently, when $x \in S_I$, or equivalently, when $I \in X_x$. For each I, let

$$W_I = \bigcup_{J \subseteq I,\ J \notin X_x} C_J.$$

Then $W_I \subseteq W_J$ for $I \subseteq J$, and W_I is not all of G. If $i \in I$ and $J \subseteq I$, then $g_j \cdot C_J$ is L_J-definable, so $g_j \cdot C_J$ is L_I-definable, so $g_i \cdot C_J$ is L_I-definable. Therefore $g_i \cdot W_I$ is L_I-definable.

Let $U'_I = U_I \setminus W_I$. This is a $\vec{g}$-system:

- If $I \subset J$, then $U_I \supseteq U_J$ and $W_I \subseteq W_J$, so $U'_I \supseteq U'_J$.
- For any I, U_I is Zariski dense in G, and W_I is Zariski closed and not all of G, so U'_I is non-empty, because G is irreducible.
- For any I and $i \in I$, $g_i \cdot U'_I$ is L_I-definable because $g_i \cdot U_I$ and $g_i \cdot W_I$ are.

We claim that *bad*(U') does not contain x. Since *bad*(U') is L-constructible, this ensures that all the realizations of p are not in *bad*(U').

To see this, note that the first k or so reduced homology groups of X_x vanish (by Theorem 2.1), so it suffices to show that $X_{U',x} = X_x$. Note that

$$I \in X_{U',x} \iff x \in U'_I \iff x \in U_I \wedge \bigwedge_{J \subseteq I,\ J \notin X_x} \perp.$$

So $I \in X_{U',x}$ if and only if $x \in U_I$ and for every $J \subseteq I$, $J \in X_x$. Given that X_x is an abstract simplicial complex, this last clause is equivalent to $I \in X_x$. Moreover,

$$I \in X_x \iff x \in S_I \implies x \in U_I,$$

so we conclude $I \in X_{U',x} \iff I \in X_x$, for arbitrary I. □

Lemma 3.4. *Let $\mathcal{F}$ be a sheaf from a Godement resolution (more precisely, an infinite product of skyscraper sheaves). Then the first k cohomology groups of the following sequence vanish:*

$$0 \to \Gamma(G, \mathcal{F}) \to \lim_{U} \prod_{i} \Gamma(U_i, \mathcal{F}) \to \lim_{U} \prod_{i<j} \Gamma(U_{ij}, \mathcal{F}) \to \cdots \tag{6}$$

where the limits are direct limits over the poset of $\vec{g}$-systems.

Proof. As an infinite product of skyscraper sheaves, there is basically a map from (geometric?) points of G to abelian groups

$$p \mapsto M_p$$

such that for any $U \subset G$,

$$\Gamma(U, \mathcal{F}) = \prod_{p \in U} M_p$$

Alternatively, we can write

$$\Gamma(U, \mathcal{F}) = \prod_{p \in G} M_{U,p},$$

where $M_{U,p}$ is M_p if $p \in U$, and 0 otherwise.

Now (6) is the direct limit over $\vec{g}$-systems U, of the product over points $p \in G$, of the sequence

$$0 \to M_p \to \prod_{i} M_{U_i,p} \to \prod_{i<j} M_{U_{ij},p} \to \cdots \tag{7}$$

In the category of abelian groups, direct limits and infinite products preserve exact sequences, so the ith cohomology group of (6) is the direct limit of the product over p of the ith cohomology group of (7). Now (7) is just the reduced simplicial cochains of $X_{U,p}$ with coefficieints in M_p, so, the ith cohomology group of (6) is nothing but

$$\lim_{U} \prod_{p \in G} \tilde{H}^i(X_{U,p}, M_p). \tag{8}$$

So: we need to show that (8) vanishes. Let c be a non-zero element of $\prod_{p \in G} \tilde{H}^i(X_{U,p}, M_p)$ for some U. We need to find $U' \geq U$ such that $c|_{U'}$ vanishes. It suffices to show that we can find some $U' \geq U$ such that

$$\overline{bad(c|_{U'})} \subsetneq \overline{bad(c)}, \tag{9}$$

where the overlines denote Zariski closure over L, and where $bad(c)$ denotes the set of p such that $c_p \in \tilde{H}^i(X_{U,p}, M_p)$ does not vanish. Because if we can get (9), then we can iterate and use Noetherian induction.

Let V be one of the irreducible components of $\overline{bad(c)}$. By Lemma 3.3, we can find some $U' \geq U$ such that $bad(U')$ does not contain the generic type of V. Note that

$$bad(c|_{U'}) \subseteq bad(c) \cap bad(U'). \tag{10}$$

Indeed, for any point p, $(c|_{U'})_p$ is the image of c_p under the natural map $\tilde{H}^i(X_{U,p}, M_p) \to \tilde{H}^i(X_{U',p}, M_p)$ induced by the inclusion $X_{U',p} \subset X_{U,p}$, so if c_p vanishes, so does $(c|_{U'})_p$. And if $X_{U',p}$ has no reduced homology up to level $k+1$, then its $\leq k$th reduced cohomology with coefficients in M_p vanish, by universal coefficients or something.

Let W be the union of the irreducible components of $\overline{bad(c)}$ other than V (possibly $W = \emptyset$). Then

$$bad(U') \cap \overline{bad(c)} = (bad(U') \cap W) \cup (bad(U') \cap V) \subseteq W \cup (bad(U') \cap V).$$

Because closure preserves unions,

$$\overline{bad(U') \cap \overline{bad(c)}} \subseteq W \cup \overline{bad(U') \cap V}.$$

Now $bad(U') \cap V$ is an L-constructible set which is not Zariski dense in V, so

$$\overline{bad(U') \cap V} \subsetneq V$$

$$W \cup \overline{bad(U') \cap V} \subsetneq W \cup V = \overline{bad(c)}.$$

Meanwhile, by (10),

$$\overline{bad(c|_{U'})} \subseteq \overline{bad(U') \cap \overline{bad(c)}}.$$

So (9) holds. $\square$

Lemma 3.5. *Let $M \to I^0 \to I^1 \to \cdots$ be an injective resolution of the constant sheaf M on G. For each $\vec{g}$-system U, let $C_U^\bullet$ be the totalization of the double complex*

$$0 \to \prod_i \Gamma(U_i, I^\bullet) \to \prod_{i<j} \Gamma(U_{ij}, I^\bullet) \to \cdots$$

If $U \leq U'$, we get a map of complexes $C_U^\bullet \to C_{U'}^\bullet$. Then: the first k cohomology groups of $\lim_U C_U^\bullet$ *agree with the cohomology groups $H^i(G, M)$.*

Proof. The functor sending an injective resolution $I^\bullet$ to $C_U^\bullet$ preserves chain homotopies, I'm pretty sure, so the choice of the injective resolution doesn't matter (any two injective resolutions of M are chain homotopy equivalent). So we may assume that $I^\bullet$ is a Godement resolution.

Consider the double complex

$$0 \to \Gamma(G, I^\bullet) \to \lim_U \prod_i \Gamma(U_i, I^\bullet) \to \lim_U \prod_{i<j} \Gamma(U_{ij}, I^\bullet) \to \cdots$$

Each row of this complex is exact, up to the kth column, by Lemma 3.4. So the totalization of this complex is exact up to the kth column or so. But this totalization is the mapping cone of the natural map from $\Gamma(G, I^\bullet)$ to $\lim_U C_U^\bullet$. $\square$

4 Connection to Group Cohomology

Recall that if K is a field, the category of etale sheaves (of sets) on $\operatorname{Spec} K$ is equivalent to the category of sets with continuous action of the profinite group $\operatorname{Gal}(K)$. By a $\operatorname{Gal}(K)$-module, we will always mean an abelian group with a continuous linear action of $\operatorname{Gal}(K)$, or equivalently, a sheaf of abelian groups on $\operatorname{Spec} K$. Note that any submodule of a finitely generated $\operatorname{Gal}(K)$-module is finitely generated.

Given any morphism f of sites, there are pushforward and pullback functors f_* and f^* on the sheaves of sets, as well as on the sheaves of abelian groups. In both cases, f^* is left adjoint to f_*, and f^* is exact (preserves finite limits).[7] In the abelian case, f_* preserves injective objects because of its exact left adjoint.

Lemma 4.1. *Let V be an irreducible variety over an algebraically closed field K. Then the pullback functor from (etale) sheaves on V to (etale) sheaves on $\operatorname{Spec} K(V)$ sends injectives to injectives.*

Proof. Let $\mathcal{I}$ be an injective sheaf on V. Let $\mathfrak{C}$ be the class of injections $M \to N$ of $\operatorname{Gal} K(V)$ modules such that

$$\operatorname{Hom}(N, f^*\mathcal{I}) \to \operatorname{Hom}(M, f^*\mathcal{I})$$

is surjective. We want to show that $\mathfrak{C}$ contains all injections. Because $\mathfrak{C}$ is closed under pushouts and transfinite compositions, it suffices to consider injections $M \to N$ with N (and hence M) finitely generated.[8]

If U is any open subvariety, the pullback i^* along the inclusion $i : U \to V$ has an exact left adjoint $i_!$, extension by zero. It follows that i^* (restriction) preserves injectives. So we may safely replace V with an open.

Because M and N are finitely generated, there is some finite quotient Q of $\operatorname{Gal} K(V)$ through which the action of $\operatorname{Gal} K(V)$ on N factors. Then

$$\pi_1^{et}(\operatorname{Spec} K(V)) = \operatorname{Gal} K(V) \to Q$$

factors through $\pi_1^{et}(\operatorname{Spec} K(V)) \to \pi_1^{et}(U)$ for some Zariski open U. It follows that M and N (and the injection between them) can be extended to locally constant sheaves on U. Replacing V with U, we may assume that $M \hookrightarrow N$ comes from an inclusion of locally constant sheaves $\mathcal{M} \hookrightarrow \mathcal{N}$ on V, by pulling back to $\operatorname{Spec} K(V)$.

Now, because $\mathcal{M}$ and $\mathcal{N}$ are locally constant with finitely generated stalks, it turns out[9] that

$$\operatorname{Hom}(M, f^*\mathcal{I}) = \varinjlim_U \operatorname{Hom}(\mathcal{M}|_U, \mathcal{I}|_U),$$

where the limit is a direct limit over Zariski opens, and similarly for N. Since $\mathcal{M}|_U \to \mathcal{N}|_U$ is injective and the object $\mathcal{I}|_U$ is an injection, it follows that

$$\operatorname{Hom}(\mathcal{N}|_U, \mathcal{I}|_U) \to \operatorname{Hom}(\mathcal{M}|_U, \mathcal{I}|_U)$$

[7] Todo: check the claims I just made.

[8] I'm basically just quoting the proof of the Baer criterion for injectivity, or the small object argument.

[9] I think? This argument is much less straightforward than I would have expected.

is a surjection. Taking the direct limit, it follows that

$$\mathrm{Hom}(N, f^*\mathcal{I}) \to \mathrm{Hom}(M, f^*\mathcal{I})$$

is surjective, which was what we wanted to show. $\square$

Given a $\vec{g}$-system U, we can view each U_I as a scheme over L_I, and find scheme-theoretic maps $U_J \to U_I$ for $I \subset J$, in such a way that

$$\begin{array}{ccc} U_J & \longrightarrow & \mathrm{Spec}\, L_J \\ \downarrow & & \downarrow \\ U_I & \longrightarrow & \mathrm{Spec}\, L_I \end{array}$$

commutes, and base change to L turns the system of U_I and maps between them into the system of open subvarieties of G that we had previously.

(In the case where the U_I are affine, we do the following: for each I let B_I be the ring of polynomial functions $P : U_I \to \mathbb{A}^1$ such that the map

$$g_i \cdot x \to P(x)$$

is an L_I-definable map from $g_i \cdot U_I$ to $\mathbb{A}^1$, for $i \in I$. Whether or not P has this property does not depend on the choice of i. Each B_I is an L_I-algebra, and for $I \subset J$ we have maps $B_I \to B_J$ compatible with $L_I \to L_J$. Now take Spec of everything.

The reason for all this confusion is that U_I is isomorphic to an L_I-definable set, but is not embedded in G in an L_I-definable way.)

Henceforth we will view $\vec{g}$-systems in this way.

Confusingly, we can also do this to the trivial $\vec{g}$-system $G_I = G$. Let G_I denote the scheme gotten in this way–it is canonically a torsor for the group scheme G base changed from K to L. Let $(U_I)_L$ and $(G_I)_L$ denote the base changes of U_I and G_I to L. Then $(G_I)_L$ is canonically G base-changed to L, and $(U_I)_L$ is an open subscheme of $(G_I)_L$.

Fix some injective resolution $M \to \mathcal{I}^\bullet$ of M in the category of (etale) sheaves on G_L (= G base-changed to L.) So far, we have related the etale cohomology of G to the cohomology groups of the total complex of the double complex

$$0 \to \lim_U \prod_i \Gamma(U_i, \mathcal{I}^\bullet) \to \lim_U \prod_{i<j} \Gamma(U_{ij}, \mathcal{I}^\bullet) \to \cdots . \tag{11}$$

Let K_I be the fraction field of U_I; so $\mathrm{Spec}\, K_I$ is essentially S_I. Equivalently, K_I is the field generated by $g_i \cdot t$ and L_I. (The element t was chosen a long time ago to be generic over all the $g_1, \ldots, g_n$. We may as well take it generic over L.) If g_I is the map $(G_I)_L = G_L \to G_I$, then (11) is isomorphic to

$$0 \to \lim_U \prod_i \Gamma(U_i, g_{i,*}\mathcal{I}^\bullet) \to \lim_U \prod_{i<j} \Gamma(U_{ij}, g_{ij,*}\mathcal{I}^\bullet) \to \cdots \tag{12}$$

If f_I denotes the inclusion of $\operatorname{Spec} K_I$ into G_I, then

$$\lim_{U} \Gamma(U_I, \mathcal{F}) = \Gamma(\operatorname{Spec} K_I, f_I^* \mathcal{F})$$

for any sheaf $\mathcal{F}$ on G_I. In particular, (12) is isomorphic to

$$0 \to \prod_i \Gamma(\operatorname{Spec} K_i, f_i^* g_{i,*} \mathcal{I}^\bullet) \to \prod_{i<j} \Gamma(\operatorname{Spec} K_{ij}, f_{ij}^* g_{ij,*} \mathcal{I}^\bullet) \to \cdots \tag{13}$$

Let $\mathcal{F}_I^\bullet$ denote $f_I^* g_{I,*} \mathcal{I}^\bullet$. Then $\mathcal{F}_I^i$ is an injective $\operatorname{Gal}(K_I)$-module, by Lemma 4.1 and the fact that pushforwards preserve injections. However, $g_{I,*}$ needn't be an exact functor, so we don't know that the chain complex

$$0 \to f_I^* g_{I,*} M = M \to \mathcal{F}_I^0 \to \mathcal{F}_I^1 \to \cdots \tag{14}$$

is exact.

At any rate, we have converted our chain complex to the (totalization of the) following:

$$0 \to \prod_i \Gamma(\operatorname{Spec} K_i, \mathcal{F}_i^\bullet) \to \prod_{i<j} \Gamma(\operatorname{Spec} K_{ij}, \mathcal{F}_{ij}^\bullet) \to \cdots \tag{15}$$

Also, recall that $\operatorname{Gal}(K_I)$ is what we were calling Gal_I earlier. Let $C_I^i(-)$ denote the ith continuous cochains functor from Gal_I-modules to abelian groups. This functor is exact. For any Gal_I-module N, there is a chain complex

$$0 \to N^{\operatorname{Gal}_I} \to C_I^0(N) \to C_I^1(N) \to \cdots \tag{16}$$

which is exact if N is injective. Note that the fixed-points functor $(-)^{\operatorname{Gal}_I}$ is essentially the same thing as the global sections functor on etale sheaves over $\operatorname{Spec} K_I$.

The totalization of the double complex (15) maps to the totalization of the *triple* complex

$$0 \to \prod_i C_i^\bullet(\mathcal{F}_i^\bullet) \to \prod_{i<j} C_{ij}^\bullet(\mathcal{F}_{ij}^\bullet) \to \cdots \tag{17}$$

because of (16). In fact, this is a quasi-isomorphism, because the $\mathcal{F}_I^\bullet$'s are injectives, so that each row

$$0 \to \prod_{I,\ |I|=j} \Gamma(\operatorname{Spec} K_I, \mathcal{F}_I^k) \to \prod_{I,\ |I|=j} C_I^0(\mathcal{F}_I^k) \to \prod_{I,\ |I|=j} C_I^1(\mathcal{F}_I^k) \to \cdots$$

of the mapping cone is exact.

Meanwhile, there is a map *to* (17) *from* the totalization of

$$0 \to \prod_i C_i^\bullet(M) \to \prod_{i<j} C_{ij}^\bullet(M) \to \cdots, \tag{18}$$

because of (14). Now (18) is just our model-theoretic definition from the introduction. So we need to show that the map from (18) to (17) is a quasi-ismorphism.

For the usual reasons[10] it suffices to prove the following:

Lemma 4.2. *For fixed I, the natural map from $C_I^\bullet(M)$ to the totalization of $C_I^\bullet(\mathcal{F}_I^\bullet)$, induced by (14), is a quasi-isomorphism.*

Proof. Write g_I and f_I and $\mathcal{F}_I^\bullet$ as g and f and $\mathcal{F}^\bullet$ for simplicity. (So g is the map $G_L \to G_I$, and f_I is the inclusion of $\operatorname{Spec} K_I$ into G_I, and $\mathcal{F}^\bullet$ is $f^* g_* \mathcal{I}^\bullet$.)

Fix some injective resolution $0 \to M \to \mathcal{J}^\bullet$ of M in the category of etale sheaves on G_I. Because $g_*\mathcal{I}^\bullet$ are injective, we can find a map of chain complexes $\mathcal{J}^\bullet \to g_*\mathcal{I}^\bullet$ such that

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & 0 & & \\ 0 \arrow[r] & \mathcal{J}^0 \arrow[r] \arrow[d] & \mathcal{J}^1 \arrow[r] \arrow[d] & \mathcal{J}^2 \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & g_*\mathcal{I}^0 \arrow[r] & g_*\mathcal{I}^1 \arrow[r] & g_*\mathcal{I}^2 \arrow[r] & \cdots \end{tikzcd} \tag{19}$$

is a factorization of $M \to g_*\mathcal{I}^\bullet$.[11]

For each open subvariety U of G_I, we know that

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(U, \mathcal{J}^0) \arrow[r] \arrow[d] & \Gamma(U, \mathcal{J}^1) \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & \Gamma(U, g_*\mathcal{I}^0) \arrow[r] & \Gamma(U, g_*\mathcal{I}^1) \arrow[r] & \cdots \end{tikzcd} \tag{20}$$

is a quasi-isomorphism, because the induced map on cohomology groups is exactly the natural map from the etale cohomology groups of U to those of $U_L := U \times_{\operatorname{Spec} L_I} \operatorname{Spec} L$. As $\operatorname{Spec} L_I$ and $\operatorname{Spec} L$ are algebraically closed fields and U is a variety, and M is torsion and prime to the characteristic, it follows from results in etale cohomology that this map on etale cohomology groups is an isomorphism.

Passing to the direct limit over U in (20), we get that

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(\operatorname{Spec} K_I, f^*\mathcal{J}^0) \arrow[r] \arrow[d] & \Gamma(\operatorname{Spec} K_I, f^*\mathcal{J}^1) \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & \Gamma(U, f^*g_*\mathcal{I}^0) \arrow[r] & \Gamma(U, f^*g_*\mathcal{I}^1) \arrow[r] & \cdots \end{tikzcd} \tag{21}$$

[10] If I have a map of double complexes $C^{\bullet,\bullet} \to D^{\bullet,\bullet}$, to check that the induced map on the totalizations is a quasi-isomorphism, it suffices to check that the induced map $C^{i,\bullet} \to D^{i,\bullet}$ on each row is a quasi-isomorphism. This has to do with the fact that we can check whether a map of complexes is a quasi-isomorphism by checking that the mapping cone is exact, and the fact that the totalization of a double complex is exact whenever every row is.

[11] This should be true because $M \to \mathcal{J}^\bullet$ is an anodyne cofibration, and $g_*\mathcal{I}^\bullet$ is a fibrant object, in the injective model structure on cochain complexes. At any rate, M maps injectively into $\mathcal{J}^0$ so we should be able to find $\mathcal{J}^0 \to g_*\mathcal{I}^0$, by injectivity of $g_*\mathcal{I}^0$. Subsequent maps can be found by induction.

is a quasi-isomorphism.

Now, we have six complexes (or double complexes) related as follows:

$$\begin{tikzcd} \Gamma(\operatorname{Spec} K_I, f^*\mathcal{J}^\bullet) \arrow[r] \arrow[d] & C_I^\bullet(f^*\mathcal{J}^\bullet) \arrow[d] & C_I^\bullet(M) \arrow[l] \arrow[d] \\ \Gamma(\operatorname{Spec} K_I, \mathcal{F}^\bullet) \arrow[r] & C_I^\bullet(\mathcal{F}^\bullet) & C_I^\bullet(M) \arrow[l] \end{tikzcd} \tag{22}$$

The vertical arrows come from the map $f^*\mathcal{J}^\bullet \to f^*g_*\mathcal{I}^\bullet = \mathcal{F}^\bullet$ obtained by applying f^* to (19). The horizontal arrows on the left (pointing to the right) come from (16). The horizontal arrows on the right (pointing left) come from (14) and (19).

We claim that all six arrows are quasi-isomorphisms.

- The leftmost vertical arrow is a quasi-isomorphism by (21).
- The rightmost vertical arrow is a quasi-isomorphism because it is the identity map.
- The two horizontal arrows on the left (pointing right) are quasi-isomorphisms because (16) is exact whenever the module is injective, and all the $\mathcal{F}^\bullet$ and $f^*\mathcal{J}^\bullet$'s are injective (the latter by Lemma 4.1).
- The middle vertical arrow is a quasi-isomorphism, because the other three sides of the left square are quasi-isomorphisms.
- The top right arrow (pointing left) is a quasi-isomorphism, because the functors $C^i(-)$ are exact, and $0 \to M \to f^*\mathcal{J}^0 \to f^*\mathcal{J}^1 \to \cdots$ is an exact sequence (as f^* is an exact functor).
- The bottom right arrow is a quasi-isomorphism because the other three morphisms in the square on the right are. This is what we wanted to prove.

$\square$