Central extensions of Groups of Finite Morley Rank

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1 The objective

Work in the setting of a group of finite Morley rank (but perhaps consider other groups interpretable within this structure). In particular, NFCP holds, Morley rank equals Lascar rank, and rank is definable in families.

Work in a monster model. "Definable" will mean "definable with parameters" by default.

Let G be a definable group and A be a finite abelian group. Let $H^2_{def}(G, A)$ denote the set of definable central extensions of G by A up to definable isomorphism. That is, it is the set of exact sequences $1 \to A \to H \to G \to 1$ with H definable, all the maps definable, and with $A \in Z(H)$, where we consider two such sequences to be equivalent if there is a homomorphism $H \to H'$ (necessarily an isomorphism) such that

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & A \arrow[r] & H' \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

commutes. There is a natural group structure on $H^2_{def}(G, A)$, where the sum of two extensions H and H' is $H \times_G H'/A$, where A maps into $H \times_G H'$ by $a \mapsto (a, -a)$. The negative of $1 \to A \to H \to G \to 1$ is the same sequence but with the map from $A \to H$ negated.

TODO: check that this still works in the definable setting.

If $G' \to G$ is a definable homomorphism of groups, there is a homomorphism $H^2_{def}(G, A) \to H^2_{def}(G', A)$ which sends a definable central extension $1 \to A \to H \to G \to 1$ to the top row of the pullback diagram

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, equal] & H \times_G G' \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & A \arrow[r] & H \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

This is functorial.

Here is the main goal:

Theorem 1.1. *Let G be a connected group of finite Morley rank. Then there is a finite set of primes S such that if A is a finite abelian group with $|A|$ prime to S, and if $H \to G$ is a finite central extension with H connected, then $H^2_{def}(H, A)$ is finite.*

2 Short exact sequences

Theorem 2.1. *Suppose G is a (definable) group and N is a connected normal subgroup. Suppose that A is finite. Then the natural sequence*

$$H^2_{def}(G/N, A) \to H^2_{def}(G, A) \to H^2_{def}(N, A)$$

is exact.

Proof. The fact that the composition vanishes is easy (the map $N \to G/N$ is the trivial map, which also factors through 1, and $H^2_{def}(1, A) = 0$). Now suppose that $1 \to A \to H \xrightarrow{\pi} G \to 1$ is a central extension of G that splits when pulled back to N. Then there is a definable section of $H \times_G N \to N$. Let N' be the image of this definable section. Then N' is exactly the connected component of $H \times_G N$. Since $H \times_G N$ is a normal subgroup of H (being the kernel of $H \to G \to G/N$), N' is a normal subgroup of H. Since $N' < H \times_G N$, we get a surjection

$$H/N' \twoheadrightarrow H/(H \times_G N) \cong G/N.$$

This map's kernel is $(H \times_G N)/N'$ which is canonically A, so we get a short exact sequence $1 \to A \to H/N' \to G/N \to 1$. This is a central extension. After checking all the details, one finds that this central extension, when pulled back to G, yields the original central extension. Indeed, one gets $H/N' \times_{G/N} G$. This is the group of pairs (x, g) where x is a coset of N', and where gN equals the image of the coset x in G. But since $N' \to G$ is injective, there is a unique element y in the coset x which lies directly above g. The map $(x, g) \to y$ is definable and is the desired isomorphism from $H/N' \times_{G/N} G$ to H.

TODO: check the details.

□

3 The abelian case

Let A be a (definable) abelian group. For each ℓ, let $A[\ell]$ denote the ℓ-torsion of A. If $A[\ell]$ is finite, then the multiplication-by-ℓ map $A \to A$ has finite fibers, so its iterates also have finite fibers. Therefore $A[\ell^n]$ is also finite.

Lemma 3.1. *Let A be a (definable) abelian group. Then there are only finitely many primes ℓ such that $A[\ell]$ is infinite.*

Proof. If $\ell_1, \ldots, \ell_n$ are distinct primes with $A[\ell_i]$ infinite, then

$$A \geq \oplus_{i=1}^n A[\ell_i].$$

The Morley rank of the right hand side is at least n, so n is bounded by $RM(A)$. □

Lemma 3.2. *If A is connected and $H \to A$ is a definable central extension by a finite group, then H is abelian.*

Proof. The commutator of any two elements of H must be in $K = \ker(H \to A)$. Since K is finite, every element of H has a finite conjugacy class. So every element's centralizer has finite index in H. As H is connected, every element of H is central, so H is abelian. $\square$

Lemma 3.3. *Let ℓ be a prime such that $A[\ell]$ is finite. Suppose that A is connected. Then every definable central extension of A by $\mathbb{Z}/\ell$ is either split or connected. In the connected case, it is definably isomorphic to A/V, where V is a codimension 1 subspace in the $\mathbb{F}_\ell$-vector space $A[\ell]$, and where the map $A/V \to A$ is the one induced by multiplication by ℓ.*

Proof. Suppose that $1 \to \mathbb{Z}/\ell \to H \to A \to 1$ is a central extension. Let H^0 be the connected component of H. Then $\mathbb{Z}/\ell$ and H^0 are both normal subgroups of H. As $RM(H^0) = RM(H) = RM(A)$ and $H^0 \to H \to A$ has finite fibers, the image of H^0 in A must have finite index, hence must be all of A. So H^0 and $\mathbb{Z}/\ell$ generate H. If $H^0 \cap \mathbb{Z}/\ell$ is trivial, then H is canonically $H^0 \times \mathbb{Z}/\ell$, i.e., the sequence splits (definably).

Otherwise, $H^0 \cap \mathbb{Z}/\ell = \mathbb{Z}/\ell$, so H^0 contains $\mathbb{Z}/\ell$. Since the two groups generate H, it follows that $H = H^0$, i.e., H is connected. Then H is itself an abelian group. The kernel of $H \to A$ is $\mathbb{Z}/\ell$, which must be contained in $H[\ell]$. This gives a commutative diagram

$$\begin{tikzcd} H \arrow[r, two heads] \arrow[d, two heads] & H \arrow[d, two heads] \\ A \arrow[r, two heads] \arrow[ur, dashed, two heads] & A \end{tikzcd}$$

where the horizontal arrows are multiplication by ℓ and the vertical arrows are both the given map $H \to A$. All the maps are surjective because we assumed that A has finite ℓ-torsion (which implies the same for H). We can fill in the dashed diagonal arrow, because the kernel of multiplication by ℓ from $H \to H$ contains $K = \ker(H \to A)$. From the diagram, we see that H is a quotient of A by some subgroup of $A[\ell]$. $\square$

Consequently, if A is a connected abelian group with finite ℓ-torsion, and if A' is a connected central extension of A, then A' is also abelian, also has finite ℓ-torsion, and $H^2_{def}(A', \mathbb{Z}/\ell)$ is finite.

4 The nearly simple case

Suppose G is a definable group which has no infinite normal (definable) subgroup. In particular, G is connected. Let A be a minimal subgroup of G. Then A is connected and abelian (Reineke's theorem). Let S_1 be a strongly minimal subset of A. Write S_1 as a disjoint union of G-indecomposable sets. Let S_2 be the unique one of these which is still infinite. So S_2 is also strongly minimal. Let $S = S_3$ be obtained from S by (left) translation by the inverse of an element of S_2. Then S is still contained in A, is still G-indecomposable, and contains 1. By Zilber's indecomposability, S generates A. Also, the conjugates of S generate G. Let $g_1, \ldots, g_n$ be such that $S^{g_1}, \ldots, S^{g_n}$ generate A.

Lemma 4.1. *There is an m such that if $1 \to M \to H \to G \to 1$ is a definable finite central extension whose class lies in the kernel of*

$$H^2_{def}(G, M) \to \bigoplus_{i=1}^{n} H^2_{def}(A^{g_i}, M)$$

and such that H is connected, then $|M| \leq m$.

Proof. Let $r = RM(G)$. By the proof of Zilber's indecomposability theorem, we can find $i_1, \ldots, i_r \in \{1, \ldots, n\}$ such that the natural map

$$\mu : S := \prod_{j=1}^{r} S^{g_{i_j}} \to G$$

$$(s_1, \ldots, s_r) \mapsto \prod_{j=1}^{r} s_j$$

has image that is generic in G.

Let g be generic in G. Then $\mu^{-1}(g)$ is finite, since $RM(S) \leq RM(G)$. It is non-empty by assumption. Let m be the cardinality of $\mu^{-1}(g)$. This does not depend on g.

Now suppose that $1 \to M \to H \xrightarrow{\pi} G \to 1$ is a definable finite central extension, which splits when pulled back to A^{g_i} for any i. For each j, let $\lambda_j : A^{g_{i_j}} \to \pi^{-1}(A^{g_{i_j}})$ be the given definable section. Consider the map

$$\nu : S \to H$$

sending

$$(s_1, \ldots, s_r) \mapsto \prod_{j=1}^{r} \lambda_j(s_j)$$

Note that $\pi \circ \nu = \mu$. Since H is connected, the image of the generic type of S under ν must be the generic type of H. In particular, any generic in H is in the image of ν. Now let g be generic in G. Then $\pi^{-1}(g)$ has cardinality $|M|$. Each element of $\pi^{-1}(g)$ is generic in H, so the set

$$\mu^{-1}(g) = \nu^{-1}(\pi^{-1}(g))$$

has at least $|M|$ elements. But it also has at most m elements, so $|M| \leq m$. $\square$

Lemma 4.2. *Let ℓ be a prime such that A has no ℓ-torsion. Then $H^2_{def}(G, \mathbb{Z}/\ell)$ is finite.*

Proof. Since $\bigoplus_{i=1}^{n} H^2_{def}(A^{g_i}, \mathbb{Z}/\ell)$ is finite, it suffices to show that

$$H^2_{def}(G, \mathbb{Z}/\ell) \to \bigoplus_{i=1}^{n} H^2_{def}(A^{g_i}, \mathbb{Z}/\ell)$$

has finite kernel.

It is an $\mathbb{F}_\ell$-vector space. Suppose it had dimension at least the constant m from the previous lemma. Then we could find $v_1, \ldots, v_m \in H^2_{def}(G, \mathbb{Z}/\ell)$ linearly independent, each becoming trivial when pulled back to A^{g_i}. The linear independence implies that the sum

$$v_1 \oplus \cdots \oplus v_m \in H^2_{def}(G, \mathbb{F}_\ell^m)$$

is not in $H^2_{def}(G, V)$ for any linear subspace $V \subset \mathbb{F}_\ell^m$.

Now let $1 \to \mathbb{F}_\ell^m \to H \to G \to 1$ be the corresponding central extension. Then H is not connected, by choice of m (as $\ell^m > m$). Let H^0 be the connected component of H. For the usual reasons, H^0 surjects onto G (as both groups are connected, have the same rank, and the map from H^0 to G has finite fibers). So H^0 and $\mathbb{F}_\ell^m$ generate H. Both are normal subgroups, so H/H^0 is identified with $\mathbb{F}_\ell^m/V$, where $V = \mathbb{F}_\ell^m \cap H^0$.

Let $\lambda : \mathbb{F}_\ell^m \to \mathbb{F}_\ell^m/V$ be the canonical surjection. Then $H^2_{def}(G, \lambda)$ sends our given central extension to one that splits, I think. This contradicts linear independence. $\square$

Now we can prove

Theorem 4.3. *Let G be a connected group of finite Morley rank. Then there is a finite set of primes S such that for N prime to S, $H^2_{def}(G, A)$ is finite for $|A| = N$.*

Proof. First of all $H^2_{def}(G, -)$ should be half-exact, allowing us to restrict to the case where A is $\mathbb{Z}/\ell$ for ℓ a prime not in S.

We prove by induction on the rank of G that there is some finite set S_G of primes such that for ℓ outside S_G, $H^2_{def}(G, \mathbb{Z}/\ell)$ is finite.

The 0-dimensional case is easy.

Suppose G has an infinite normal subgroup. Let N be its connected component. Then $H^2_{def}(G, \mathbb{Z}/\ell)$ sits in an exact sequence between $H^2_{def}(N, \mathbb{Z}/\ell)$ and $H^2_{def}(G/N, \mathbb{Z}/\ell)$, so we can take $S_G = S_{G/N} \cup S_N$.

Now suppose that G has no infinite definable normal subgroup. Then by the above comments, we can find an abelian subgroup A such that if $A[\ell]$ is finite, then $H^2_{def}(G, \mathbb{Z}/\ell)$ is finite. As there are only finitely many ℓ such that $A[\ell]$ is infinite, we are done. $\square$

Suppose G' is a connected finite central extension of G. And N is a connected normal subgroup of G. Then the connected component of the preimage of N in G' is a finite cover of N, and the quotient of G' by that group is a finite cover of G/N, so everything checks out.

Really, we could have just been talking about "finite covers" the whole time.