Geometric irreducibility and Zariski closure are definable in families

Will Johnson

December 27, 2014

1 Irreducibility in Projective Space

Let $\mathbb{C}$ be a monster model of ACF. For $\vec{x} \in \mathbb{P}^n(\mathbb{C})$, let $\mathbb{P}_{\vec{x}}$ be the $n-1$-dimensional projective space of lines through $\vec{x}$, and let $\pi_{\vec{x}} : \mathbb{P}^n \setminus \{\vec{x}\} \to \mathbb{P}_{\vec{x}}$ be the projection.

Lemma 1.1. *Let A be a small set of parameters, and suppose $\vec{x} \in \mathbb{P}^n(\mathbb{C})$ is generic over A. Suppose V is an A-definable Zariski closed subset of $\mathbb{P}^n$, of codimension* greater than *1. Then $\pi_{\vec{x}}(V) \subset \mathbb{P}_{\vec{x}}$ is well-defined, Zariski closed, of codimension one less than the codimension of V. Moreover, $\pi_{\vec{x}}(V)$ is irreducible if and only if V is irreducible.*

Proof. Replacing A with $\operatorname{acl}(A)$, we may assume A is algebraically closed, implying that the irreducible components of V are also A-definable.

Since $\vec{x}$ is generic, and V has codimension *at least* 1, $\vec{x} \notin V$ so $\pi_{\vec{x}}(V)$ is well-defined. It is Zariski closed because $\mathbb{P}^n$ is a complete variety, so V is complete and the image of V under any morphism of varieties is closed.

Claim 1.2. *Let C be any irreducible component of V, and let $\vec{c} \in V$ realize the generic type of C, over $A\vec{x}$. Then $\vec{c}$ is the sole preimage in V of $\pi_{\vec{x}}(\vec{c})$.*

Proof. The generic type of C is A-definable, so $\vec{c} \mathop{\smile\!\!\!\!\!\!\,|}_A \vec{x}$, and therefore $RM(\vec{x}/A\vec{c}) = RM(\vec{x}/A) = n$. Suppose for the sake of contradiction that there was a second point $\vec{d} \in V$, $\vec{d} \neq \vec{c}$, satisfying

$$\pi_{\vec{x}}(\vec{d}) = \pi_{\vec{x}}(\vec{c}).$$

This means exactly that the three points $\vec{c}$, $\vec{d}$, and $\vec{x}$ are collinear. Then $\vec{x}$ is on the 1-dimensional line determined by $\vec{c}$ and $\vec{d}$, so

$$RM(\vec{x}/A\vec{c}\vec{d}) \leq 1.$$

But then

$$n = RM(\vec{x}/A\vec{c}) \leq RM(\vec{x}\vec{d}/A\vec{c}) = RM(\vec{x}/A\vec{c}\vec{d}) + RM(\vec{d}/A\vec{c}) \leq 1 + RM(V) < n,$$

by the codimension assumption. $\square$

Using the claim, we see that $\pi_{\vec{x}}(V)$ and V have the same dimension (= Morley rank). Indeed, let $\vec{v} \in V$ have Morley rank $RM(V)$ over $A\vec{x}$. Then $\vec{v}$ realizes the generic type of *some* irreducible component C, so by the claim, $\vec{v}$ is interdefinable over $A\vec{x}$ with $\pi_{\vec{x}}(\vec{v})$. But then

$$RM(\pi_{\vec{x}}(V)) \geq RM(\pi_{\vec{x}}(\vec{v})/A\vec{x}) = RM(\vec{v}/A\vec{x}) = RM(V),$$

and the reverse inequality is obvious. So the codimension of $\pi_{\vec{x}}(V)$ is indeed one less.

Let $C_1, \ldots, C_m$ enumerate the irreducible components of V. (Possibly $m = 1$.) Each of the images $\pi_{\vec{x}}(C_i)$ is a Zariski closed subset of $\mathbb{P}_{\vec{x}}$, for the same reason that $\pi_{\vec{x}}(V)$ is, and each image is irreducible, on general grounds. If $\pi_{\vec{x}}(C_i) \subseteq \pi_{\vec{x}}(C_j)$ for some $i \neq j$, then the generic type of C_i would have the same image under $\pi_{\vec{x}}$ as some point in C_j, contradicting the Claim. So $\pi_{\vec{x}}(C_i) \not\subseteq \pi_{\vec{x}}(C_j)$ for $i \neq j$. It follows that the images $\pi_{\vec{x}}(C_i)$ are the irreducible components of

$$\pi_{\vec{x}}(V) = \bigcup_{i=1}^{m} \pi_{\vec{x}}(C_i).$$

Therefore, $\pi_{\vec{x}}(V)$ and V have the same number of irreducible components, proving the last point of the lemma. $\square$

Theorem 1.3. *Let $X_{\vec{a}} \subseteq \mathbb{P}^n$ be a definable family of Zariski closed subsets of $\mathbb{P}^n$. Then the set of $\vec{a}$ for which $X_{\vec{a}}$ is irreducible, is definable.*

Proof. Dimension is definable in families, because ACF is strongly minimal. So we may assume that all (non-empty) $X_{\vec{a}}$ have the same (co)dimension. We proceed by induction on codimension, allowing n to vary.

For the base case of codimension 1, note that

1. The family of Zariski closed subsets of $\mathbb{P}^n$ is ind-definable, i.e., a small union of definable families, because the Zariski closed subsets are exactly the zero sets of finitely-generated ideals.

2. Using 1, the family of *reducible* Zariski closed subsets of $\mathbb{P}^n$ is also ind-definable, because a definable set is a *reducible* Zariski closed set if and only if it is the union of two incomparable Zariski closed sets.

3. Whether or not a polynomial in $\mathbb{C}[x_1, \ldots, x_{n+1}]$ is irreducible, is definable in terms of the coefficients, because we only need to quantify over lower-degree polynomials.

4. A hypersurface in $\mathbb{P}^n$ is irreducible if and only if it is the zero-set of an irreducible homogeneous polynomial. It follows by 3 that the family of irreducible codimension 1 closed subsets of $\mathbb{P}^n$ is ind-definable.

5. By 2 (resp. 4), the set of $\vec{a}$ such that $X_{\vec{a}}$ is reducible (resp. irreducible) is ind-definable. Since these two sets are complementary, both are definable, proving the base case.

For the inductive step, suppose that irreducibility is definable in families of codimension one less than $X_{\vec{a}}$. By choosing an isomorphism between $\mathbb{P}_{\vec{x}}$ and $\mathbb{P}^{n-1}$, one easily verifies the definability of the set of $(\vec{x}, \vec{a})$ such that $\pi_{\vec{x}}(X_{\vec{a}})$ is irreducible and has codimension one less.

By Lemma 1.1, $X_{\vec{a}}$ is irreducible if and only if $(\vec{x}, \vec{a})$ lies in this set, for generic $\vec{x}$. Definability of types in stable theories then implies definability of the set of $\vec{a}$ such that $X_{\vec{a}}$ is irreducible. $\square$

Corollary 1.4. *The family of irreducible closed subsets of $\mathbb{P}^n$ is ind-definable.*

Proof. The family of closed subsets is ind-definable, and by Theorem 1.3 we can select the irreducible ones within any definable family. $\square$

Corollary 1.5. *The family of pairs $(X, \overline{X})$ with X definable and $\overline{X}$ its Zarisk-closure, is ind-definable.*

Proof. By quantifier elimination in ACF, any definable set X can be written as a union of sets of the form $C \cap U$ with C closed and U open. Replacing V with a union of irreducibles, and distributing, we can write X as a union $\bigcup_{i=1}^m C_i \cap U_i$, with C_i Zariski closed and U_i Zariski open. We may assume that $C_i \cap U_i \neq \emptyset$ for each i, or equivalently, that $C_i \setminus U_i \neq C_i$.

In any topological space, closure commutes with unions, so

$$\overline{X} = \bigcup_{i=1}^n \overline{C_i \cap U_i}.$$

Now $\overline{C_i \cap U_i} \subseteq \overline{C_i} = C_i$, and

$$C_i = \overline{C_i \cap U_i} \cup (C_i \setminus U_i),$$

so by irreducibility of C_i, $\overline{C_i \cap U_i} = C_i$. Therefore,

$$\overline{X} = \bigcup_{i=1}^n C_i.$$

Corollary 1.4 implies the ind-definability of the family of pairs

$$\left(\bigcup_{i=1}^n \overline{C_i \cap U_i}, \bigcup_{i=1}^n C_i \right)$$

with C_i irreducible closed, U_i open, and $C_i \cap U_i \neq \emptyset$. We have seen that this is the desired family of pairs. $\square$

The following corollary is an easy consequence:

Corollary 1.6. *Let $X_{\vec{a}}$ be a definable family of subsets of $\mathbb{P}^n$. Then the Zariski closures $\overline{X_{\vec{a}}}$ are also a definable family.*

2 Irreducibility in Affine Space

Theorem 2.1. *Let $X_{\vec{a}}$ be a definable family of subsets of affine n-space.*

1. *The family of Zariski closures $\overline{X_{\vec{a}}}$ is also definable.*

2. *The set of $\vec{a}$ such that $\overline{X_{\vec{a}}}$ is irreducible is definable. More generally, the number of irreducible components of $\overline{X_{\vec{a}}}$ is definable in families (and bounded in families).*

3. *Dimension and Morley degree of $X_{\vec{a}}$ are definable in $\vec{a}$.*

4. *If each $\overline{X_{\vec{a}}}$ is a hypersurface given by the irreducible polynomial $F_{\vec{a}}(x_1, \ldots, x_n)$, then the degree of $F_{\vec{a}}$ in each x_i is definable in $\vec{a}$. In fact, the polynomials $F_{\vec{a}}$ have bounded total degree and the family of $F_{\vec{a}}$ (up to scalar multiples) is definable.*

Proof.

1. Embed $\mathbb{A}^n$ into $\mathbb{P}^n$. Then the Zariski closure of $X_{\vec{a}}$ within $\mathbb{A}^n$ is the intersection of $\mathbb{A}^n$ with the closure within $\mathbb{P}^n$. Use Corollary 1.6.

2. The number of irreducible components of the Zariski closure is the same whether we take the closure in $\mathbb{A}^n$ or $\mathbb{P}^n$. This proves the first sentence. The first sentence yields the ind-definability of the family of irreducible Zariski closed subsets of $\mathbb{A}^n$, from which the second statement is an exercise in compactness.

3. We may assume $X_{\vec{a}}$ is closed, since taking the closure changes neither Morley rank nor Morley degree. The family of d-dimensional Zariski irreducible closed subsets of $\mathbb{A}^n$ is ind-definable, making this an exercise in compactness.

4. Whether or not an n-variable polynomial is irreducible is definable in the coefficients, because to check reducibility one only needs to quantify over the (definable) set of lower-degree polynomials. This makes the family of irreducible polynomials ind-definable. Therefore, the set of pairs $(\vec{a}, F_{\vec{a}})$ where $F_{\vec{a}}$ cuts out $\overline{X_{\vec{a}}}$, is ind-definable. For any given $\vec{a}$, all the possibilities for $F_{\vec{a}}$ are essentially the same, differing only by scalar multiples. So the total degree of $F_{\vec{a}}$ only depends on $\vec{a}$, and compactness yields a bound on the total degree. This in turn makes the set of pairs $(\vec{a}, F_{\vec{a}})$ definable.

$\square$