Amalgamating valued fields

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1 Preliminaries

Recall some facts...

Lemma 1.1 (Nakayama). Let (A, \mathfrak{m}) be a local ring and M be a finitely-generated A-module such that $A = \mathfrak{m}A$. Then A = 0.

Proof. Let g_1, \ldots, g_n be a minimal set of generators of M. If n > 0, then by assumption we can write

$$g_1 = \sum_{i=1}^n m_i g_i$$

with $m_i \in \mathfrak{m}$. Then

$$(1-m_i)g_1 = \sum_{i=2}^n m_i g_i.$$

But $1 - m_i$ is invertible by locality, so g_1 is generated by the other generators, contradicting minimality..

Lemma 1.2. Let R be a valuation ring. Then any finitely generated torsion-free R-module M is free.

Proof. Let g_1, \ldots, g_n be a minimal set of generators. Then g_1, \ldots, g_n freely generate M. If not, then there are some m_i , not all zero, such that

$$\sum_{i=1}^{n} m_i g_i = 0.$$

Let j be such that $val(m_j)$ is minimal. Then $m_i/m_j \in R$ for each i, and

$$m_j \cdot \sum_{i=1}^n \frac{m_i}{m_j} g_i = \sum_{i=1}^n m_i g_i = 0.$$

As M is torsion-free,

$$\sum_{i=1}^{n} \frac{m_i}{m_j} g_i = 0, \text{ so that } g_j = -\sum_{i \neq j} \frac{m_i}{m_j} g_i,$$

contradicting minimality.

Lemma 1.3. Let R be a valuation ring. Then any torsion-free R-module M is flat.

Proof. M can be written as a direct limit of its finitely-generated submodules, which are torsion-free, hence flat. A direct limit of flat modules is flat (Because \otimes preserves direct limits, and direct limits of exact sequences are exact.)

Lemma 1.4. Let $f : A \to R$ be an injective ring homomorphism. Then every minimal prime of A is a pullback $f^{-1}(\mathfrak{q})$ of some prime \mathfrak{q} in R.

Proof. Let \mathfrak{p} be any minimal prime of A. The complement of \mathfrak{p} is closed under multiplication, so its image $S = f(A \setminus \mathfrak{p})$ is closed as well. By injectivity of $f, 0 \notin S$. So the localization $S^{-1}R$ is nonzero, and has at least one prime ideal, whose pullback to R is a prime ideal \mathfrak{q} not intersecting $S = f(A \setminus \mathfrak{p})$. Then

$$x \in f^{-1}(\mathfrak{q}) \implies f(x) \in \mathfrak{q} \implies f(x) \notin f(A \setminus \mathfrak{p}) \implies x \notin A \setminus \mathfrak{p} \implies x \in \mathfrak{p}.$$

for $x \in R$. Thus

 $f^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}.$

By minimality of \mathfrak{p} , equality holds.

2 Extending valuation rings

If K is a field, we can look at the set of pairs (R, \mathfrak{p}) , where R is a subring of K and \mathfrak{p} is a prime ideal of R. We make this a poset by setting

$$(R,\mathfrak{p}) \leq (S,\mathfrak{q})$$

if

$$R \subseteq S \text{ and } \mathfrak{p} = R \cap \mathfrak{q}, \qquad or \ equivalently, \qquad \mathfrak{p} \subseteq \mathfrak{q} \text{ and } (R \setminus \mathfrak{p}) \subseteq (S \setminus \mathfrak{q})$$

Zorn's lemma applies to this poset, so each pair lies below some maximal pair.

Lemma 2.1. The maximal pairs are exactly the valuation rings $(\mathcal{O}, \mathfrak{m})$ in K.

Proof. First suppose (A, \mathfrak{m}) is maximal. If $A_{\mathfrak{m}}$ denotes the localization of A at \mathfrak{m} , then

$$(A,\mathfrak{m}) \leq (A_{\mathfrak{m}},\mathfrak{m}A_{\mathfrak{m}}).$$

As (A, \mathfrak{m}) is maximal, we see that $A = A_{\mathfrak{m}}$ is a local ring and \mathfrak{m} is its maximal ideal. Claim 2.2. For any $x \in K \setminus A$, we have $\mathfrak{m}A[x] = A[x]$. *Proof.* If not, let $\mathfrak{n} \supseteq \mathfrak{m} A[x]$ be a maximal ideal of A[x]. Then we have a contradiction:

 $(A, \mathfrak{m}) < (A[x], \mathfrak{n})$

because ker $(A \to A[x]/\mathfrak{n})$ contains, hence equals, the maximal ideal \mathfrak{m} of A.

Finally, we show that A is a valuation ring. If not, there is $x \in K^{\times}$ such that neither x nor x^{-1} is in A. By the claim, $\mathfrak{m}A[x] = A[x]$ and $\mathfrak{m}A[x^{-1}] = A[x^{-1}]$. The first of these means that

$$1 = \sum_{i=0}^{n} m_i x^{-i}$$

for some $m_i \in \mathfrak{m}$. Rearranging, we see that

$$x^{i} = m_{0}x^{i} + m_{1}x^{i-1} + \dots + m_{n-1}x + m_{n},$$
$$(1 - m_{0})x^{i} = m_{1}x^{i-1} + \dots + m_{n}.$$

Since $(1 - m_0)$ is a unit in A, it follows that x is integral over A, so A[x] is finitely generated as an A-module. Now $\mathfrak{m}A[x] = A[x]$, so by Nakayama's lemma A[x] = 0, which is absurd.

Conversely, if $(\mathcal{O}, \mathfrak{m})$ is a valuation ring, and $(\mathcal{O}, \mathfrak{m}) < (A, \mathfrak{p})$, then taking $x \in A \setminus \mathcal{O}$, we have $x^{-1} \in \mathfrak{m} \subseteq \mathfrak{p}$. Then $x^{-1} \in A^{\times} \cap \mathfrak{p} = \emptyset$, a contradiction. \square

Observation 2.3. If K/F is an extension of fields, then valuation data $(\mathcal{O}_K, \mathfrak{m}_K)$ on K extends valuation data ($\mathcal{O}_F, \mathfrak{m}_F$) on F if and only if

$$(\mathcal{O}_F, \mathfrak{m}_F) \le (\mathcal{O}_K, \mathfrak{m}_K). \tag{1}$$

Proof. If $(\mathcal{O}'_F, \mathfrak{m}'_F)$ is the restriction of $(\mathcal{O}_K, \mathfrak{m}_K)$ to F, then (1) is equivalent to

$$(\mathcal{O}_F, \mathfrak{m}_F) \le (F \cap \mathcal{O}_K, F \cap \mathfrak{m}_K) = (\mathcal{O}'_F, \mathfrak{m}'_F)$$
(2)

Since $(\mathcal{O}_F, \mathfrak{m}_F)$ is maximal, " \leq " in (2) can equivalently be replaced with "=".

3 Amalgamating valued fields

Theorem 3.1. Let (K_0, \mathcal{O}_{K_0}) be a valued field, and (K_i, \mathcal{O}_{K_i}) be two valued field extensions for i = 1, 2. Then there is a valued field (L, \mathcal{O}_L) and a diagram of valued fields



Proof. For simplicity, we denote \mathcal{O}_{K_i} by \mathcal{O}_i and similarly \mathfrak{m}_i and k_i . The ring $k_1 \otimes_{k_0} k_2$ is nonzero, so it has a prime ideal \mathfrak{p}_0 . Let $\mathfrak{p} \in \operatorname{Spec} R$ be the pullback of \mathfrak{p}_0 along the map

$$R := \mathcal{O}_1 \otimes_{\mathcal{O}_0} \mathcal{O}_2 \to k_1 \otimes_{k_0} k_2$$

For i = 1, 2, we have a commuting square of sets

$$\operatorname{Spec} \mathcal{O}_{i} \longleftarrow \operatorname{Spec} k_{i} = \{*\}$$

$$\uparrow \qquad \qquad \uparrow$$

$$\operatorname{Spec} R = \operatorname{Spec} \mathcal{O}_{1} \otimes_{\mathcal{O}_{0}} \mathcal{O}_{2} \longleftarrow \operatorname{Spec} k_{1} \otimes_{k_{0}} k_{2}$$

The pullback of \mathfrak{p} to \mathcal{O}_i is in the image of Spec $k_i \to \operatorname{Spec} \mathcal{O}_i$, so it must be \mathfrak{m}_i .

Let \mathfrak{q} be any minimal prime of R below \mathfrak{p} . Now \mathcal{O}_1 and K_2 are flat \mathcal{O}_0 -modules by Lemma 1.3, so the natural map

$$\mathcal{O}_1 \otimes_{\mathcal{O}_0} \mathcal{O}_2 \hookrightarrow \mathcal{O}_1 \otimes_{\mathcal{O}_0} K_2 \hookrightarrow K_1 \otimes_{\mathcal{O}_0} K_2$$

is an injection. By Lemma 1.4, there is some prime \mathfrak{q}_0 in $K_1 \otimes_{\mathcal{O}_0} K_2$ which pulls back to \mathfrak{q} . Now for i = 1, 2, we have a commuting square of sets

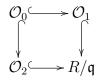
$$\operatorname{Spec} \mathcal{O}_{i} \xleftarrow{} \operatorname{Spec} K_{i} = \{*\}$$

$$\uparrow \qquad \qquad \uparrow$$

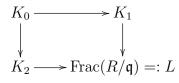
$$\operatorname{Spec} R = \operatorname{Spec} \mathcal{O}_{1} \otimes_{\mathcal{O}_{0}} \mathcal{O}_{2} \xleftarrow{} \operatorname{Spec} K_{1} \otimes_{\mathcal{O}_{0}} K_{2}$$

The pullback of \mathfrak{q} to \mathcal{O}_i is in the image of Spec $K_i \to \operatorname{Spec} \mathcal{O}_i$, so it must be (0).

Therefore each of the maps $\mathcal{O}_i \to R/\mathfrak{q}$ is injective, so the commuting square of domains



yields a commuting square of fields



Then for i = 1, 2,

$$(\mathcal{O}_i, \mathfrak{m}_i) \leq (R/\mathfrak{q}, \mathfrak{p}/\mathfrak{q})$$

because \mathfrak{m}_i is the pullback of \mathfrak{p} to \mathcal{O}_i . By Zorn's lemma and Lemma 2.1, we can find a valuation ring \mathcal{O}_L on L such that

$$(\mathcal{O}_i,\mathfrak{m}_i) \leq (R/\mathfrak{q},\mathfrak{p}/\mathfrak{q}) \leq (\mathcal{O}_L,\mathfrak{m}_L).$$

Then by Observation 2.3, the valuation structure on L extends those on K_1 and K_2 .

4 Getting quantifier-elimination

Recall the following fact:

Fact 4.1. Suppose T is a universal theory with the amalgamation property, T' is a theory extending T, and every model of T can be embedded into a model of T'. Suppose also that every model of T' is 1-existentially closed: for every inclusion of models $M \leq N$ of T', every non-empty quantifier-free M-definable subset of N^1 intersects M. Then T' has quantifier elimination and is the model completion of T.

We would like to apply this in the case where T' is ACVF, and T is the theory of domains with a divisibility predicate arising from a valuation on their fraction field. Given the previous sections, it is not hard to see that T has the amalgamation property, and models of T embed into models of T'. So, to obtain quantifier elimination in T', one merely needs to prove 1-existential closedness. This can be proved by analyzing quantifier-free definable subsets of the home sort, and showing that they are all boolean combinations of balls.