Amalgamating valued fields

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1 Preliminaries

Recall some facts...

Lemma 1.1 (Nakayama)**.** *Let $(A, \mathfrak{m})$ be a local ring and M be a finitely-generated A-module such that $A = \mathfrak{m}A$. Then $A = 0$.*

Proof. Let $g_1, \ldots, g_n$ be a minimal set of generators of M. If $n > 0$, then by assumption we can write

$$g_1 = \sum_{i=1}^{n} m_i g_i$$

with $m_i \in \mathfrak{m}$. Then

$$(1 - m_i) g_1 = \sum_{i=2}^{n} m_i g_i.$$

But $1 - m_i$ is invertible by locality, so g_1 is generated by the other generators, contradicting minimality.. □

Lemma 1.2. *Let R be a valuation ring. Then any finitely generated torsion-free R-module M is free.*

Proof. Let $g_1, \ldots, g_n$ be a minimal set of generators. Then $g_1, \ldots, g_n$ freely generate M. If not, then there are some m_i, not all zero, such that

$$\sum_{i=1}^{n} m_i g_i = 0.$$

Let j be such that $\mathrm{val}(m_j)$ is minimal. Then $m_i/m_j \in R$ for each i, and

$$m_j \cdot \sum_{i=1}^{n} \frac{m_i}{m_j} g_i = \sum_{i=1}^{n} m_i g_i = 0.$$

As M is torsion-free,

$$\sum_{i=1}^{n} \frac{m_i}{m_j} g_i = 0, \text{ so that } g_j = -\sum_{i \neq j} \frac{m_i}{m_j} g_i,$$

contradicting minimality. □

Lemma 1.3. *Let R be a valuation ring. Then any torsion-free R-module M is flat.*

Proof. M can be written as a direct limit of its finitely-generated submodules, which are torsion-free, hence flat. A direct limit of flat modules is flat (Because $\otimes$ preserves direct limits, and direct limits of exact sequences are exact.) □

Lemma 1.4. *Let $f : A \to R$ be an injective ring homomorphism. Then every minimal prime of A is a pullback $f^{-1}(\mathfrak{q})$ of some prime $\mathfrak{q}$ in R.*

Proof. Let $\mathfrak{p}$ be any minimal prime of A. The complement of $\mathfrak{p}$ is closed under multiplication, so its image $S = f(A \setminus \mathfrak{p})$ is closed as well. By injectivity of f, $0 \notin S$. So the localization $S^{-1}R$ is nonzero, and has at least one prime ideal, whose pullback to R is a prime ideal $\mathfrak{q}$ not intersecting $S = f(A \setminus \mathfrak{p})$. Then

$$x \in f^{-1}(\mathfrak{q}) \implies f(x) \in \mathfrak{q} \implies f(x) \notin f(A \setminus \mathfrak{p}) \implies x \notin A \setminus \mathfrak{p} \implies x \in \mathfrak{p}.$$

for $x \in R$. Thus

$$f^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}.$$

By minimality of $\mathfrak{p}$, equality holds. □

2 Extending valuation rings

If K is a field, we can look at the set of pairs $(R, \mathfrak{p})$, where R is a subring of K and $\mathfrak{p}$ is a prime ideal of R. We make this a poset by setting

$$(R, \mathfrak{p}) \leq (S, \mathfrak{q})$$

if

$$R \subseteq S \text{ and } \mathfrak{p} = R \cap \mathfrak{q}, \qquad \textit{or equivalently,} \qquad \mathfrak{p} \subseteq \mathfrak{q} \text{ and } (R \setminus \mathfrak{p}) \subseteq (S \setminus \mathfrak{q}).$$

Zorn's lemma applies to this poset, so each pair lies below some maximal pair.

Lemma 2.1. *The maximal pairs are exactly the valuation rings $(\mathcal{O}, \mathfrak{m})$ in K.*

Proof. First suppose $(A, \mathfrak{m})$ is maximal. If $A_\mathfrak{m}$ denotes the localization of A at $\mathfrak{m}$, then

$$(A, \mathfrak{m}) \leq (A_\mathfrak{m}, \mathfrak{m}A_\mathfrak{m}).$$

As $(A, \mathfrak{m})$ is maximal, we see that $A = A_\mathfrak{m}$ is a local ring and $\mathfrak{m}$ is its maximal ideal.

Claim 2.2. *For any $x \in K \setminus A$, we have $\mathfrak{m}A[x] = A[x]$.*

Proof. If not, let $\mathfrak{n} \supseteq \mathfrak{m}A[x]$ be a maximal ideal of $A[x]$. Then we have a contradiction:

$$(A, \mathfrak{m}) < (A[x], \mathfrak{n})$$

because $\ker(A \to A[x]/\mathfrak{n})$ contains, hence equals, the maximal ideal $\mathfrak{m}$ of A. $\square$

Finally, we show that A is a valuation ring. If not, there is $x \in K^\times$ such that neither x nor x^{-1} is in A. By the claim, $\mathfrak{m}A[x] = A[x]$ and $\mathfrak{m}A[x^{-1}] = A[x^{-1}]$. The first of these means that

$$1 = \sum_{i=0}^{n} m_i x^{-i}$$

for some $m_i \in \mathfrak{m}$. Rearranging, we see that

$$x^i = m_0 x^i + m_1 x^{i-1} + \cdots + m_{n-1} x + m_n,$$

$$(1 - m_0) x^i = m_1 x^{i-1} + \cdots + m_n.$$

Since $(1 - m_0)$ is a unit in A, it follows that x is integral over A, so $A[x]$ is finitely generated as an A-module. Now $\mathfrak{m}A[x] = A[x]$, so by Nakayama's lemma $A[x] = 0$, which is absurd.

Conversely, if $(\mathcal{O}, \mathfrak{m})$ is a valuation ring, and $(\mathcal{O}, \mathfrak{m}) < (A, \mathfrak{p})$, then taking $x \in A \setminus \mathcal{O}$, we have $x^{-1} \in \mathfrak{m} \subseteq \mathfrak{p}$. Then $x^{-1} \in A^\times \cap \mathfrak{p} = \emptyset$, a contradiction. $\square$

Observation 2.3. *If K/F is an extension of fields, then valuation data $(\mathcal{O}_K, \mathfrak{m}_K)$ on K extends valuation data $(\mathcal{O}_F, \mathfrak{m}_F)$ on F if and only if*

$$(\mathcal{O}_F, \mathfrak{m}_F) \leq (\mathcal{O}_K, \mathfrak{m}_K). \tag{1}$$

Proof. If $(\mathcal{O}'_F, \mathfrak{m}'_F)$ is the restriction of $(\mathcal{O}_K, \mathfrak{m}_K)$ to F, then (1) is equivalent to

$$(\mathcal{O}_F, \mathfrak{m}_F) \leq (F \cap \mathcal{O}_K, F \cap \mathfrak{m}_K) = (\mathcal{O}'_F, \mathfrak{m}'_F) \tag{2}$$

Since $(\mathcal{O}_F, \mathfrak{m}_F)$ is maximal, "$\leq$" in (2) can equivalently be replaced with "$=$". $\square$

3 Amalgamating valued fields

Theorem 3.1. *Let $(K_0, \mathcal{O}_{K_0})$ be a valued field, and $(K_i, \mathcal{O}_{K_i})$ be two valued field extensions for $i = 1, 2$. Then there is a valued field $(L, \mathcal{O}_L)$ and a diagram of valued fields*

$$\begin{array}{ccc} K_0 & \longrightarrow & K_1 \\ \downarrow & & \downarrow \\ K_2 & \longrightarrow & L \end{array}$$

Proof. For simplicity, we denote $\mathcal{O}_{K_i}$ by $\mathcal{O}_i$ and similarly $\mathfrak{m}_i$ and k_i. The ring $k_1 \otimes_{k_0} k_2$ is nonzero, so it has a prime ideal $\mathfrak{p}_0$. Let $\mathfrak{p} \in \operatorname{Spec} R$ be the pullback of $\mathfrak{p}_0$ along the map

$$R := \mathcal{O}_1 \otimes_{\mathcal{O}_0} \mathcal{O}_2 \to k_1 \otimes_{k_0} k_2.$$

For $i = 1, 2$, we have a commuting square of sets

$$\begin{array}{ccc} \operatorname{Spec} \mathcal{O}_i & \longleftarrow & \operatorname{Spec} k_i = \{*\} \\ \uparrow & & \uparrow \\ \operatorname{Spec} R = \operatorname{Spec} \mathcal{O}_1 \otimes_{\mathcal{O}_0} \mathcal{O}_2 & \longleftarrow & \operatorname{Spec} k_1 \otimes_{k_0} k_2 \end{array}$$

The pullback of $\mathfrak{p}$ to $\mathcal{O}_i$ is in the image of $\operatorname{Spec} k_i \to \operatorname{Spec} \mathcal{O}_i$, so it must be $\mathfrak{m}_i$.

Let $\mathfrak{q}$ be any minimal prime of R below $\mathfrak{p}$. Now $\mathcal{O}_1$ and K_2 are flat $\mathcal{O}_0$-modules by Lemma 1.3, so the natural map

$$\mathcal{O}_1 \otimes_{\mathcal{O}_0} \mathcal{O}_2 \hookrightarrow \mathcal{O}_1 \otimes_{\mathcal{O}_0} K_2 \hookrightarrow K_1 \otimes_{\mathcal{O}_0} K_2$$

is an injection. By Lemma 1.4, there is some prime $\mathfrak{q}_0$ in $K_1 \otimes_{\mathcal{O}_0} K_2$ which pulls back to $\mathfrak{q}$. Now for $i = 1, 2$, we have a commuting square of sets

$$\begin{array}{ccc} \operatorname{Spec} \mathcal{O}_i & \longleftarrow & \operatorname{Spec} K_i = \{*\} \\ \uparrow & & \uparrow \\ \operatorname{Spec} R = \operatorname{Spec} \mathcal{O}_1 \otimes_{\mathcal{O}_0} \mathcal{O}_2 & \longleftarrow & \operatorname{Spec} K_1 \otimes_{\mathcal{O}_0} K_2 \end{array}$$

The pullback of $\mathfrak{q}$ to $\mathcal{O}_i$ is in the image of $\operatorname{Spec} K_i \to \operatorname{Spec} \mathcal{O}_i$, so it must be (0).

Therefore each of the maps $\mathcal{O}_i \to R/\mathfrak{q}$ is injective, so the commuting square of domains

$$\begin{array}{ccc} \mathcal{O}_0 & \hookrightarrow & \mathcal{O}_1 \\ \downarrow & & \downarrow \\ \mathcal{O}_2 & \hookrightarrow & R/\mathfrak{q} \end{array}$$

yields a commuting square of fields

$$\begin{array}{ccc} K_0 & \longrightarrow & K_1 \\ \downarrow & & \downarrow \\ K_2 & \longrightarrow & \operatorname{Frac}(R/\mathfrak{q}) =: L \end{array}$$

Then for $i = 1, 2$,

$$(\mathcal{O}_i, \mathfrak{m}_i) \leq (R/\mathfrak{q}, \mathfrak{p}/\mathfrak{q})$$

because $\mathfrak{m}_i$ is the pullback of $\mathfrak{p}$ to $\mathcal{O}_i$. By Zorn's lemma and Lemma 2.1, we can find a valuation ring $\mathcal{O}_L$ on L such that

$$(\mathcal{O}_i, \mathfrak{m}_i) \leq (R/\mathfrak{q}, \mathfrak{p}/\mathfrak{q}) \leq (\mathcal{O}_L, \mathfrak{m}_L).$$

Then by Observation 2.3, the valuation structure on L extends those on K_1 and K_2. $\square$

4 Getting quantifier-elimination

Recall the following fact:

Fact 4.1. *Suppose T is a universal theory with the amalgamation property, T' is a theory extending T, and every model of T can be embedded into a model of T'. Suppose also that every model of T' is 1-existentially closed: for every inclusion of models $M \leq N$ of T', every non-empty quantifier-free M-definable subset of N^1 intersects M. Then T' has quantifier elimination and is the model completion of T.*

We would like to apply this in the case where T' is ACVF, and T is the theory of domains with a divisibility predicate arising from a valuation on their fraction field. Given the previous sections, it is not hard to see that T has the amalgamation property, and models of T embed into models of T'. So, to obtain quantifier elimination in T', one merely needs to prove 1-existential closedness. This can be proved by analyzing quantifier-free definable subsets of the home sort, and showing that they are all boolean combinations of balls.