Solutions to the Midterm

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1. (a) Let a_n be a sequence of real numbers and let L be a real number. Carefully define what it means to say that $\lim_{n\to\infty} a_n = L$.

   Solution. For every $\epsilon > 0$, there is a natural number N such that for all $n > N$, we have $|a_n - L| < \epsilon$. □

   (b) Now let a_n be a decreasing sequence of real numbers, and suppose a_n is bounded below. Prove carefully... that a_n converges.

   Proof. By assumption the set $S = \{a_n : n \in \mathbb{N}\}$ is bounded below. By the completeness axiom, S has a greatest lower bound L. We claim that

$$\lim_{n\to\infty} a_n = L \tag{1}$$

   so the sequence converges.

   We must show that for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that if $n > N$, then $|a_n - L| < \epsilon$. Let $\epsilon > 0$. Then $L + \epsilon > L$ so $L + \epsilon$ is not a lower bound of S. Therefore, some element of S is less than $L + \epsilon$. So there is some N such that $a_N < L + \epsilon$. We claim that if $n > N$, then $|a_n - L| < \epsilon$. Suppose $n > N$. As the sequence is decreasing,

$$a_n < a_N < L + \epsilon.$$

   Also, as L is a lower bound of S and $a_n \in S$,

$$L \leq a_n.$$

   But $\epsilon > 0$, so $L - \epsilon < L$, and thus

$$L - \epsilon < a_n.$$

   So

$$L - \epsilon < a_n < L + \epsilon, \text{ or equivalently } |a_n - L| < \epsilon.$$

   So we showed that whenever $n > N$, we have $|a_n - L| < \epsilon$.

   So, we showed that there is some N such that if $n > N$, then $|a_n - L| < \epsilon$.

As ϵ was an arbitrary positive number, we showed that for every $\epsilon > 0$, there was an N such that if $n > N$, then $|a_n - L| < \epsilon$.

So, we showed that $\lim_{n\to\infty} a_n = L$. It follows that $\lim_{n\to\infty} a_n$ exists, or equivalently, a_n is a convergent sequence. □

(c) Now let a_n be defined by $a_1 = 2$ and $a_{n+1} = \frac{a_n}{2}$. Show that $\lim_{n\to\infty} a_n = 0$.

Proof. We prove by induction on n that $a_n > a_{n+1} > 0$. For the base case, $n = 1$, and so $a_n = a_1 = 2$ and $a_{n+1} = a_2 = 1$, and indeed $2 > 1 > 0$.

For the inductive step, suppose $n > 1$ and

$$a_{n-1} > a_n > 0.$$

Dividing by two, we get

$$\frac{a_{n-1}}{2} > \frac{a_n}{2} > \frac{0}{2}.$$

But this is the same as

$$a_n > a_{n+1} > 0,$$

completing the inductive step.

Therefore, by induction we know that for all n, $a_n > a_{n+1}$ (so the sequence is decreasing), and $a_n > 0$ (so the sequence is bounded below by 0). Therefore, the previous problem applies, and

$$\lim_{n\to\infty} a_n = L \tag{2}$$

for some L.

By some limit laws

$$\frac{L}{2} = \frac{\lim_{n\to\infty} a_n}{2} = \lim_{n\to\infty} \frac{a_n}{2}.$$

But $\frac{a_n}{2} = a_{n+1}$, so

$$\frac{L}{2} = \lim_{n\to\infty} a_{n+1}.$$

By one of the secret limit laws,

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n,$$

so

$$\frac{L}{2} = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L.$$

So

$$\frac{L}{2} = L.$$

Multiplying both sides by 2 and subtracting L from both sides, we get that

$$0 = L.$$

Combining this with (2), we conclude that

$$\lim_{n\to\infty} a_n = 0.$$

□

2. State the alternating series test.

Solution. If $b_1, b_2, \ldots$ is a sequence such that

(a) $b_n > 0$ for all n
(b) $b_{n+1} \leq b_n$ for all n
(c) $\lim_{n\to\infty} b_n = 0$

then the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$$

converges. □

3. Show that the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{10n+5^n}$ converges.

Proof. For any n,

$$0 < 5^n < 10n + 5^n,$$

so for any n,

$$0 < \frac{1}{10n+5^n} < \frac{1}{5^n}.$$

By the comparison test,

$$\sum_{n=1}^{\infty} \frac{1}{10n+5^n}$$

converges if

$$\sum_{n=1}^{\infty} \frac{1}{5^n}$$

converges, which is true, because it's a geometric series with common ration $1/5$, and $|1/5| < 1$. □

(b) $\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!}$ diverges

Proof. Note that

$$\frac{2^n n!}{(n+2)!} = \frac{2^n n!}{(n+2)(n+1)n!} = \frac{2^n}{(n+2)(n+1)} = \frac{2^n}{n^2+3n+2}.$$

By a couple applications of L'Hopital's rule,

$$\lim_{x\to\infty} \frac{2^x}{x^2+3x+2} = \lim_{x\to\infty} \frac{(\ln 2)2^x}{2x+3} = \lim_{x\to\infty} \frac{(\ln 2)(\ln 2)2^x}{2} = \infty.$$

So $\lim_{n\to\infty} \frac{2^n n!}{(n+2)!} = \infty$, and the series diverges by the Test for Divergence. □

(c) The series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ converges.

Proof. Note that

$$(\ln n)^{\ln n} = \left(e^{\ln\ln n}\right)^{\ln n} = e^{(\ln\ln n)(\ln n)}.$$

If n is sufficiently big, then $\ln\ln n > 2$, and $\ln n$ is positive, so

$$(\ln\ln n)(\ln n) > 2\ln n$$

As the exponential function is increasing,

$$(\ln n)^{\ln n} = e^{(\ln\ln n)(\ln n)} > e^{2\ln n} = \left(e^{\ln n}\right)^2 = n^2.$$

So, for n sufficiently big,

$$0 < \frac{1}{(\ln n)^{\ln n}} < \frac{1}{n^2}.$$

Now $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges because it's a p-series. So by the comparison test,

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$$

converges as well. □

4. Give an example of a series that is conditionally convergent.

Solution.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

□

5. Calculate the integrals

(a) $\int_0^{\pi/2} \sin(x)\left(1+\cos^2(x)\right)\, dx$

Solution. If $u = \cos x$, then $du = -\sin(x)\,dx$, so

$$\int \sin(x)\left(1+\cos^2(x)\right)\,dx = -\int (1+u^2)\,du = -u - \frac{u^3}{3} + C = -\cos(x) - \frac{\cos^3(x)}{3} + C.$$

So

$$\begin{aligned}\int_0^{\pi/2} \sin(x)\left(1+\cos^2(x)\right)\,dx &= \left[-\cos(x) - \frac{\cos^3 x}{3}\right]_{x=0}^{x=\pi/2} \\ &= \left(-\cos(\pi/2) - \frac{(\cos\pi/2)^3}{3}\right) - \left(-\cos 0 - \frac{(\cos 0)^3}{3}\right) \\ &= (0-0) + \left(1 + \frac{1^3}{3}\right) = \frac{4}{3}.\end{aligned}$$

□

(b) $\int \frac{\ln(\tan x)}{\sin x \cos x}\,dx$

Solution. Let $u = \ln(\tan x)$. Then

$$du = \frac{1}{\tan x}\sec^2 x\,dx = \frac{\cos x}{\sin x}\frac{1}{\cos^2 x}dx = \frac{dx}{\sin x \cos x}.$$

So

$$\int \frac{\ln(\tan x)}{\sin x \cos x}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}\left(\ln(\tan x)\right)^2 + C.$$

□

(c) $\int \frac{6}{x^2+2x-8}\,dx$

Solution. We use partial fractions to write

$$\frac{6}{x^2+2x-8} = \frac{6}{(x+4)(x-2)} = \frac{A}{x+4} + \frac{B}{x-2}.$$

So, we want

$$\frac{A}{x+4} + \frac{B}{x-2} = \frac{A(x-2)+B(x+4)}{(x+4)(x-2)} \stackrel{?}{=} \frac{6}{(x+4)(x-2)}.$$

So, we need

$$A(x-2) + B(x+4) = 6.$$

Setting $x = 2$, we see that $B = 1$, and setting $x = -4$, we see that $A = -1$. So

$$\frac{6}{x^2+2x-8} = \frac{-1}{x+4} + \frac{1}{x-2}.$$

Thus

$$\int \frac{6}{x^2+2x-8}\,dx = (-1)\int \frac{dx}{x+4} + \int \frac{dx}{x-2} = -\ln|x+4| + \ln|x-2| + C.$$

□

6. Consider a real-valued function $f(x)$.

   (a) Write down a formula approximating $\int_3^5 f(x)\,dx$ obtained by using

      i. The midpoint rule with $n = 4$.

      Solution.
      $$\frac{1}{2}\left(f(3.25) + f(3.75) + f(4.25) + f(4.75)\right)$$
      □

      ii. The trapezoidal rule with $n = 4$.

      Solution.
      $$\frac{1}{4}\left(f(3) + 2f(3.5) + 2f(4) + 2f(4.5) + f(5)\right)$$
      □

      iii. Simpson's rule with $n = 4$.

      Solution.
      $$\frac{1}{6}\left(f(3) + 4f(3.5) + 2f(4) + 4f(4.5) + f(5)\right)$$
      □

   (b) State which of these three is typically most accurate, and which is typically least accurate.

   Solution. Typically, Simpson's rule is the most accurate, and the trapezoidal rule is the least accurate. □