Some review problems for Midterm 1

February 16, 2015

1 Problems

Here are some review problems to practice what we've learned so far. Problems a,b,c,d and p,q,r,s aren't from the book, and are a little tricky. You should maybe think of them more as *general facts you should know* rather than as practice problems. The remaining problems are from the book, and are practice problems from the reviews at the ends of chapters 11 and 7.

1.1 Series problems

(a) If $\sum_n a_n$ converges, and $|b_n| \leq a_n$ for all n, then $\sum_n b_n$ converges. Why?

(b) Suppose n is really big. Sort the following from least to greatest. Don't justify your answer unless you want to.

$$2^n, e^n, n^2, n^{-1}, n \log n, n, \log n, \sqrt{n}, \tan^{-1} n, e^{n^2}, \frac{n}{\log n}$$

If you only know how to sort some of these expressions, that's okay too. (Some of them are tricky to deal with).

(c) Suppose $\sum_n a_n$ converges and the a_n are all nonnegative. Show that $\sum (\sin n) a_n$ converges.

(d) Suppose that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists and is less than 1. Then

$$\sum_{n=1}^{\infty} n^4 a_n$$

converges. Why?

(11.R.13) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{n^3}{5^n}$$

(11.R.15) Determine whether the series is convergent or divergent

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

(11.R.17) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{\cos 3n}{1+(1.2)^n}$$

(11.R.18) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$$

(11.R.21) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1}$$

(11.R.34) For what values of x does the series $\sum_{n=1}^{\infty} (\ln x)^n$ converge?

1.2 Integration problems

(p) Integrate

$$\int \sec x \, dx$$

(q) Integrate

$$\int \sec^3 x \, dx$$

(r) Recall that $\cos 2x = \cos^2 x - \sin^2 x$. Derive the other two formulas for $\cos 2x$. Then derive the formulas for $\sin^2 x$ and $\cos^2 x$. Then evaluate the indefinite integrals

$$\int \cos^2 x \, dx \text{ and } \int \sin^2 x \, dx$$

(7.R.13) Evaluate the integral

$$\int e^{\sqrt[3]{x}} \, dx$$

(7.R.15) Evaluate the integral

$$\int \frac{x-1}{x^2+2x} \, dx$$

(7.R.17) Evaluate the integral

$$\int x \sec x \tan x \, dx$$

1.3 Challenge Integration Problems

These ones are only intended for people who found the previous sections *too easy*. I don't expect problems like this to appear on the exam.

(*) Find the area of a circle, as follows: Write out the corresponding integral. Do the indefinite integral by a trig substitution and double angle formulas. Use this to evaluate the definite integral.

(7.4.59) "The German mathematician Karl Weierstrass (1815-1897) noticed that the substitution $t = \tan(x/2)$ will convert any rational function of $\sin x$ and $\cos x$ into an ordinary rational function of t."

1. If $t = \tan(x/2)$, $-\pi < x < \pi$, sketch a right triangle or use trigonometric identities to show that
$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}} \qquad \text{and} \qquad \sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}.$$
2. Show that
$$\cos x = \frac{1-t^2}{1+t^2} \qquad \text{and} \qquad \sin x = \frac{2t}{1+t^2}$$
3. Show that
$$dx = \frac{2}{1+t^2}\, dt$$

(7.4.60) Use the substitution in Exercise 59 to transform the integrand into a rational function of t and then evaluate the integral.

$$\int \frac{dx}{1-\cos x}$$

2 Hints

2.1 Series problems

(a) Combine the comparison test (page 722) with absolute convergence (Theorem 3 on page 733). See example 3 on page 733 for an example.

(b) Generally speaking, exponential functions are bigger than polynomials which are bigger than logarithms. Also note that a couple expressions on the list *aren't* going to ∞ as $n \to \infty$. These will be the smallest. Incidentally, the way to prove any of these comparisons is to use L'Hospital's rule.

(c) This is an instance of (a).

(d) Use the ratio test.

(11.R.13) Try to bound n^3 in terms of an exponential, perhaps? Or, even better, use (d) above.

(11.R.15) The integral test applies!

(11.R.17) Somehow the 1.2^n is all that matters. Use something like (d) or (a) above to forget about the cosine, and limit comparison to forget about the "1+".

(11.R.18) Use the root test.

(11.R.21) Use the alternating series test. The bulk of the work goes into showing that some sequence is decreasing.

(11.R.34) The answer here is a couple lines long, and doesn't require any clever tricks–the problem can just be done "directly" because the series is geometric, and we know when geometric series converge.

2.2 Integration problems

(p) There's no strategy for doing this; you just have to memorize it. See formula 1 on page 475.

(q) This too requires a trick, namely, integration by parts with $u = \sec x$ and $v = \tan x$. See example 8 on page 475.

(r) The other two formulas for $\cos 2x$ are

$$\cos 2x = 2\cos^2 x - 1 \tag{1}$$

$$\cos 2x = 1 - 2\sin^2 x \tag{2}$$

Both of these are derived from the given formula, using the pythagorean identitiy $\cos^2 x + \sin^2 x = 1$. The formulas for $\cos^2 x$ and $\sin^2 x$ are the ones on the top of page 472, which are gotten by solving (1) and (2) for $\cos^2 x$ and $\sin^2 x$, respectively.

(7.R.13) Make the subsitution $x = u^3$ or equivalently, $u = \sqrt[3]{x}$, to turn this into something to which integration by parts can be applied.

(7.R.15) Use partial fractions! (The rational function here is *proper* so we don't have to do the preliminary long division step. See page 485). The denominator factors as $(x + 2)$ times x, so we are going to write the integrand as *something* divided by $x + 2$, plus *something* divided by x.

(7.R.17) Use integration by parts: one thing here is especially easy to integrate.

2.3 Challenge Integration Problems

(*) The corresponding integral is

$$2\int_{-1}^{1}\sqrt{1-x^2}\,dx. \tag{3}$$

So the problem boils down to evaluating

$$2\int\sqrt{1-x^2}\,dx.$$

Since we see a $\sqrt{1-x^2}$, we make the substitution $x=\sin\theta$, in accordance with the table on page 478. You end up with a $\cos^2\theta$, to which you must apply the double angle formulas. Ultimately you end up with a peculiar formula for the antiderivative of $\sqrt{1-x^2}$, which you can use to evaluate the definite integral (3).

(7.4.59)

1. Use the identity $\sec^2 = 1+\tan^2$ to turn the expression $\sqrt{1+t^2}$ into something simpler.
2. We have double angle formulas which express cosine and sine of twice $x/2$ in terms of cosine and sine of $x/2$.
3. Take differentials of both sides of $t=\tan(x/2)$ and fiddle around with things, I guess. Or, just solve for x in terms of t and take the derivative of the resulting function.

(7.4.60) After making the substitutions, you should have this:

$$\int\frac{2/(1+t^2)}{1-\frac{1-t^2}{1+t^2}}\,dt$$

Then simplify the integrand.

3 Solutions

3.1 Series problems

(a) If $\sum_n a_n$ converges, and $|b_n|\le a_n$ for all n, then $\sum_n b_n$ converges. Why?

Solution. Note that $0\le|b_n|\le a_n$ for every n, so by the comparison test, $\sum_n|b_n|$ converges. Therefore, $\sum_n b_n$ *absolutely* converges, hence converges. □

(b) Suppose n is really big. Sort the following from least to greatest. Don't justify your answer unless you want to.

$$2^n, e^n, n^2, n^{-1}, n\log n, n, \log n, \sqrt{n}, \tan^{-1} n, e^{n^2}, \frac{n}{\log n}$$

If you only know how to sort some of these expressions, that's okay too. (Some of them are tricky to deal with).

Solution. For n really big,

$$n^{-1} < \tan^{-1} n < \log n < \sqrt{n} < \frac{n}{\log n} < n < n \log n < n^2 < 2^n < e^n < e^{n^2}$$

For example, $\sqrt{n} < \frac{n}{\log n}$ because

$$\lim_{x\to\infty} \frac{x/\log x}{\sqrt{x}} = \lim_{x\to\infty} \frac{x}{\sqrt{x}} \cdot \frac{1}{\log x} = \lim_{x\to\infty} \frac{x^{1/2}}{\log x},$$

which, by L'Hospital's, is

$$\lim_{x\to\infty} \frac{1/2x^{-1/2}}{x^{-1}} = \frac{1}{2} \lim_{x\to\infty} x^{1-1/2} = \infty.$$

Since $x/\log x$ divided by $\sqrt{x}$ approaches $+\infty$, eventually $x/\log x$ must exceed $\sqrt{x}$. □

(c) Suppose $\sum_n a_n$ converges and the a_n are all nonnegative. Show that $\sum(\sin n)a_n$ converges.

Solution. The sine function always takes values between -1 and 1, so $|\sin n| \leq 1$. As the a_n are nonnegative,

$$0 \leq |\sin n| a_n \leq a_n.$$

So by the comparison test,

$$\sum_n |\sin n| a_n = \sum_n |(\sin n) a_n|$$

converges. This means that

$$\sum_n (\sin n) a_n$$

converges *absolutely*, so it also converges. □

(d) Suppose that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists and is less than 1. Then

$$\sum_{n=1}^{\infty} n^4 a_n$$

converges. Why?

Solution. By the ratio test, it suffices to show that

$$\lim_{n\to\infty} \left| \frac{(n+1)^4 a_{n+1}}{n^4 a_n} \right|$$

exists and is less than 1. Now, this is the same thing as

$$\lim_{n\to\infty} \frac{(n+1)^4}{n^4} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^4}{n^4} \cdot \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

by the product law for limits. And

$$\lim_{n\to\infty} \frac{(n+1)^4}{n^4} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^4 = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^4 = 1.$$

Therefore

$$\lim_{n\to\infty} \left|\frac{(n+1)^4 a_{n+1}}{n^4 a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^4}{n^4} \cdot \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

which is less than 1, by assumption. □

(11.R.13) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{n^3}{5^n}$$

Solution. The series is convergent, by the ratio test:

$$\lim_{n\to\infty} \frac{(n+1)^3/5^{n+1}}{n^3/5^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \frac{5^n}{5^{n+1}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^3 \frac{1}{5} = \frac{1}{5}$$

which is less than 1. □

(11.R.15) Determine whether the series is convergent or divergent

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

Solution. We claim that the integral test applies, using the function $f(x) = \frac{1}{x\sqrt{\ln x}}$. If $x > 1$, then $\ln x$ is positive, so $f(x) > 0$. We also need to check that $f(x)$ is eventually decreasing, so we calculate:

$$f'(x) = \frac{-1}{\left(x\sqrt{\ln x}\right)^2} \left(x \frac{1}{2\sqrt{\ln x}} \frac{1}{x} + \sqrt{\ln x}\right)$$

As long as $x > 1$, everything there will be positive *except* for the -1, so $f'(x)$ itself is negative. Therefore f is decreasing. (Alternatively, we could argue that $\ln x$ is increasing, so $\sqrt{\ln x}$ is increasing. And x is increasing, so $x\sqrt{\ln x}$ is increasing since both are positive. Thus the reciprocal $\frac{1}{x\sqrt{\ln x}}$ is *decreasing*.)

At any rate, the integral test applies, and we calculate the limit

$$\lim_{N\to\infty}\int_2^N \frac{dx}{x\sqrt{\ln x}} = \cdots = ?$$

Well, the indefinite integral is not so hard, if we make the u-substitution $u = \ln x$

$$\int \frac{dx}{x\sqrt{\ln x}} = \int \frac{1}{\sqrt{\ln x}} \cdot \frac{dx}{x} = \int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{\ln x} + C.$$

So

$$\int_2^N \frac{dx}{x\sqrt{\ln x}} = 2\sqrt{\ln x} - 2\sqrt{\ln 2},$$

which goes to ∞ in the limit as $x \to \infty$. So, since

$$\lim_{N\to\infty}\int_2^N f(x)\,dx = \lim_{N\to\infty}\int_2^N \frac{dx}{x\sqrt{\ln x}} = \infty,$$

it follows by the integral test that

$$\sum_{n=2}^{\infty} f(n) = \sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}} \text{ diverges.}$$

□

(11.R.17) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{\cos 3n}{1+(1.2)^n}$$

Solution. The series converges. First of all, note that for any n,

$$1 + (1.2)^n > 1.2^n,$$

so

$$\frac{1}{1+(1.2)^n} < \frac{1}{1.2^n}.$$

And both sides are positive. Since $\sum_{n=1}^{\infty} \frac{1}{1.2^n}$ converges (it's a geometric series), the comparison test tells us that

$$\sum_{n=1}^{\infty} \frac{1}{1+(1.2)^n} \text{ also converges.}$$

Meanwhile, $\cos 3n$ is always between -1 and 1, so for any n,

$$0 \le \frac{|\cos 3n|}{1+(1.2)^n} \le \frac{1}{1+(1.2)^n}.$$

By the comparison test *again*, it follows that

$$\sum_{n=1}^{\infty} \frac{|\cos 3n|}{1+(1.2)^n} \text{ also converges.}$$

Now

$$\left|\frac{\cos 3n}{1+(1.2)^n}\right| = \frac{|\cos 3n|}{1+(1.2)^n}$$

so

$$\sum_{n=1}^{\infty} \frac{\cos 3n}{1+(1.2)^n}$$

absolutely converges. Therefore it converges. □

(11.R.18) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$$

Solution. There are nth powers floating around, so we use the root test.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^{2n}}{(1+2n^2)^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{(n^2)^n}}{\sqrt[n]{(1+2n^2)^n}} = \lim_{n\to\infty} \frac{n^2}{1+2n^2} = \frac{1}{2},$$

which is less than 1, so the series converges. □

(11.R.21) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1}$$

Solution. This series converges, by the alternating series test. To apply the alternating series test, it suffices to show that the sequence

$$b_n = \frac{\sqrt{n}}{n+1}$$

is eventually decreasing, and goes to 0 in the limit. The latter is not that hard, since

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = \lim_{n\to\infty} \frac{1}{n^{1/2}+n^{-1/2}} = \text{“}\frac{1}{\infty}\text{”} = 0,$$

using the fact that $\lim_{n\to\infty} n^{1/2} + n^{-1/2} = \infty$. Alternatively, one could write

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = \left(\lim_{n\to\infty} \frac{\sqrt{n}}{n}\right) \cdot \left(\lim_{n\to\infty} \frac{n}{n+1}\right) = \left(\lim_{n\to\infty} n^{-1/2}\right) \cdot \lim_{n\to\infty} \frac{1}{1+1/n} = 0 \cdot 1 = 0.$$

Next, we check that the function $f(x) = \sqrt{x}x + 1$ is eventually decreasing, by taking its derivative:

$$f'(x) = \frac{(x+1)\frac{1}{2\sqrt{x}} - \sqrt{x}}{(x+1)^2}.$$

The denominator is always positive (for big enough x), so what about the numerator? Well,

$$\frac{(x+1)}{2\sqrt{x}} - \sqrt{x} = \frac{(x+1)/2}{\sqrt{x}} - \frac{x}{\sqrt{x}} = \frac{(x+1)/2 - x}{\sqrt{x}} = \frac{1/2 - x/2}{\sqrt{x}}.$$

This is negative when x is sufficiently large, because $1/2 - x/2$ will be negative, and $\sqrt{x}$ will be positive. Thus, $f'(x) < 0$ for $x \gg 0$, and therefore $f(x)$ is eventually decreasing. It follows that eventually, the sequence b_n is decreasing, so the alternating series test applies, and

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n+1} \text{ does converge.}$$

□

(11.R.34) For what values of x does the series $\sum_{n=1}^{\infty}(\ln x)^n$ converge?

Solution. This is a geometric series with common ratio $\ln x$, so it converges exactly when

$$-1 < \ln x < 1,$$

or equivalently, when

$$e^{-1} < x < e^{1}.$$

□

3.2 Integration problems

(p) Integrate

$$\int \sec x \, dx$$

Solution. Rewrite it as

$$\int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\, dx = \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x}\, dx.$$

Now we make the u substitution

$$u = \sec x + \tan x$$
$$du = (\sec^2 x + \sec x \tan x)\, dx$$

So

$$\int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x}\, dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C.$$

So the final answer is

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C.$$

□

(q) Integrate

$$\int \sec^3 x\, dx$$

Proof. For some reason the trick here is to do integration by parts, with

$$\begin{aligned} u &= \sec x \\ du &= \sec x \tan x\, dx dv \qquad\qquad = \sec^2 x\, dx \\ v &= \tan x \end{aligned}$$

Thus

$$\begin{aligned} \int \sec^3 x\, dx = \int (\sec x)(\sec^2 x\, dx) = \int u\, dv = uv - \int v\, du \\ = (\sec x)(\tan x) - \int \tan x(\sec x \tan x)\, dx. \end{aligned}$$

So

$$\begin{aligned} \int \sec^3 x\, dx &= \sec x \tan x - \int \sec x \tan^2 x\, dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)\, dx \\ &= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx. \end{aligned}$$

Adding $\int \sec^3 x\, dx$ to both sides, we get

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx = \sec x \tan x + \ln|\sec x + \tan x|.$$

Then, dividing both sides by 2 and adding the constant of integration, we get the bizarre answer

$$\int \sec^3 x\, dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C.$$

□

(r) Recall that $\cos 2x = \cos^2 x - \sin^2 x$. Derive the other two formulas for $\cos 2x$. Then derive the formulas for $\sin^2 x$ and $\cos^2 x$. Then evaluate the indefinite integrals

$$\int \cos^2 x\, dx \text{ and } \int \sin^2 x\, dx$$

Solution. If we add the equations

$$1 = \cos^2 x + \sin^2 x \tag{4}$$

$$\cos 2x = \cos^2 x - \sin^2 x \tag{5}$$

we get

$$1 + \cos 2x = 2\cos^2 x. \tag{6}$$

If we instead subtract Equation (5) from Equation (4), we instead get

$$1 - \cos 2x = 2\sin^2 x. \tag{7}$$

Solving these for $\cos 2x$, we get

$$\cos 2x = 2\cos^2 x - 1$$

and

$$\cos 2x = 1 - 2\sin^2 x.$$

If we instead divide (6) and (7) by 2, we get the formulas for $\cos^2 x$ and $\sin^2 x$:

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

Using these, we can do the integrals

$$\int \cos^2 x\, dx = \frac{1}{2}\int (1 + \cos 2x)\, dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$$

$$\int \sin^2 x\, dx = \frac{1}{2}\int (1 - \cos 2x)\, dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C$$

□

(7.R.13) Evaluate the integral

$$\int e^{\sqrt[3]{x}}\, dx$$

Solution. We do a u-substitution setting $u = \sqrt[3]{x}$. Thus $x = u^3$, and $dx = 3u^2\,du$, so that

$$\int e^{\sqrt[3]{x}}\,dx = \int e^u 3u^2\,du = 3\int u^2 e^u\,du$$

To evaluate this integral, we do integration by parts:

$$\int 3u^2 e^u\,du = (3u^2 e^u) - \int 6ue^u\,du,$$

where "u" was $3u^2$ and "dv" was $e^u\,du$ so that "v" was e^u. Next, we use integration by parts again:

$$\int 6ue^u\,du = (6ue^u) - \int 6e^u\,du = 6ue^u - 6e^u + C.$$

So

$$\int 3u^2 e^u\,du = 3u^2 e^u - \int 6ue^u\,du = 3u^2 e^u - 6ue^u + 6e^u + C = e^u(3u^2 - 6u + 6) + C.$$

Replacing u with the original expression, we get

$$\int e^{\sqrt[3]{x}}\,dx = e^{\sqrt[3]{x}}(3x^{2/3} - 6x^{1/3} + 6) + C.$$

□

Let's double check our answer:

$$\begin{aligned}\frac{d}{dx}e^{\sqrt[3]{x}}(3x^{2/3} - 6x^{1/3} + 6) &= e^{\sqrt[3]{x}}\frac{1}{3}x^{-2/3}(3x^{2/3} - 6x^{1/3} + 6) + e^{\sqrt[3]{x}}(2x^{-1/3} - 2x^{-2/3}) \\ &= e^{\sqrt[3]{x}}(1 - 2x^{-1/3} + 2x^{-2/3} + 2x^{-1/3} - 2x^{-2/3}) = e^{\sqrt[3]{x}}.\end{aligned}$$

(7.R.15) Evaluate the integral

$$\int \frac{x-1}{x^2+2x}\,dx$$

Solution. We use the strategy of partial fractions, so we need to write

$$\frac{x-1}{x^2+2x} = \frac{x-1}{(x+2)x}$$

as

$$\frac{A}{x+2} + \frac{B}{x}$$

for some constants $A, B \in \mathbb{R}$. We want the following identity to hold

$$\frac{x-1}{(x+2)x} = \frac{A}{x+2} + \frac{B}{x}$$

for all x. Clearing denominators, this turns in to

$$x - 1 = A \cdot x + B \cdot (x+2) = (A+B)x + 2B.$$

This will work if $A + B = 1$ and $2B = -1$. So B must be $-1/2$ and $A = 3/2$. Thus, we hope that

$$\frac{x-1}{x(x+2)} \stackrel{?}{=} \frac{3/2}{x+2} - \frac{1/2}{x} = \frac{3x - 1(x+2)}{2(x+2)x} = \frac{2x-2}{2(x+2)x} = \frac{x-1}{x(x+2)}.$$

Great!

Now we can integrate

$$\begin{aligned}\int \frac{x-1}{x^2+2x}\,dx = \int \left(\frac{3/2}{x+2} - \frac{1/2}{x}\right) dx = \frac{3}{2}\int \frac{dx}{x+2} - \frac{1}{2}\int \frac{dx}{x} \\ = \frac{3}{2}\ln(x+2) - \frac{1}{2}\ln x.\end{aligned}$$

□

(7.R.17) Evaluate the integral

$$\int x \sec x \tan x \, dx$$

Proof. We use integration by parts, with

$$\begin{aligned} u &= x \\ du &= dx \\ dv &= \sec x \tan x \, dx \\ v &= \sec x \end{aligned}$$

So

$$\begin{aligned}\int x \sec x \tan x \, dx &= \int u \, dv = uv - \int v \, du \\ &= x \sec x - \int \sec x \, dx = x \sec x - \ln|\sec x + \tan x| + C\end{aligned}$$

□

3.3 Challenge Integration Problems

(*) Find the area of a circle, as follows: Write out the corresponding integral. Do the indefinite integral by a trig substitution and double angle formulas. Use this to evaluate the definite integral.

Solution. If we have a circle of radius r, the corresponding integral is

$$\int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx$$

Let's focus on the indefinite integral, for simplicity:

$$\int 2\sqrt{r^2 - x^2}\, dx$$

This looks like we should do a trig substitution, so we do:

$$x = r \sin\theta$$

$$dx = r\cos\theta\, d\theta$$

$$\sqrt{r^2 - x^2} = \sqrt{r^2 - r^2\sin^2\theta} = \sqrt{r^2\cos^2\theta} \approx r\cos\theta.$$

(Let's ignore issues like $\cos\theta$ vs $|\cos\theta|$, and just check at the end that whatever antiderivative we got is correct.)

So, the integral can be rewritten as

$$\int 2\sqrt{r^2 - x^2}\, dx = \int 2r\cos\theta r\cos\theta\, d\theta = r^2 \int 2\cos^2\theta\, d\theta.$$

Now we use the double angle formula

$$2\cos^2\theta = 1 + \cos 2\theta$$

to rewrite this as

$$r^2 \int (1 + \cos 2\theta)\, d\theta = r^2(\theta + \frac{1}{2}\sin 2\theta).$$

Using the double angle formula for sine, we note

$$\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta,$$

so that ultimately

$$r^2 \int 2\cos^2\theta\, d\theta = r^2 \int (1 + \cos 2\theta)\, d\theta = r^2(\theta + \sin\theta\cos\theta).$$

Now, we need to convert this back into a formula involving x. Since x was $r\sin\theta$, we see that

$$\sin\theta = \frac{x}{r}$$

$$\theta = \sin^{-1}\frac{x}{r}$$

Also, we decided that $\sqrt{r^2 - x^2} = r\cos\theta$, so

$$\cos\theta = \frac{\sqrt{r^2 - x^2}}{r}.$$

So, putting everything together, we hope that

$$\int 2\sqrt{r^2 - x^2}\, dx = r^2 \left(\sin^{-1}\frac{x}{r} + \frac{x}{r}\frac{\sqrt{r^2 - x^2}}{r} \right) = r^2 \sin^{-1}\frac{x}{r} + x\sqrt{r^2 - x^2}.$$

To check this, let's take the derivative:

$$\frac{d}{dx}\left(r^2 \sin^{-1}\frac{x}{r} + x\sqrt{r^2 - x^2} \right) =$$

$$r^2 \frac{1}{\sqrt{1 - x^2/r^2}}\frac{1}{r} + \sqrt{r^2 - x^2} + x\frac{1}{2\sqrt{r^2 - x^2}}(-2x) =$$

$$\frac{r^2}{r\sqrt{1 - x^2/r^2}} + \frac{r^2 - x^2}{\sqrt{r^2 - x^2}} + \frac{-x^2}{\sqrt{r^2 - x^2}} =$$

$$\frac{r^2}{\sqrt{r^2 - x^2}} + \frac{r^2 - 2x^2}{\sqrt{r^2 - x^2}} = \frac{2r^2 - 2x^2}{\sqrt{r^2 - x^2}} = 2\sqrt{r^2 - x^2}.$$

Whew.

Now, we can evalute the definite integral:

$$\begin{aligned}
\int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx &= \left[r^2 \sin^{-1}\frac{x}{r} + x\sqrt{r^2 - x^2} \right]_{x=-r}^{x=r} \\
&= \left(r^2 \sin^{-1}\frac{r}{r} + r\sqrt{r^2 - r^2} \right) - \left(r^2 \sin^{-1}\frac{-r}{r} + r\sqrt{r^2 - (-r)^2} \right) \\
&= r^2 \sin^{-1} 1 + 0 - r^2 \sin^{-1}(-1) - 0 = r^2\frac{\pi}{2} - r^2\frac{-\pi}{2} \\
&= \pi r^2.
\end{aligned}$$

□

(7.4.59) 1. If $t = \tan(x/2)$, $-\pi < x < \pi$, sketch a right triangle or use trigonometric identities to show that

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1 + t^2}} \qquad \text{and} \qquad \sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1 + t^2}}.$$

Proof. Note

$$1 + t^2 = 1 + \tan^2(x/2) = \sec^2(x/2).$$

Since $-\pi < x < \pi$, the value $x/2$ is between $-\pi/2$ and $\pi/2$, so the corresponding angle is in the first or fourth quadrant. Therefore $\cos(x/2)$ and $\sec(x/2)$ are positive. So

$$\sqrt{1+t^2} = \sqrt{\sec^2(x/2)} = |\sec(x/2)| = \sec(x/2).$$

And then

$$\frac{1}{\sqrt{1+t^2}} = \frac{1}{\sec(x/2)} = \cos(x/2).$$

Meanwhile,

$$\sin(x/2) = \tan(x/2)\cos(x/2) = t\frac{1}{1+t^2} = \frac{t}{1+t^2}$$

as claimed. □

2. Show that

$$\cos x = \frac{1-t^2}{1+t^2} \qquad \text{and} \qquad \sin x = \frac{2t}{1+t^2}$$

Proof. Well, by the double angle formulas, applied to the angle $x/2$,

$$\begin{aligned}\cos x = \cos(x/2)^2 - \sin(x/2)^2 &= \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2 \\ &= \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2}.\end{aligned}$$

And

$$\sin x = 2\sin(x/2)\cos(x/2) = 2\frac{1}{\sqrt{1+t^2}}\frac{t}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$$

□

3. Show that

$$dx = \frac{2}{1+t^2}\,dt$$

Proof. Well, we know that

$$t = \tan(x/2)$$

so

$$x = 2\tan^{-1} t$$

and then

$$\frac{dx}{dt} = \frac{2}{1+t^2}$$

so that

$$dx = \frac{2\,dt}{1+t^2}.$$

□

(7.4.60) Use the substitution in Exercise 59 to transform the integrand into a rational function of t and then evaluate the integral.

$$\int \frac{dx}{1-\cos x}$$

Solution. Replacing dx with $\frac{2\,dt}{1+t^2}$ and $\cos x$ with $\frac{1-t^2}{1+t^2}$, we get

$$\int \frac{dx}{1-\cos x} = \int \frac{\frac{2}{1+t^2}\,dt}{1-\frac{1-t^2}{1+t^2}}$$

The integrand is now

$$\frac{2/(1+t^2)}{1-\frac{1-t^2}{1+t^2}}.$$

Multiplying numerator and denominator by $1+t^2$, this becomes

$$\frac{2}{(1+t^2)-(1-t^2)} = \frac{2}{2t^2} = t^{-2}.$$

So

$$\int \frac{dx}{1-\cos x} = \int \frac{\frac{2}{1+t^2}\,dt}{1-\frac{1-t^2}{1+t^2}} = \int t^{-2}\,dt =$$

$$= -t^{-1} + C = \frac{-1}{\tan(x/2)} + C.$$

□