Limits of Sequences

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Recall that $\mathbb{N}$ denotes the natural numbers $\{1, 2, 3, 4, \ldots\}$, and $\mathbb{R}$ denotes the real numbers, i.e., all the numbers (except for imaginary and complex numbers).[1]

1 The limit of a sequence

Let $a_1, a_2, \ldots$ be a sequence of real numbers, and let L be some real number. Here's the official definition of what it means for the sequence to converge to the number L:

Definition. *"$\lim_{n\to\infty} a_n = L$" means that for every positive number $\epsilon > 0$, there is a natural number $N \in \mathbb{N}$, such that $n > N$ implies $|a_n - L| < \epsilon$.*

Here's another way of phrasing this:

Definition. *"$\lim_{n\to\infty} a_n = L$" means that for every positive number $\epsilon > 0$, there is a natural number $N \in \mathbb{N}$, such that for every larger natural number $n > N$, we have $|a_n - L| < \epsilon$.*

1.1 Close to L

If we think of ϵ as a fixed constant, what does that last bit mean? Well,

$$|a_n - L| < \epsilon$$

could also be written as

$$L - \epsilon < a_n < L + \epsilon$$

or

$$a_n \in (L - \epsilon, L + \epsilon).$$

Here, $\in$ indicates set-membership: a_n is a member of the open interval $(L - \epsilon, L + \epsilon)$. These three statements are logically equivalent. Informally, each means that a_n is "close" to L.

[1] Symbols like $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ are also called "blackboard bold" letters, and they generally have set meanings within mathematics, unlike letters like N, n, L, ϵ which are variables.

1.2 Almost all

Next, what does it mean that

> there is some N such that for every $n > N$, something-or-other holds

? It means that something-or-other holds for *all sufficiently large* n, or for *almost all* n, or for *all but finitely many* n, or that it "eventually" holds.

For example, if we have some sequence of integers $a_1, a_2, \ldots$, the following statements are all equivalent:

- For all sufficiently large n, the nth term a_n is an even number.
- All but finitely many terms in the sequence $a_1, a_2, \ldots$ are even numbers.
- After throwing out a finite initial segment of the sequence, all the remaining terms are even numbers.
- As n gets bigger, eventually all terms a_n are even numbers.
- There is a natural number N such that if $n > N$, then a_n is an even number.
- There is a natural number N such that "$n > N$" implies "a_n is even."

As an example, if our sequence is the Fibonacci numbers

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

then all of these statements are *false*. But if our sequence is the sequence of factorial numbers

$$1, 2, 6, 24, 120, 720, 5040, \ldots$$

then all of these statements are *true*.

Informally, we could think of all the above equivalent statements as saying that "almost all" terms in the sequence are even.

NOW, replace "is an even number" with "is close (within ϵ) to L" in the above discussion. We see that the somewhat opaque statement

> There is an N such that $n > N$ implies $|a_n - L| < \epsilon$

really just means

> "Almost all" terms in the sequence $a_1, a_2, \ldots$ are "close" (within ϵ) to L.

1.3 Putting things together

The original definition of

$$\lim_{n\to\infty} a_n = L,$$

was

> For every positive number $\epsilon > 0$, there is a natural number $N \in \mathbb{N}$ such that $n > N$ implies $|a_n - L| < \epsilon$.

We can basically paraphrase this as follows

> For every positive number $\epsilon > 0$, almost all terms of the sequence are within ϵ of L.

The notion of "close to L" is vague, but we can think of the positive number ϵ as a *choice* of what "close" means. So, we could paraphrase the definition further, as

> For any choice of what it means to be "close to L," almost all terms in the sequence are "close to L."

1.4 An example

Let a_n be π rounded to n decimal places. So

$$\begin{aligned} a_1 &= 3 \\ a_2 &= 3.1 \\ a_3 &= 3.14 \\ a_4 &= 3.141 \\ a_5 &= 3.1415 \\ &\vdots \end{aligned}$$

This sequence converges to π. This means that for any ϵ, eventually all terms are within ϵ of π. For example,

- If $\epsilon = 0.001$, then "close to π" means that your distance from π is less than 0.001. So the first few terms aren't close to π, but 3.141 and all subsequent terms are close to π.
- If $\epsilon = 10$, then "close to π" means within ten of π (between -6.86 and 13.14). *All* terms of the sequence are distance less than 10 from π.

This same sequence does *not* converge to, say, 10.[2] Let's see why $\lim_{n\to\infty} a_n \neq 10$.

[2] The notation $\lim_{n\to\infty} a_n = L$ suggests that a sequence can converge to at most one number—it shouldn't be able to converge to π and also converge to 10. This *turns out* to be true, but requires a non-obvious argument.

Well, the terms in the sequence *do* get closer and closer to 10. Each number a_n is closer to 10 than the previous number a_{n-1} was. However, they don't get *arbitrarily close* to 10. If we decide that "close to 10" means within 1 of 10 (i.e., between 9 and 11), then it's *not* true that almost all terms are "close to 10." In fact, none of the terms are close to 10. On the other hand, if we decided that "close to 10" meant within 8 of 10 (i.e., between 2 and 18), then *all* the terms are "close to 10." So the choice of ϵ matters! But in order for the limit to be 10, all values of ϵ need to work. All notions of "closeness" need to work.[3]

2 Game semantics

Here is another way of thinking about what

$$\lim_{n\to\infty} a_n = L$$

means.

Assume that we have some fixed sequence $a_1, a_2, \ldots$. If you like, you can take the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \ldots,$$

so the nth term a_n is n^{-1}.

Also, assume there's some fixed number L, such as 0 or 1.

Now consider the following game between two players, Abelard and Eloise:

1. First, Abelard chooses a positive number ϵ
2. Then, Eloise chooses a natural number N
3. Then Abelard chooses a natural number n which is required to be bigger than N.
4. If $|a_n - L| < \epsilon$, then Eloise wins. Otherwise, Abelard wins.

For example, with the sequence $a_n = n^{-1}$ and the number $L = 1$, the game might go as follows:

1. Abelard chooses $\epsilon = 1/2$.
2. Eloise chooses $N = 1$.
3. Abelard chooses $n = 100$.
4. Now
$$|a_n - L| = |a_{100} - 1| = |1/100 - 1| = 99/100 \not< 1/2 = L,$$
so Eloise loses and Abelard wins.

[3]The exact reason why a sequence can't converge to two different numbers L and L' is the following: one can always define "close" in such a way that no number can be close to *both* L and L'. Then there is no way that almost all terms of the sequence can be close to L, and almost all can be close to L'.

The point of this game is that...

- $\lim_{n\to\infty} a_n = L$ means that Eloise has a (guaranteed) winning strategy
- $\lim_{n\to\infty} a_n \neq L$ means that Abelard has a (guaranteed) winning strategy.

To prove that $\lim_{n\to\infty} a_n = L$, one essentially must give a winning strategy for Eloise. This involves deciding how to choose the natural number N, in terms of Abelard's first move ϵ. Eloise can't control what happens after she chooses N, so the only way she can guarantee a win is if $n > N$ entirely implies $|a_n - L| < \epsilon$.

A proof that $\lim_{n\to\infty} a_n = L$ generally has the form

> Let $\epsilon > 0$ be given. Put $N =$[SOMETHING]. Suppose $n > N$... [Logic]... Then $|a_n - L| < \epsilon$. So $n > N$ implies $|a_n - L| < \epsilon$. Therefore $\lim_{n\to\infty} a_n = L$. $\square$

The trick here is figuring out what "[SOMETHING]" should be. Figuring this out essentially amounts to choosing Eloise's strategy. The part marked as "[Logic]" is the justification that this strategy works.

There may be many different strategies which work, and this is part of what makes ϵ, N-proofs difficult to write. A general guideline is to try and solve the inequality $|a_n - L| < \epsilon$ for n, and then choose N larger than all *non*-solutions.

2.1 Why ϵ is small, and N is big

We often informally say that ϵ is a tiny number and N is a big number. The precise reason we say this is basically the following. In the above game, Abelard has no reason to choose ϵ to be big. Making ϵ smaller only makes his life easier—it makes it harder for Eloise to win. Ideally, Abelard would choose the smallest positive number possible, but there isn't one. Nevertheless, he *may as well* decide to always choose ϵ to be less than 1, or less than 0.1, or less than 10^{-100}.

Likewise, Eloise would like to choose N to be as large as possible. Choosing a larger and larger value of N has no effect other than restricting Abelard's choices in the next turn of the game. So, we can basically assume that Eloise only chooses values of N which are bigger than 100, or bigger than 10^{100}.