Answers to some problems from 7.1

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In class I recommended a few medium-difficulty problems on integration by parts, from section 7.1 of the textbook. Some people requested solutions, so here they are:

7.1.13

$$\int t \sec^2 2t \, dt$$

Solution. Since this is a polynomial times a non-inverse trig function, it seems like we probably want to be differentiating t and integrating $\sec^2 2t$, especially since the latter sounds easy to integrate.

So, take $u = t$ and $dv = \sec^2 2t \, dt$. Then $du = dt$, and

$$v = \int \sec^2 2t \, dt.$$

To integrate this, we do a substitution: $\theta = 2t$. Then $dt = d\theta/2$, so

$$v = \int \sec^2 2t \, dt = \int \frac{\sec^2 \theta \, d\theta}{2} = \frac{1}{2} \tan \theta = \frac{1}{2} \tan 2t.$$

So we have

$$\begin{aligned} u &= t & dv &= \sec^2 2t \, dt \\ du &= dt & v &= \frac{1}{2} \tan 2t \end{aligned}$$

Thus

$$\begin{aligned} \int t \sec^2 2t \, dt &= \int u \, dv = uv - \int v \, du = \frac{1}{2} t \tan 2t - \int \frac{1}{2} \tan 2t \, dt \\ &= \frac{t \tan 2t}{2} - \frac{1}{4} \ln |\sec 2t| + C, \end{aligned}$$

where we have used the fact that $\int \tan x \, dx = \ln |\sec x|$. □

7.1.24

$$\int_0^1 (x^2+1)e^{-x}\,dx$$

Solution. We use integration by parts, letting $u = (x^2+1)$ and $dv = e^{-x}\,dx$. We take $v = -e^{-x}$, so

$$\begin{aligned} u &= (x^2+1) & dv &= e^{-x}\,dx \\ du &= 2x\,dx & v &= -e^{-x} \end{aligned}$$

Thus

$$\begin{aligned} \int_0^1 (x^2+1)e^{-x}\,dx &= \left[(x^2+1)(-e^{-x})\right]_0^1 + \int_0^1 2x\cdot e^{-x}\,dx \\ &= -(1^2+1)e^{-1} + (0^2+1)e^0 + \int_0^1 2x\cdot e^{-x}\,dx \\ &= -2/e + 1 + \int_0^1 2x\cdot e^{-x}\,dx. \end{aligned}$$

To evaluate the integral $\int_0^1 2x\cdot e^{-x}$, we use integration by parts **again**. This time, we take $u = 2x$, and $dv = e^{-x}$. So

$$\begin{aligned} u &= 2x & dv &= e^{-x}\,dx \\ du &= 2\,dx & v &= -e^{-x} \end{aligned}$$

Thus

$$\begin{aligned} \int_0^1 2x\cdot e^{-x}\,dx = \left[2x(-e^{-x})\right]_0^1 + \int_0^1 2e^{-x}\,dx = \left[-2xe^{-x}\right]_0^1 + \left[-2e^{-x}\right]_0^1 \\ = -2e^{-1} - 0 + -2e^{-1} + 2 = -4/e + 2. \end{aligned}$$

So putting everything together,

$$\int_0^1 (x^2+1)e^{-x}\,dx = -2/e + 1 - 4/e + 2 = -6/e + 3.$$

□

7.1.26

$$\int_4^9 \frac{\ln y}{\sqrt{y}}dy$$

Solution. We use integration by parts, with $u = \ln y$ and $dv = \frac{dy}{\sqrt{y}}$. Thus

$$\begin{aligned} u &= \ln y & dv &= \frac{dy}{\sqrt{y}} \\ du &= \frac{dy}{y} & v &= 2\sqrt{y}, \end{aligned}$$

and so

$$\begin{aligned} \int_4^9 \ln y \frac{dy}{\sqrt{y}} &= [2\sqrt{y}\ln y]_4^9 - \int_4^9 2\frac{\sqrt{y}}{y}dy = 2\sqrt{9}\ln 9 - 2\sqrt{4}\ln 4 - \int_4^9 \frac{2}{\sqrt{y}}dy \\ &= 6\ln 9 - 4\ln 4 - [4\sqrt{y}]_4^9 = 6\ln 9 - 4\ln 4 + 8 - 12 \\ &= 6\ln 9 - 4\ln 4 - 4 \end{aligned}$$

□

7.1.30

$$\int_1^{\sqrt{3}} \tan^{-1}(1/x)\,dx$$

Solution. We do integration by parts, using $u = \tan^{-1}(1/x)$ and $dv = dx$. So $v = x$ and

$$du = \frac{1}{1+(1/x)^2}\frac{-1}{x^2}dx = \frac{-1}{x^2+1}dx.$$

So

$$\begin{aligned} u &= \tan^{-1}(1/x) & dv &= dx \\ du &= \frac{-1}{1+x^2} & v &= x \end{aligned}$$

Thus

$$\int \tan^{-1}(1/x)\,dx = x\tan^{-1}(1/x) + \int \frac{x\,dx}{1+x^2}.$$

Making the substitution $t = 1 + x^2$, so that $dt = 2x\,dx$, the latter integral becomes

$$\int \frac{x\,dx}{1+x^2} = \int \frac{dt}{2t} = \frac{1}{2}\ln t + C = \frac{1}{2}\ln(1+x^2) + C.$$

Therefore, the original integral becomes

$$\int \tan^{-1}(1/x)\,dx = x\tan^{-1}(1/x) + \int \frac{x\,dx}{1+x^2} = x\tan^{-1}(1/x) + \frac{1}{2}\ln(1+x^2) + C.$$

Thus, the *definite* integral becomes

$$\begin{aligned}
\int_1^{\sqrt{3}} \tan^{-1}(1/x)\,dx &= \left[x\tan^{-1}(1/x) + \frac{1}{2}\ln(1+x^2)\right]_1^{\sqrt{3}} \\
&= \sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{2}\ln(1+3) - 1\tan^{-1}(1) - \frac{1}{2}\ln(1+1^2) \\
&= \sqrt{3}\frac{\pi}{6} + \frac{\ln 4}{2} - \frac{\pi}{4} - \frac{\ln 2}{2}.
\end{aligned}$$

□

7.1.33

$$\int \cos x \ln(\sin x)\,dx$$

Solution. We do integration by parts, with $u = \ln(\sin x)$ and $dv = \cos x\,dx$. Then

$$\begin{aligned}
u &= \ln(\sin x) & dv &= \cos x\,dx \\
du &= \frac{\cos x}{\sin x}dx & v &= \sin x
\end{aligned}$$

Thus

$$\begin{aligned}
\int \ln(\sin x)\cos x\,dx &= \sin x\ln(\sin x) - \int \sin x\frac{\cos x}{\sin x}dx \\
&= \sin x\ln(\sin x) - \int \cos x\,dx = \sin x\ln(\sin x) - \sin x + C.
\end{aligned}$$

□