1 Power series to memorize

See Table 1 on Page 762.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$
$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$
$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$
$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

The first two and last one have radius of convergence 1, while the others have radius of convergence ∞.

Also, here are some power series which are handy to know, but can be derived on the fly if necessary, by integrating...

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$
$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Both have radius of convergence 1.

2 Rules for manipulating power series

If

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

then

$$
\begin{aligned}
f'(x) &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots \\
\int_0^x f(t)\,dt &= a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \cdots \\
f(0) &= a_0 \\
f(-x) &= a_0 - a_1x + a_2x^2 - a_3x^3 + a_4x^4 - \cdots \\
f(2x) &= a_0 + 2a_1x + 4a_2x^2 + 8a_3x^3 + \cdots \\
f(\alpha x) &= a_0 + \alpha a_1x + \alpha^2a_2x^2 + \alpha^3a_3x^3 + \cdots \\
f(x^2) &= a_0 + a_1x^2 + a_2x^4 + a_3x^6 + \cdots \\
f(x^k) &= a_0 + a_1x^k + a_2x^{2k} + a_3x^{3k} + \cdots \\
\frac{f(x) + f(-x)}{2} &= a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \cdots \\
\frac{f(x) - f(-x)}{2} &= a_1 + a_3x^3 + a_5x^5 + a_7x^7 + \cdots \\
-f(x) &= -a_0 - a_1x - a_2x^2 - \cdots \\
\alpha \cdot f(x) &= \alpha a_0 + \alpha a_1x + \alpha a_2x^2 + \alpha a_3x^3 + \cdots \\
xf(x) &= a_0x + a_1x^2 + a_2x^3 + \cdots \\
\frac{f(x)}{x} &= \frac{a_0}{x} + a_1 + a_2x + a_3x^2 + a_4x^3 + \cdots
\end{aligned}
$$

For the most part, these operations don't change the radii of convergence, though they may change the intervals of convergence.

If, in addition $g(x) = b_0 + b_1x + b_2x^2 + \cdots$, then

$$
\begin{aligned}
f(x) + g(x) &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots \\
f(x) - g(x) &= (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + \cdots
\end{aligned}
$$

If $f(x)$ is a function and we want a power series in $(x - a)$, look at $g(t) = f(t + a)$. If you can find a power series

$$g(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + \cdots ,$$

for $g(t)$ in the variable t, this gives you a power series expansion

$$f(x) = g(x - a) = a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots$$

in $(x - a)$.

If f is infinitely differentiable, then *usually*[1]

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots ,$$

at least for values of x within the interval of convergence of the series on the right.

[1] In fact, it works as long as f is infinitely differentiable, and given by a single formula rather than a piecewise definition. This is difficult to prove, however.