Challenge Problems

DIS 203 and 210

March 6, 2015

Choose one of the following problems, and work on it in your group. Your goal is to convince me that your answer is correct. Even if your answer isn't completely correct, I may be able to clarify what you need to fix.

1. Determine the value of

$$\sum_{k=1}^{\infty} \frac{(e-2)^k}{k(k+2)}$$

Solution. Consider the power series

$$f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k(k+2)}.$$

This converges for $|x| < 1$ by the ratio test:

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{k_n} \right| = \lim_{k\to\infty} \left| \frac{x^{k+1}/((k+1)(k+3))}{x^k/(k(k+2))} \right| = \lim_{k\to\infty} |x| \frac{k(k+2)}{(k+1)(k+3)} = |x|.$$

We're interested in $f(e-2)$, which exists, because $|e-2| \approx 0.718 < 1$.

Per the hint, we break the coefficient up into partial fractions:

$$\frac{1}{k(k+2)} = \frac{A}{k} + \frac{B}{k+2}.$$

$$1 = A(k+2) + Bk$$

Set $k = 0$, see that $A = \frac{1}{2}$. Set $k = -2$, see that $B = \frac{-1}{2}$. So

$$\frac{1}{k(k+2)} = \frac{1/2}{k} + \frac{-1/2}{k+2}$$

and

$$f(x) = \sum_{k=1}^{\infty} x^k \frac{1}{k(k+2)} = \sum_{k=1}^{\infty} x^k \left(\frac{1/2}{k} - \frac{1/2}{k+2} \right)$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{2}\frac{x^k}{k} - \frac{1}{2}\frac{x^k}{k+2} \right) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{x^k}{k} - \frac{1}{2}\sum_{k=1}^{\infty} \frac{x^k}{k+2}.$$

So we're concerned with the series

$$\sum_{k=1}^{\infty} \frac{x^k}{k} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

and

$$\sum_{k=1}^{\infty} \frac{x^k}{k+2} = \frac{x}{3} + \frac{x^2}{4} + \frac{x^3}{5} + \frac{x^4}{6} + \cdots$$

Per the hint, we also note that

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots,$$

so integrating term by term,

$$\int_0^x \frac{dt}{1-t} = -\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

which is one of the two series we care about. The other is

$$\begin{aligned} \frac{x}{3} + \frac{x^2}{4} + \frac{x^3}{5} + \frac{x^4}{6} + \cdots = \frac{\left(\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \cdots\right)}{x^2} &= \\ = \frac{\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) - x - \frac{x^2}{2}}{x^2} \\ = \frac{-\ln(1-x) - x - x^2/2}{x^2}. \end{aligned}$$

So we have formulas for $\sum_{k=1}^{\infty} \frac{x^k}{k}$ and $\sum_{k=1}^{\infty} \frac{x^k}{k+2}$. Combining them, we get

$$f(x) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{x^k}{k} - \frac{1}{2}\sum_{k=1}^{\infty} \frac{x^k}{k+2} = \frac{-\ln(1-x)}{2} + \frac{\ln(1-x) + x + x^2/2}{2x^2}.$$

Substituting $x = e - 2$, and using the fact that $1 - x = 1 - (e-2) = 3 - e$, we get the final value:

$$\sum_{k=1}^{\infty} \frac{(e-2)^k}{k(k+2)} = \frac{-\ln(3-e)}{2} + \frac{\ln(3-e) + (e-2) + (e-2)^2/2}{2(e-2)^2}.$$

□

2. Suppose that $f(x)$ is a function which is differentiable for all x. Show that there is some x such that

$$f'(x) \neq e^{f(x)} \tan^{-1} x$$

Proof. Suppose not. Then for every x,

$$f'(x) = e^{f(x)} \tan^{-1} x.$$

Multiplying both sides by $e^{-f(x)}$, we get

$$e^{-f(x)} f'(x) = \tan^{-1} x.$$

By the chain rule, the left hand side is the derivative of $-e^{-f(x)}$. Integrating both sides, we get that

$$-e^{-f(x)} = \int \tan^{-1} x \, dx$$

To integrate arctangent, use integration by parts, with $u = \tan^{-1} x$, $dv = dx$, $v = x$, and $du = \frac{dx}{1+x^2}$. So

$$\int \tan^{-1} x \, dx = \int u \, dv = uv - \int v \, du = x \tan^{-1} x - \int \frac{xdx}{1+x^2}$$

The latter integral can be evaluated by doing a substitution, $t = 1 + x^2$, so $dt = 2xdx$ and

$$\int \frac{xdx}{1+x^2} = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \ln |t| = \frac{1}{2} \ln(1+x^2)$$

so

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C.$$

So, we see that

$$-e^{-f(x)} = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$$

for all values of x.

Now, we'll get a contradiction by choosing x sufficiently large. We claim that the right hand side looks asymptotically like $\frac{\pi}{2} x$, for $x \gg 0$. Indeed,

$$\lim_{x\to\infty} \frac{x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C}{x} = \lim_{x\to\infty} \left(\tan^{-1} x - \frac{\ln(1+x^2)}{2x} + \frac{C}{x} \right)$$

$$= \lim_{x\to\infty} \tan^{-1} x - \lim_{x\to\infty} \frac{\ln(1+x^2)}{2x} + \lim_{x\to\infty} \frac{C}{x} = \frac{\pi}{2} - \lim_{x\to\infty} \frac{\ln(1+x^2)}{2x} + 0$$

assuming that $\lim_{x\to\infty} \frac{\ln(1+x^2)}{2x}$ exists. And indeed, we can evaluate it by L'Hospital's rule:

$$\lim_{x\to\infty} \frac{\ln(1+x^2)}{2x} = \lim_{x\to\infty} \frac{\frac{1}{1+x^2} 2x}{2} = \lim_{x\to\infty} \frac{x}{1+x^2} = 0,$$

so that

$$\lim_{x\to\infty} \frac{x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C}{x} = \frac{\pi}{2} - 0 + 0 = \frac{\pi}{2}.$$

So, there is an N such that if $x > N$, then

$$\frac{x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C}{x} = \frac{-e^{-f(x)}}{x} \text{ is within } 0.001 \text{ of } \pi/2.$$

By taking x to be bigger than both N and 0, we see that

$$\frac{-e^{-f(x)}}{x} > \frac{\pi}{2} - 0.001 > 0$$

so that

$$-e^{-f(x)} > 0,$$

which is false. □

3. Suppose $\sum_n c_n x^n$ and $\sum_n d_n x^n$ are two power series in x. Suppose the radius of convergence of the first one is 2, and the radius of convergence of the second one is 3. What is the radius of convergence of the sum $\sum_n (c_n + d_n)x^n$?

   Solution. It is 2. If $|x| < 2$, then both series converge, so their sum converges. If $|x|$ is strictly between 2 and 3, then the first series diverges, and the second series converges, so their sum diverges.[1] In particular, we know that the power series $\sum_n (c_n + d_n)x^n$

   - Converges for all x such that $|x| < 2$
   - Diverges for all x such that $2 < |x| < 3$. In particular, it diverges on the intervals $(-3, -2)$ and $(2, 3)$.

   The first fact implies that the radius of convergence R must be at least 2, since the interval of convergence contains $(-2, 2)$. The second fact implies that the radius of convergence R can't be any bigger than 2. Indeed, if $R > 2$, then $(-R, R)$ would intersect $(2, 3)$ and $(-3, -2)$, which is false, since the series converges on $(-R, R)$, and diverges on $(2, 3)$ and $(-3, -2)$.

   So the radius is at most 2 and at least 2. Therefore it is 2. □

4. Write down a series which converges to

$$\int_{1.25}^{1.5} \frac{\ln x}{x-1}\,dx$$

[1] We have used the fact that the sum $\sum_n (a_n + b_n)$ of a divergent series $\sum_n a_n$ and a convergent series $\sum_n b_n$ is *divergent*. To see this, suppose for the sake of contradiction that the sum converged. Then since differences of convergent series are convergent, and $\sum_n (a_n + b_n)$ and $\sum_n b_n$ are both convergent, their difference $\sum_n [(a_n + b_n) - b_n] = \sum_n a_n$ is convergent, contradicting the assumption that $\sum_n a_n$ was not convergent.

Solution. Following the hint, we do a change of variables $t = x - 1$, so $x = 1 + t$ and

$$\int_{1.25}^{1.5} \frac{\ln x}{x-1}\,dx = \int_{0.25}^{0.5} \frac{\ln(1+t)}{t}\,dt.$$

The power series for $\ln(1+t)$ is gotten by integrating the power series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

$$\ln(1+t) = C + t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots$$

Setting $t = 0$ on both sides reveals $C = 0$. So

$$\ln(1+t) = t - \frac{t^2}{2} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^n}{n}.$$

We can divide by t, to get a series for $\ln(1+t)/t$:

$$\frac{\ln(1+t)}{t} = 1 - \frac{t}{2} + \frac{t^2}{3} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n+1}.$$

This series has radius of convergence 1, by using the ratio test in the usual fashion.[2] Since 0.25 and 0.5 are within the radius of convergence, we can integrate term by term

$$\int_{0.25}^{0.5} \frac{\ln(1+t)}{t}\,dt = \int_{0.25}^{0.5} 1\,dt - \int_{0.25}^{0.5} \frac{t}{2}\,dt + \int_{0.25}^{0.5} \frac{t^2}{3}\,dt - \cdots =$$

$$\sum_{n=0}^{\infty} (-1)^n \int_{0.25}^{0.5} \frac{t^n\,dt}{n+1} = \sum_{n=0}^{\infty} (-1)^n \left[\frac{t^{n+1}}{(n+1)^2}\right]_{t=0.5}^{t=0.5}$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{0.5^{n+1}}{(n+1)^2} - \frac{0.25^{n+1}}{(n+1)^2}\right).$$

If we let $m = n + 1$, we can rewrite this in the slightly simpler form

$$\sum_{m=1}^{\infty} (-1)^{m-1} \frac{0.5^m - 0.25^m}{m^2}$$

□

5. Suppose $f(x)$ is a differentiable function, with $f'(0) = 1$, and suppose that for every x and y,

$$f(x+y) = f(x)f(y).$$

Show that $f(x) = e^x$ for all x.

[2] The ratio between consecutive terms is $-t\frac{n+1}{n+2}$, whose absolute value approaches $|t|$ in the limit. So if $|t| < 1$, the series converges.

Proof. For any constant a, we have the identity

$$f(x+a) = f(x)f(a).$$

Differentiating both sides with respect to x,

$$f'(x+a) = f'(x)f(a),$$

since $f(a)$ is a constant. Now, setting $x = 0$, we get

$$f'(a) = f'(0)f(a) = 1 \cdot f(a) = f(a).$$

As a was arbitrary, it follows that

$$f'(x) = f(x)$$

for all x.

This could alternatively be seen as follows:

First note that

$$f(0) = f(0+0) = f(0)f(0)$$

so

$$0 = f(0)^2 - f(0) = f(0)(f(0) - 1),$$

and so $f(0)$ is 0 or 1. If it were 0, then for any x,

$$f(0+x) = f(x)f(0) = f(x) \cdot 0 = 0,$$

so f would be the constant function 0, contradicting the assumption that $f'(0) = 1$. So $f(0) = 1$.

Now for any x,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x)f(\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} f(x)\frac{f(\Delta x) - 1}{\Delta x}.$$

Since $f(x)$ is a constant, this is

$$f(x) \lim_{\Delta x \to 0} \frac{f(\Delta x) - 1}{\Delta x} = f(x) \lim_{\Delta x \to 0} \frac{f(\Delta x + 0) - f(0)}{\Delta x} = f(x)f'(0) = f(x).$$

So $f'(x) = f(x)$ for any x.

Regardless of how we got that, we now know that f solves the linear differential equation

$$f'(x) - f(x) = 0$$

Multiplying by the factor of integration $e^{\int -dx} = e^{-x}$, we get

$$e^{-x}f'(x) - e^{-x}f(x) = 0$$

The left hand side is the derivative of $e^{-x}f(x)$, so $e^{-x}f(x)$ must be a constant. Thus

$$e^{-x}f(x) = k$$

$$f(x) = ke^x.$$

So $f'(x) = ke^x$ and $f'(0) = 1 = ke^0 = k$, so $k = 1$. Thus $f(x) = ke^x = e^x$ for all x. □