Challenge Problems

DIS 203 and 210

March 5, 2015

Choose one of the following problems, and work on it in your group. Your goal is to convince me that your answer is correct. Even if your answer isn't completely correct, I may be able to clarify what you need to fix.

1. Determine the value of
$$\sum_{k=1}^{\infty} \frac{(e-2)^k}{k(k+2)}$$
   Hints:
   - Replace $e-2$ with x. You'll need to figure out what function that series represents.
   - Break $\frac{1}{k(k+2)}$ into partial fractions, as $A/k + B/(k+2)$ for some constants A and B. Use this to pretend that the denominator was k or $k+2$, rather than $k(k+2)$.
   - Take the power series for $1/(1-x)$, integrate it, and fiddle around with the results.
   - At some point you'll need to reindex some series.
2. Suppose that $f(x)$ is a function which is differentiable for all x. Show that there is some x such that
$$f'(x) \neq e^{f(x)} \tan^{-1} x$$
   Hints:
   - I'm asking you to show that f doesn't solve some differential equation.
   - Try solving the equation. You'll need integration by parts.
   - Do a proof by contradiction.
   - When x is really big, $\ln(1+x^2)$ is approximately $2 \ln x$, which is a lot smaller than $\frac{\pi}{2}x$.

3. Suppose $\sum_n c_n x^n$ and $\sum_n d_n x^n$ are two power series in x. Suppose the radius of convergence of the first one is 2, and the radius of convergence of the second one is 3. What is the radius of convergence of the sum $\sum_n (c_n + d_n) x^n$?

   Hints:

   - If you're not sure what the answer should be, try writing down some examples.
   - Use the fact that sums and differences of convergent series are convergent.
   - Show that the sum of a divergent series and a convergent series is divergent.

4. Write down a series which converges to

$$\int_{1.25}^{1.5} \frac{\ln x}{x-1}\, dx$$

   Hints:

   - Write down a power series for the integrand
   - Write down a power series for the indefinite integral
   - Either do a change of variables $u = x - 1$, or use powers of $x - 1$.

5. Suppose $f(x)$ is a differentiable function, with $f'(0) = 1$, and suppose that for every x and y,

$$f(x+y) = f(x)f(y).$$

   Show that $f(x) = e^x$ for all x.

   Hints:

   - Show that $f'(x) = f(x)$.
   - Remember how we took the derivative of $\exp(x)$. Do the same calculations with f instead of exp.
   - Or, take the given equation, replace y with a constant, differentiate with respect to x, and then set $x = 0$.