December 2-4 Review

DIS 313/315

December 5, 2014

1. What is $\int_a^b r\,dx$? *(Note: r and a and b are supposed to be constants, not depending on x.)*
2. What is $\int_a^0 (x+a)dx$?
3. Differentiate $\sqrt{1+(\cos x)^{\sin x}}$.
4. Differentiate x^{2^x}.
5. Evaluate the indefinite integral
$$\int e^{\cos 2x+\sin^2 x}\sin x\cos x\,dx$$
6. Evaluate
$$\int_0^5 e^{\int_0^x t\,dt} x\,dx$$
7. If $f(x)$ is continuous, what is
$$\lim_{x\to 5}\frac{\int_5^x f(t)dt}{x-5}?$$
8. Evaluate
$$\lim_{x\to 0}\int_x^{2x}\frac{dt}{t}$$
9. The *sine integral function* is defined as follows
$$\mathrm{Si}(x)=\int_0^x \frac{\sin t}{t}\,dt.$$
   (a) What is the derivative of the sine integral function?
   (b) Evaluate the following indefinite integrals in terms of Si
$$\int \frac{\sin x\,dx}{x} \qquad\qquad \int \sin e^x\,dx$$
   (Hint: for the second, do a u-substitution $u = e^x$.)

10. Show, using $\epsilon - \delta$, that

$$\lim_{x\to 5} \frac{1}{x-4} = 1$$

11. True or False?

   (a) If f is integrable, then f is bounded.

   (b) If f is integrable, then f is continuous.

   (c) If $\lim_{x\to a} g(x) = L$, then $\lim_{x\to a} \cos g(x) = \cos L$.

   (d) If $\lim_{x\to 5} f(x)$ exists, then

   $$\lim_{x\to 5} f(x) = \lim_{x\to 0} f(5-x).$$

   (e) If $\lim_{x\to 5^+} f(x)$ exists, then

   $$\lim_{x\to 5^+} f(x) = \lim_{x\to 0^+} f(5-x).$$

12. Show that $\int_e^a \ln x\, dx = a\ln a - a$ for $a > 0$.

13. What is

$$\int_{-5}^{5} \sin(x^3)\, dx?$$

14. What is

$$\frac{d}{dx}\int_x^{x^2} e^{t^2}\, dt?$$

15. Let $f(x)$ be a continuous function with range and domain $[0,1]$. Prove that $f(x) = x$ for some $x \in [0,1]$. (Hint: apply the intermediate value theorem to the function $f(x) - x$.)

1 Solutions

1. What is $\int_a^b r\, dx$?

   Solution. The antiderivative of the constant function r is rx, so

   $$\int_a^b r\, dx = [rx]_a^b = rb - ra.$$

   □

2. What is $\int_a^0 (x+a) dx$?

Solution. The antiderivative of $x + a$ is $x^2/2 + ax$, so

$$\int_a^0 (x+a)\,dx = \left[\frac{x^2}{2} + ax\right]_a^0 = 0 - \frac{a^2}{2} - a \cdot a = \frac{-3a^2}{2}.$$

□

3. Differentiate $\sqrt{1 + (\cos x)^{\sin x}}$.

Solution. By the chain rule,

$$\frac{d}{dx}\sqrt{1 + (\cos x)^{\sin x}} = \frac{1}{2\sqrt{1 + (\cos x)^{\sin x}}} \cdot \frac{d}{dx}\left((\cos x)^{\sin x}\right).$$

It remains to find the derivative of $(\cos x)^{\sin x}$. The general way to differentiate something like $f(x)^{g(x)}$ is to rewrite it as follows:

$$f(x)^{g(x)} = \left(e^{\ln f(x)}\right)^{g(x)} = e^{(\ln f(x))g(x)}$$

and then use the chain rule and product rule. Here, we see that

$$(\cos x)^{\sin x} = e^{(\ln \cos x)\cdot(\sin x)},$$

so

$$\begin{aligned}
\frac{d}{dx}\left((\cos x)^{\sin x}\right) &= \frac{d}{dx} e^{(\ln \cos x)\cdot(\sin x)} = e^{(\ln \cos x)\cdot(\sin x)} \cdot \frac{d}{dx}\left[(\ln \cos x)\cdot(\sin x)\right] \\
&= (\cos x)^{\sin x} \cdot \frac{d}{dx}\left[(\ln \cos x)\cdot(\sin x)\right] \\
&= (\cos x)^{\sin x} \cdot \left[\frac{1}{\cos x}(-\sin x)(\sin x) + (\ln \cos x)\cos x\right] \\
&= (\cos x)^{\sin x}\left[\cos x \ln \cos x - \frac{\sin^2 x}{\cos x}\right].
\end{aligned}$$

So putting everything together,

$$\frac{d}{dx}\sqrt{1 + (\cos x)^{\sin x}} = \frac{(\cos x)^{\sin x}\left[\cos x \ln \cos x - \frac{\sin^2 x}{\cos x}\right]}{2\sqrt{1 + (\cos x)^{\sin x}}}$$

□

4. Differentiate x^{2^x}.

Solution. Similar to the previous problem, we rewrite x^{2^x} as $e^{2^x \cdot \ln x}$. Then

$$\begin{aligned} \frac{d}{dx} x^{2^x} = \frac{d}{dx} e^{2^x \cdot \ln x} &= e^{2^x \cdot \ln x} \frac{d}{dx} \left[2^x \cdot \ln x \right] \\ &= x^{2^x} \left[(\ln 2) \cdot 2^x \cdot \ln x + \frac{2^x}{x} \right]. \end{aligned}$$

□

5. Evaluate the indefinite integral

$$\int e^{\cos 2x + \sin^2 x} \sin x \cos x \, dx$$

Solution. Note that $\cos 2x = \cos^2 x - \sin^2 x$, so that the exponent $\cos 2x + \sin^2 x$ is actually just $\cos^2 x$. Thus

$$\int e^{\cos 2x + \sin^2 x} \sin x \cos x \, dx = \int e^{\cos^2 x} \sin x \cos x \, dx$$

Now we do a u-substitution: let $u = \cos^2 x$. Then $du = -2 \cos x \sin x \, dx$, so

$$\int e^{\cos^2 x} \sin x \cos x \, dx = \int e^u \frac{du}{-2} = \frac{-1}{2} \int e^u \, du = \frac{-1}{2} e^u = \frac{-e^{\cos^2 x}}{2},$$

$+C$ if you like. □

6. Evaluate

$$\int_0^5 e^{\int_0^x t \, dt} x \, dx$$

Solution. First of all, the antiderivative of t is $t^2/2$, so

$$\int_0^x t \, dt = \left[\frac{t^2}{2} \right]_0^x = \frac{x^2}{2}.$$

Therefore

$$\int_0^5 e^{\int_0^x t \, dt} x \, dx = \int_0^5 e^{x^2/2} x \, dx.$$

To evaluate this, we do a u-substitution, and let $u = x^2/2$. Then $du = x \, dx$, so

$$\int e^{x^2/2} x \, dx = \int e^u \, du = e^u = e^{x^2/2},$$

and therefore

$$\int_0^5 e^{x^2/2} x \, dx = \left[e^{x^2/2} \right]_0^5 = e^{25/2} - e^0 = e^{25/2} - 1.$$

□

7. If $f(x)$ is continuous, what is
$$\lim_{x\to 5} \frac{\int_5^x f(t)dt}{x-5}?$$

Solution. Let $F(x)$ be some antiderivative of $f(x)$. By the fundamental theorem of calculus, $\int_5^x f(t)\,dt = F(x) - F(5)$, so

$$\lim_{x\to 5} \frac{\int_5^x f(t)\,dt}{x-5} = \lim_{x\to 5} \frac{F(x)-F(5)}{x-5} = F'(5) = f(5),$$

so the limit is exactly $f(5)$.

Alternatively, if you didn't think of writing the integral in terms of some antiderivative of f, you could use the more brute-force approach of L'Hospital's rule. Note that the function $F(x) = \int_5^x f(t)\,dt$ is differentiable (by the fundamental theorem of calculus), so it's continuous, and therefore

$$\lim_{x\to 5} F(x) = F(5) = \int_5^5 f(t)\,dt = 0.$$

So by L'Hospital's rule,
$$\lim_{x\to 5} \frac{F(x)}{x-5} = \lim_{x\to 5} \frac{F'(x)}{1}.$$
But we know what $F'(x)$ is; it's $f(x)$. So

$$\lim_{x\to 5} F'(x) = \lim_{x\to 5} f(x) = f(5),$$

because we assumed f is continuous. □

> (Technical aside: Ole Hald would probably find both these proofs fishy, because the statement we're trying to prove is really a necessary step along the way to proving the Fundamental Theorem of Calculus. All the proofs of FTC, such as the one on page 388-389 of the textbook, go through the fact that
> $$\lim_{x\to a} \frac{\int_a^x f(t)\,dt}{x-a} = f(a).$$
> So the preferred proof that $\lim_{x\to 5} \frac{\int_5^x f(t)\,dt}{x-5} = f(5)$ might go a little more like so: for any $\epsilon > 0$, continuity of f implies there's some $\delta > 0$ such that when t is within δ of 5, $f(t)$ is within ϵ of $f(5)$. That is, f only takes values in the range $(f(5)-\epsilon, f(5)+\epsilon)$ when you restrict it to $(5-\delta, 5+\delta)$. Now if $|x-5| < \delta$, then the interval $[5,x]$ or $[x,5]$ is contained in $(5-\delta, 5+\delta)$, so f is stuck between $f(5)-\epsilon$ and $f(5)+\epsilon$ on the interval between x and 5. Then by basic properties of integrals,
> $$(f(5)-\epsilon)(x-5) \le \int_5^x f(t)\,dt \le (f(5)+\epsilon)(x-5) \qquad \text{if } x > 5$$

$$(f(5)-\epsilon)(5-x) \le \int_x^5 f(t)\,dt \le (f(5)+\epsilon)(5-x) \qquad \text{if } x<5$$

Either way,

$$f(5)-\epsilon \le \frac{\int_5^x f(t)\,dt}{x-5} \le f(5)+\epsilon.$$

So, for any positive ϵ, we can make $\frac{\int_5^x f(t)\,dt}{x-5}$ be within ϵ of $f(5)$, by making $|x-5|$ be less than some δ (namely, the one coming from continuity of f.) This establishes that

$$\lim_{x\to 5} \frac{\int_5^x f(t)\,dt}{x-5} = f(5).$$

On the other hand, if you believe the FTC, or take it for granted, and just want to know what the value of the limit I gave is, rather than proving it from first principles, it's logically okay to apply FTC and L'Hospital.)

8. Evaluate

$$\lim_{x\to 0} \int_x^{2x} \frac{dt}{t}$$

Solution. For any $x \neq 0$, we have

$$\int_x^{2x} \frac{dt}{t} = [\ln|t|]_x^{2x} = \ln|2x| - \ln|x|.$$

Now $|2x| = 2|x|$, so $\ln|2x| = \ln(2|x|) = \ln 2 + \ln|x|$, so

$$\int_x^{2x} \frac{dt}{t} = \ln|2x| - \ln|x| = \ln 2 + \ln|x| - \ln|x| = \ln 2.$$

Thus the expression we're taking the limit of doesn't even depend on x, and

$$\lim_{x\to 0} \int_x^{2x} \frac{dt}{t} = \lim_{x\to 0} \ln 2 = \ln 2.$$

□

9. The *sine integral function* is defined as follows

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt.$$

(a) What is the derivative of the sine integral function?

Solution. The derivative of $\text{Si}(x)$ is $\frac{\sin x}{x}$. □

(b) Evaluate the following indefinite integrals in terms of Si

$$\int \frac{\sin x\,dx}{x} \qquad\qquad \int \sin e^x\,dx$$

Solution. The first is $\mathrm{Si}(x)$ (or $\mathrm{Si}(x) + C$, if you like).

For the second, we do a u-substitution, $u = e^x$. Then $du = e^x\,dx$, so $dx = \frac{du}{e^x}$, and

$$\int \sin e^x\,dx = \int \sin u \frac{du}{e^x} = \int \frac{\sin u}{u} du = \mathrm{Si}(u) = \mathrm{Si}(e^x).$$

□

10. Show, using $\epsilon - \delta$, that

$$\lim_{x \to 5} \frac{1}{x-4} = 1$$

Preliminary work. We'll want to make the following expression be less than ϵ:

$$\left|\frac{1}{x-4} - 1\right| = \left|\frac{1-(x-4)}{x-4}\right| = \left|\frac{5-x}{x-4}\right| = \frac{|5-x|}{|x-4|} = \frac{|x-5|}{|x-4|}.$$

We can directly control $|x-5|$, making it as small as we like (because we get to choose δ). The problem is that this is getting multiplied by the junk term $\frac{1}{|x-4|}$. We want to put some absolute bound on this junk term. Usually one does this by deciding to always take $\delta \leq 1$, though that doesn't work in this case, because this would still allow x to be something like 4.000001, for which $\frac{1}{|x-4|}$ is enormous. So, because we're dividing by $x-4$, we really need to keep x a good distance away from 4. We need to keep $x-4$ away from zero. So let's instead agree that δ will be at most $1/2$. This keeps x in the range from 4.5 to 5.5. The *smallest* that $|x-4|$ can be is $4.5 - 4 = 0.5$, so the *biggest* that $1/|x-4|$ can be is $1/0.5 = 2$.

So, if we agree that we'll always make δ be less than $1/2$, then $\frac{1}{|x-4|}$ will be at most 2. Then

$$\frac{|x-5|}{|x-4|} < \frac{\delta}{|x-4|} \leq 2\delta.$$

So, if we arrange that $2\delta \leq \epsilon$, then the quantity we *want* to be small will be small.

So, in summary, we need to make $\delta \leq 1/2$ and $2\delta \leq \epsilon$. This imposes two upper bounds on δ, namely $1/2$ and $\epsilon/2$, and we just take whichever is smaller. With this in mind, we get the following proof...

Proof. Given $\epsilon > 0$, let δ be the minimum of $\{1/2, \epsilon/2\}$, so that $\delta \leq 1/2$ and $\delta \leq \epsilon/2$. Now suppose that $0 < |x-5| < \delta$. Then first of all,

$$|x-5| < \delta \leq 1/2,$$

so

$$5 - 1/2 < x < 5 + 1/2,$$

and in particular $x > 4.5$. Therefore $|x - 4| > 0.5$, so

$$\frac{1}{|x-4|} < 2.$$

Multiplying both sides by $|x - 5|$, we see that

$$\frac{|x-5|}{|x-4|} < 2|x-5|.$$

Thus

$$\left|\frac{1}{x-4} - 1\right| = \left|\frac{x-5}{x-4}\right| = \frac{|x-5|}{|x-4|} < 2|x-5| < 2\delta \le 2\epsilon/2 = \epsilon.$$

So we've shown that

$$0 < |x-5| < \delta \implies \left|\frac{1}{x-4} - 1\right| < \epsilon.$$

□

11. True or False?

(a) If f is integrable, then f is bounded.

Solution. This is true. (Here's the rough explanation why, which you don't need to know: if f is unbounded, then there's no way to control the Riemann sums. Integrability would imply that there's some L such that, by making n big enough, $\sum_{i=0}^{n} f(x_i^*)\Delta x$ is guaranteed to be within, say, *1* of L. If f isn't bounded, then it won't be bounded on one of the intervals $[x_i, x_{i+1}]$, and since x_i^* can be any number in that interval, there's no way to keep $f(x_i^*)$ from being really enormous or really negative, since we don't get to choose x_i^*. So in fact if f isn't bounded, there *isn't* a way to ensure that the Riemann sum is in the range $[L - 1, L + 1]$, and integrability fails.) □

(b) If f is integrable, then f is continuous.

Solution. This is false. Functions with jump discontinuities, like the step function, are still integrable. (For a wilder example, look up Thomae's function.) □

(c) If $\lim_{x\to a} g(x) = L$, then $\lim_{x\to a} \cos g(x) = \cos L$.

Solution. True, because $\cos x$ is continuous. See Theorem 8 in Section 2.5 of Stewart. □

(d) If $\lim_{x\to 5} f(x)$ exists, then

$$\lim_{x\to 5} f(x) = \lim_{x\to 0} f(5-x).$$

Solution. It is indeed true that

$$\lim_{y\to 5} f(y) = \lim_{x\to 0} f(5-x).$$

We can think of y as being $5-x$. As x approaches 0, $5-x$ approaches 5. ☐

(e) If $\lim_{x\to 5^+} f(x)$ exists, then

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 0^+} f(5-x).$$

Solution. It is *not* true that

$$\lim_{y\to 5^+} f(y) \overset{no!}{=} \lim_{x\to 0^+} f(5-x).$$

If we think of y as $5-x$, then as x approaches 0 from above, $5-x$ approaches 5 *from below*. For example, when x is a tiny positive number like 0.001, the quantity $5-x$ is like 4.999. So instead, the true statement is that

$$\lim_{y\to 5^-} f(y) = \lim_{x\to 0^+} f(5-x).$$

If f has a jump discontinuity at 5, the one-sided limits won't agree, so

$$\lim_{y\to 5^+} f(y) \neq \lim_{y\to 5^-} f(y) = \lim_{x\to 0^+} f(5-x).$$

☐

12. Show that $\int_e^a \ln x\,dx = a\ln a - a$ for $a > 0$.

Proof. Both sides are differentiable functions of a, so it suffices[1] to show that they have the same derivative, and agree at at least one point. Differentiating the left hand side, we get $\ln a$, by the fundamental theorem of calculus. Differentiating the right hand side, we get

$$\frac{d}{da}(a\ln a - a) = a\frac{1}{a} + \ln a - 1 = \ln a.$$

So the two sides have the same derivative. Finally, when we plug in $a = e$, the left hand side is

$$\int_e^e \ln x\,dx = 0,$$

and the right hand side is

$$e\ln e - e = e\cdot 1 - e = 0.$$

☐

[1]by the MVT or one of its corollaries

Alternatively, you could merely observe that $x \ln x - x$ is an antiderivative of $\ln x$, so that

$$\int_e^a \ln x \, dx = [x \ln x - x]_e^a = (a \ln a - a) - (e \ln e - e) = a \ln a - a.$$

13. What is

$$\int_{-5}^{5} \sin(x^3) \, dx?$$

Solution. Zero, because $\sin(x^3)$ is an odd function. See statement 7 in Section 5.5. □

14. What is

$$\frac{d}{dx} \int_x^{x^2} e^{t^2} \, dt?$$

Solution. Let $F(x)$ be an antiderivative of e^{t^2}. Then

$$\int_x^{x^2} e^{t^2} \, dt = F(x^2) - F(x).$$

So by the chain rule,

$$\frac{d}{dx} \int_x^{x^2} e^{t^2} \, dt = \frac{d}{dx}(F(x^2) - F(x)) = F'(x^2)2x - F'(x) = 2xe^{x^4} - e^{x^2}.$$

□

15. Let $f(x)$ be a continuous function with range and domain $[0, 1]$. Prove that $f(x) = x$ for some $x \in [0, 1]$.

Proof. Let $g(x) = f(x) - x$. We want to show that $g(x)$ has some zero, because $g(x) = 0 \iff f(x) = x$. Note that $g(x)$ is a continuous function on $[0, 1]$, because differences of continuous functions are continuous.

For every $x \in [0, 1]$, we know that $f(x)$ is in $[0, 1]$ as well, because we assumed $[0, 1]$ was the range of $f(x)$. So $0 \leq f(x) \leq 1$ for $0 \leq x \leq 1$. In particular,

$$g(0) = f(0) - 0 \geq 0$$

and

$$g(1) = f(1) - 1 \leq 1 - 1 = 0.$$

If $g(0) = 0$ or $g(1) = 0$, then we've found a zero of $g(x)$, so we're done. Otherwise, $g(0) > 0$ and $g(1) < 0$. Then, by the intermediate value theorem, $g(x) = 0$ for some x between 0 and 1. So either way, $g(x)$ has a zero somewhere in the interval $[0, 1]$. □