Practice with Proofs

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For a good introduction to mathematical proofs, see the first thirteen pages of this document `http://math.berkeley.edu/~hutching/teach/proofs.pdf` by Michael Hutchings.

1. Prove $\forall x \exists y : y^2 > x$.

2. Disprove $\forall x \exists y : y^2 < x$.

3. Consider the piecewise function

$$f(x) = \begin{cases} e^x & \text{if } x \leq 0 \\ 1 + x & \text{if } x > 0 \end{cases}$$

   (a) Prove that -1 is not in the range of f. That is, show that there does not exist an x such that $f(x) = -1$.

   (b) Prove that f is continuous.

   (c) Prove that f is differentiable.

4. Prove that the composition of any two decreasing functions is increasing.

5. Prove that the sum of any two decreasing functions is decreasing.

6. Suppose f is a differentiable function on $\mathbb{R} = (\infty, \infty)$, and that $f'(x) < 0$ for all x. Prove that f is decreasing.

7. Suppose $f(x) = ax^2 + bx + c$, where $a \neq 0$. Prove that f is not one-to-one.

8. Let $g(x) = 2^x + 3^x$ for $|x| \leq 1$. (So the domain of g is $[-1, 1]$.) Prove that the range of g is exactly $[5/6, 5]$.

9. Suppose that $h(x)$ is a continuous function on all of $\mathbb{R}$, that $h(0) = 0$, and that $h(x)$ is one-to-one. Show that $h(-1)h(1) < 0$.

1 Hints

1. Given x, you need to find y such that $y^2 > x$. It might help to break into cases according to whether $x \leq 1$ or $x > 1$. A lot of the time you can take $y = x$.

2. The problem is asking you to prove that $\exists x \forall y : y^2 \geq x$. What number is less than or equal to all squares?

3. (a) Break into cases according to whether $x \leq 0$ or $x > 0$.

   (b) This is basically automatic everywhere except $x = 0$. There, compare the two-sided limits.

   (c) This is basically automatic everywhere except $x = 0$. There, use the definition of the derivative as a limit. To check the limit, calculate the two-sided limits (perhaps by rewriting them as derivatives of other functions).

4. Given decreasing functions f and g, and numbers $x_1 < x_2$, you need to show that $f(g(x_1)) < f(g(x_2))$.

5. Given decreasing functions f and g, and numbers $x_1 < x_2$, you need to show that $f(x_1) + g(x_1) > f(x_2) + g(x_2)$.

6. Prove this by contradiction, and use the mean value theorem. (What is the logical negation of the statement that f is a decreasing function? It should give you data to plug into the mean value theorem.) Also this is in the book.

7. Find the vertex of the parabola and go to the left and the right by, say, 1.

8. You need to show two things: that the range of g is contained in $[5/6, 5]$, and that it contains $[5/6, 5]$. For the first of these, use the fact that g is increasing. For the second, use the intermediate value theorem.

9. You need to show that $f(1)$ and $f(-1)$ don't have the same sign. Do a proof by contradiction: assume they have the same sign. Break into cases according to whether they're both positive, or both negative. Ultimately, you'll need to apply the intermediate value theorem to the intervals $[-1, 0]$ and $[0, 1]$, and contradict the fact that f is one-to-one.

2 Solutions

1. Prove $\forall x \exists y : y^2 > x$.
   Fix x. We need to show that there is some y whose square is greater than x. That is, we need to show that the set
   $$\{y : y^2 > x\}$$
   is non-empty. For example, if $x = 7$, we need to prove that
   $$\{y : y^2 > 7\}$$
   is non-empty. Solving the inequality $y^2 > 7$, one sees that
   $$\{y : y^2 > 7\} = (-\infty, -\sqrt{7}) \cup (\sqrt{7}, \infty)$$
   so we just need to specify a number bigger than $\sqrt{7}$ or less than $-\sqrt{7}$. We could take $y = 1000$, for example.

   Likewise
   $$\{y : y^2 > 5\} = (-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty).$$
   More generally, whenever $x \geq 0$, one can check that
   $$\{y : y^2 > x\} = (-\infty, -\sqrt{x}) \cup (\sqrt{x}, \infty).$$
   To prove that this set is non-empty, it suffices to specify a number bigger than $\sqrt{x}$. For example, $1 + \sqrt{x}$ works.

   On the other hand, when $x < 0$, all squares are bigger than x, so
   $$\{y : y^2 > x\} = (-\infty, \infty) = \mathbb{R}.$$
   So *any* value of y works. We could take $y = 0$, for example.

   From all this preliminary analysis, one can extract the following proof.

   Proof. Given x, we need to produce y such that $y^2 > x$. We break into cases according to whether $x \geq 0$ or $x < 0$. If $x \geq 0$, let $y = 1 + \sqrt{x}$. Then
   $$y^2 = (1 + \sqrt{x})^2 = 1 + 2\sqrt{x} + x \geq 1 + x > x,$$
   so there is a y such that $y^2 > x$. On the other hand, if $x < 0$, let $y = 0$. Then $y^2 = 0 > x$, so again there's a number y whose square is greater than x. Either way, $\exists y : y^2 > x$ is true. As x was arbitrary, it follows that $\forall x \exists y : y^2 > x$. □

   The same proof could be written more compactly as follows:

(of $\forall x \exists y : y^2 > x$*).* If $x \geq 0$, let $y = 1 + \sqrt{x}$. Then

$$y^2 = 1 + 2\sqrt{x} + x \geq 1 + x > x.$$

Otherwise, $0^2 = 0 > x$, so take $y = 0$. $\square$

Here's another proof, which I was thinking of when I wrote the hint.

Proof. Given x, we need to find y such that $y^2 > x$. If $x \leq 1$, then

$$x \leq 1 < 23^2,$$

so we can take $y = 23$. *Otherwise* $x > 1$. Multiplying both sides of $x > 1$ by the positive number x, we see that

$$x^2 > x,$$

so we can take $y = x$. $\square$

Alternatively, one could maybe make a case that the statement of Problem 1 is obvious.

2. Disprove $\forall x \exists y : y^2 < x$.

   Proof. Suppose for the sake of contradiction that for every x, there is a y such that $y^2 < x$. Then we can take $x = 0$, and so there must be some y such that $y^2 < 0$. But every number's square is nonnegative, so $y^2 \geq 0$, a contradiction. $\square$

   Equivalently, we could just prove the logical negation of the given statement, which is the statement $\exists x \forall y : y^2 \geq x$. (There is an x such that for every y, $y^2 \geq x$.)

   Proof. Take $x = 0$. Then for every y, $y^2 \geq 0 = x$. $\square$

3. Consider the piecewise function

$$f(x) = \begin{cases} e^x & \text{if } x \leq 0 \\ 1 + x & \text{if } x > 0 \end{cases}$$

   (a) Prove that -1 is not in the range of f. That is, show that there does not exist an x such that $f(x) = -1$.

   Proof. Suppose $f(x) = -1$. Then $x \leq 0$ or $x > 0$. In the first case, $e^x = f(x) = -1$, which is impossible since e^x is always positive. In the second case ($x > 0$), $1 + x = f(x) = -1$. But $x > 0$, so $1 + x > 1$ and in particular $1 + x$ can't be -1. Either way, we get a contradiction. $\square$

   (b) Prove that f is continuous.

Proof. Given a, we need to show that

$$\lim_{x \to a} f(x) = f(a).$$

We break into cases.

Case 1: $a < 0$. Then every value of x close enough to a will be negative, and so $f(x) = e^{-x}$ for x close enough to a. Therefore

$$\lim_{x \to a} f(x) = \lim_{x \to a} e^{-x} = e^{-a} = f(a).$$

(Technical note[1])

Case 2: $a > 0$. Then [do the same thing as in Case 1, mutatis mutandis. These two cases are the easy cases that you could almost just ignore or take for granted, if you were trying to prove this on the exam.]

Case 3: $a = 0$. We need to show that $\lim_{x \to 0} f(x) = f(0) = 1$. It suffices to show that the two one-sided limits both take the value 1. Indeed,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 + x = 1 + 0 = 1$$
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} e^{x} = e^{0} = 1$$

(Technical note.[2])

□

(c) Prove that f is differentiable.

Proof. Wow, this is more of a hassle than I realized! Let a be given, and let's show that $f(x)$ is differentiable at a. Again, the cases where $a > 0$ or $a < 0$ "obviously" work, so let's just focus on the case where $a = 0$. Then we need to prove that

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

[1] We are using the secret limit law that says that if two functions/expressions $g(x)$ and $h(x)$ take the same values for all x in a neighborhood of a, then

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x).$$

That is, the limit of $g(x)$ at a only depends on what g does at x-values really close to a. If another function does the same thing as $g(x)$ around a, then it has the same limit as g. This sort of fact has been used implicitly in many of the proofs in class, and you should feel free to use it without explanation on the exam.

[2] We're using the secret limit law that says that if two functions agree just to the right of a, then they have the same right-handed limit at a. Repeat everything in the previous footnote.

exists. In fact, it's going to equal 1, so it suffices to show that the one-sided limits are both 1. This can be checked as follows:

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{(1+h)-1}{h} = \lim_{h\to 0^+} 1 = 1$$
$$\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{e^h-1}{h} = \lim_{h\to 0} \frac{e^h-1}{h} = 1$$

□

4. Prove that the composition of any two decreasing functions is increasing.

   Proof. Let f and g be given decreasing functions. We claim that $f \circ g$ is increasing. Given numbers $x < y$, we need to show that $(f \circ g)(x) < (f \circ g)(y)$. First note that because g is decreasing,

   $$g(x) > g(y)$$

   or equivalently $g(y) < g(x)$. Meanwhile, f is decreasing. Applying the definition of "decreasing" to f and the numbers $g(y)$ and $g(x)$, we see that

   $$f(g(y)) > f(g(x))$$

   or equivalently

   $$(f \circ g)(x) < (f \circ g)(y).$$

   As x, y were arbitrary numbers satisfying $x < y$, we've shown that $f \circ g$ is increasing. (And as f and g were arbitrary decreasing functions, we've shown that the composition of any two decreasing functions is increasing.) □

5. Prove that the sum of any two decreasing functions is decreasing.

   (writing things a little more compactly than in the previous problem). Let f and g be decreasing functions, and $x < y$. As f is decreasing,

   $$f(x) > f(y)$$

   Likewise

   $$g(x) > g(y)$$

   Adding these two inequalities together, we see that

   $$(f+g)(x) = f(x) + g(x) > f(y) + g(y) = (f+g)(y),$$

   so $f + g$ is decreasing. □

6. Suppose f is a differentiable function on $\mathbb{R} = (\infty, \infty)$, and that $f'(x) < 0$ for all x. Prove that f is decreasing.

Proof. Suppose $x < y$. (We need to prove that $f(x) > f(y)$.) By the mean value theorem, there is some z (between x and y, though we won't use this), so that

$$\frac{f(y) - f(x)}{y - x} = f'(z).$$

As $f'(z) < 0$ by assumption, we see that

$$\frac{f(y) - f(x)}{y - x} < 0.$$

Multiplying both sides by the positive number $y - x$, we see that

$$f(y) - f(x) < 0.$$

After rearranging, we conclude that $f(x) > f(y)$. So f is decreasing [because $x < y$ was arbitrary]. □

7. Suppose $f(x) = ax^2 + bx + c$, where $a \neq 0$. Prove that f is not one-to-one.

Proof. Let

$$x_1 = \frac{-b}{2a} + 1 \text{ and } x_2 = \frac{-b}{2a} - 1.$$

We claim that $f(x_1) = f(x_2)$. This can probably be proven by just expanding things out. Let's do it by completing the square instead. Note that for any x,

$$f(x) = ax^2 + bx + c = a\left(x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c.$$

Letting $v = b/(2a)$ and $d = c - b^2/(4a)$, we see that for any x,

$$f(x) = a(x^2 + 2vx + v^2) + d = a(x + v)^2 + d.$$

Now $x_1 = -v + 1$ and $x_2 = -v - 1$, so

$$f(x_1) = f(-v+1) = a(-v+1+v)^2+d = a\cdot 1+d = a(-v-1+v)^2+d = f(-v-1) = f(x_2).$$

As x_2 and x_1 are not the same (they differ by the non-zero number 2), it follows that f is not one-to-one. □

There may be better proofs, this was what I thought of off the top of my head.

8. Let $g(x) = 2^x + 3^x$ for $|x| \leq 1$. (So the domain of g is $[-1, 1]$.) Prove that the range of g is exactly $[5/6, 5]$.

Proof. Let y be given. We need to show that y is in the range of g if and only if $5/6 \leq y \leq 5$.

First suppose that y is in the range of g. Then $y = g(x)$ for some $x \in [-1, 1]$. Note that 2^x and 3^x are increasing functions, so so is their sum g. Therefore,

$$5/6 = g(-1) \leq g(x) \leq g(1) = 5,$$

so $5/6 \leq y \leq 5$.

Conversely, suppose that $5/6 \leq y \leq 5$. If y equals $5/6$ or 5, then $y = g(-1)$ or $g(1)$, so y is in the range. Otherwise, y is strictly between $g(-1)$ and $g(1)$. As g is continuous, the intermediate value theorem applies, and there is some x between -1 and 1, such that $g(x) = y$. So y is in the range. □

9. Suppose that $h(x)$ is a continuous function on all of $\mathbb{R}$, that $h(0) = 0$, and that $h(x)$ is one-to-one. Show that $h(-1)h(1) < 0$.

Proof. Since h is one-to-one, $h(-1)$ and $h(1)$ can't be the same as $h(0) = 0$. So both are non-zero, and therefore their product $h(-1)h(1)$ is also non-zero. So the only way $h(-1)h(1) < 0$ can fail to hold is if $h(-1)h(1) > 0$. Assume this (for the sake of contradiction). Then $h(-1)$ and $h(1)$ have the same sign. This gives two cases:

Case 1: Both $h(1)$ and $h(-1)$ are positive. Let ϵ be some positive number smaller than both $h(1)$ and $h(-1)$. (For example, we could take ϵ to be half the minimum of $h(1)$ and $h(-1)$.) Now ϵ is between $0 = h(0)$ and $h(1)$, so by the intermediate value theorem there is some c between 0 and 1 such that $h(c) = \epsilon$. Also ϵ is between $0 = h(0)$ and $h(-1)$, so there is some d between 0 and -1 such that $h(d) = \epsilon$. Now

$$h(c) = \epsilon = h(d),$$

so by one-to-oneness, c should equal d. But this is false, since c is strictly between 0 and 1, and d is strictly between 0 and -1, so they can't be equal.[3]

Case 2: Both $h(1)$ and $h(-1)$ are negative. [Then do the same argument, but with ϵ a negative number bigger than both $h(1)$ and $h(-1)$.]

□

A more complicated version of this argument can be used to prove the following piece of trivia: any continuous one-to-one function on an interval is decreasing or increasing.

[3] If this argument seems weird, drawing a picture may help. The point is that in going from $h(0) = 0$ to the positive numbers $h(1)$ and $h(-1)$, the continuous function h has to run over the same numbers on both sides of 0, and this violates one-to-one-ness.