Solution to 4.4.89

November 5, 2014

Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Use the definition of derivative to compute $f'(0)$.

Solution. By definition

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}.$$

The expression inside the limit is only evaluated values of x which don't equal zero, so we can replace $f(x)$ with e^{-1/x^2}. And $f(0)$ is just 0, so

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{e^{-1/x^2}}{x}.$$

For mysterious reasons, we rewrite this as

$$f'(0) = \lim_{x \to 0} \frac{e^{-1/x^2}}{x} = \lim_{x \to 0} \frac{x^{-1}}{e^{1/x^2}}.$$

We claim that

$$\lim_{x \to 0^+} \frac{x^{-1}}{e^{1/x^2}} \stackrel{?}{=} \lim_{x \to 0^-} \frac{x^{-1}}{e^{1/x^2}} = 0. \tag{1}$$

We will calculate both these limits using the $\frac{\infty}{\infty}$-version of l'Hôpital's rule. First we check that l'Hôpital's rule applies:

- Both x^{-1} and e^{1/x^2} are differentiable on $(-\infty, 0) \cup (0, \infty)$, and the derivative of e^{1/x^2} is

$$\frac{-2e^{1/x^2}}{x^3},$$

  which does not equal zero anywhere.
- As $x \to 0^+$, the quantity x^{-1} goes to $+\infty$, and $1/x^2$ goes to $+\infty$, and e^{1/x^2} goes to $+\infty$. So as $x \to 0^+$, both the numerator and denominator approach $\pm\infty$.

- As $x \to 0^-$, the quantity x^{-1} goes to $-\infty$, and $1/x^2$ goes to $+\infty$, and e^{1/x^2} goes to $+\infty$. So as $x \to 0^0$, both the numerator and denominator approach $\pm\infty$.

So l'Hôpital's rule applies, and we can make the calculations:

$$\lim_{x\to 0^+} \frac{x^{-1}}{e^{1/x^2}} = \lim_{x\to 0^+} \frac{-1/x^2}{-2e^{1/x^2}/x^3} = \lim_{x\to 0^+} \frac{1}{2} x e^{-1/x^2} = \frac{1}{2}\left(\lim_{x\to 0^+} x\right)\left(\lim_{x\to 0^+} e^{-1/x^2}\right) = \frac{1}{2}\cdot 0\cdot 0 = 0.$$

$$\lim_{x\to 0^-} \frac{x^{-1}}{e^{1/x^2}} = \lim_{x\to 0^-} \frac{-1/x^2}{-2e^{1/x^2}/x^3} = \lim_{x\to 0^-} \frac{1}{2} x e^{-1/x^2} = \frac{1}{2}\left(\lim_{x\to 0^-} x\right)\left(\lim_{x\to 0^-} e^{-1/x^2}\right) = \frac{1}{2}\cdot 0\cdot 0 = 0.$$

So (1) is true. Then because the one-sided limits agree,

$$f'(0) = \lim_{x\to 0} \frac{x^{-1}}{e^{1/x^2}} = 0.$$

□

(b) The function f has derivatives of all orders that are defined on $\mathbb{R}$.

Proof. We will need the following variant of the limit that came up in the previous problem:

Lemma 0.1. *For any integer N,*

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^N} = 0.$$

Proof. There's probably a way to do this inductively. Instead, we do the following.

First suppose that $N = 2$. Then we need to show that

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^N} = \lim_{x\to 0} \frac{e^{-1/x^2}}{x^2} \stackrel{?}{=} 0.$$

We can rewrite the left hand side as

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^2} = \lim_{x\to 0} \frac{x^{-2}}{e^{1/x^2}}.$$

Rewritten in this way, the $\frac{\infty}{\infty}$-form of l'Hôpital's rule applies. Indeed, x^{-2} and e^{1/x^2} both go to $+\infty$ as $x \to 0$, and both are differentiable for nonzero numbers, and the derivative of e^{1/x^2} never vanishes, as we saw in part (a).

So we can apply l'Hôpital:

$$\lim_{x\to 0} \frac{x^{-2}}{e^{1/x^2}} = \lim_{x\to 0} \frac{-2/x^3}{-2e^{1/x^2}/x^3} = \lim_{x\to 0} e^{-1/x^2} = 0.$$

So we have handled the case where $N = 2$.

Now let N be arbitrary. Let k be a positive integer greater than $N/2$, so that $2k > N$. By the $N = 2$ case just proven, we know

$$\lim_{y\to 0} \frac{e^{-1/y^2}}{y^2/k} = \lim_{y\to 0} k \frac{e^{-1/y^2}}{y^2} = k \lim_{y\to 0} \frac{e^{-1/y^2}}{y^2} = k \cdot 0 = 0.$$

Making the change of variables $x = y/\sqrt{k}$, so that $y^2 = kx^2$, we get

$$\lim_{x\to 0} \frac{e^{-1/(kx^2)}}{x^2} = \lim_{y\to 0} \frac{e^{-1/y^2}}{y^2/k} = 0.$$

Then

$$\lim_{x\to 0} \left(\frac{e^{-1/(kx^2)}}{x^2} \right)^k = \left(\lim_{x\to 0} \frac{e^{-1/(kx^2)}}{x^2} \right)^k = 0^k = 0.$$

But

$$\left(\frac{e^{-1/(kx^2)}}{x^2} \right)^k = \frac{\left(e^{-1/(kx^2)}\right)^k}{(x^2)^k} = \frac{e^{-1/x^2}}{x^{2k}}.$$

So

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^{2k}} = \lim_{x\to 0} \left(\frac{e^{-1/(kx^2)}}{x^2} \right)^k = 0.$$

Finally,

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^N} = \lim_{x\to 0} x^{2k-N} \frac{e^{-1/x^2}}{x^{2k}} = \left(\lim_{x\to 0} x^{2k-N} \right) \left(\lim_{x\to 0} \frac{e^{-1/x^2}}{x^{2k}} \right) = 0^{2k-N} \cdot 0 = 0.$$

This completes the proof of the Lemma. □

Next, we prove by induction on n the following statement: the nth derivative $f^{(n)}$ of f exists on all of $\mathbb{R}$, and has the following form

$$f^{(n)}(x) = \begin{cases} p(x)x^{-k}e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

for $p(x)$ some polynomial and k some integer, depending on n but not on x.

For the base case, we take $n = 0$. Then the zeroth derivative $f^{(0)}$ is just f, which has the prescribed form. (Take $p(x) = 1$ and $k = 0$.)

Now suppose that $n > 0$ and $f^{(n-1)}$ exists and has the prescribed form. So

$$f^{(n-1)}(x) = \begin{cases} p(x)x^{-k}e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

We need to show that the derivative $f^{(n)}$ of $f^{(n-1)}$ exists and has this form as well. When $x \neq 0$, the derivative of $f^{(n-1)}(x)$ is the same as the derivative of $p(x)x^{-k}e^{-1/x^2}$, which by the product rule is

$$\begin{aligned}\frac{d}{dx}p(x)x^{-k}e^{-1/x^2} &= p'(x)x^{-k}e^{-1/x^2} - p(x)\cdot kx^{-k-1}e^{-1/x^2} - p(x)x^{-k}2x^{-3}e^{-1/x^2} \\ &= \left(p'(x)x^3 - kp(x)x^2 - 2p(x)\right)x^{-(k+3)}e^{-1/x^2}.\end{aligned}$$

The expression $p'(x)x^3 - kp(x)x^2 - 2p(x)$ is a polynomial, and $k+3$ is a nonnegative integer, so we've established that $f(n) = (f^{(n-1)})'$ has the desired form when $x \neq 0$.

It remains to check the value and existence of $f^{(n)}(x)$ at $x = 0$, i.e., to show that $f^{(n)}(0) = 0$. We need to show that

$$f^{(n)}(0) = \lim_{x\to 0}\frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x-0} \stackrel{?}{=} 0.$$

By the inductive hypothesis, $f^{(n-1)}(0) = 0$, and $f^{(n-1)}(x) = p(x)x^{-k}e^{-1/x^2}$. So we can rewrite the limit as

$$\lim_{x\to 0}\frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x-0} = \lim_{x\to 0}\frac{p(x)x^{-k}e^{-1/x^2} - 0}{x} = \lim_{x\to 0}p(x)x^{-k-1}e^{-1/x^2}$$

Now we can apply the product rule for limits, to see

$$f^{(n)}(0) = \lim_{x\to 0}p(x)x^{-k-1}e^{-1/x^2} = \left(\lim_{x\to 0}p(x)\right)\left(\lim_{x\to 0}\frac{e^{-1/x^2}}{x^{k+1}}\right).$$

Polynomials are continuous, so $\lim_{x\to 0}p(x) = p(0)$, which is some number. And by the Lemma,

$$\lim_{x\to 0}\frac{e^{-1/x^2}}{x^{k+1}} = 0.$$

Thus

$$f^{(n)}(0) = \left(\lim_{x\to 0}p(x)\right)\left(\lim_{x\to 0}\frac{e^{-1/x^2}}{x^{k+1}}\right) = p(0)\cdot 0 = 0.$$

So we see that $f^{(n)}$ exists everywhere, and has the following form

$$f^{(n)}(x) = \begin{cases} q(x)x^{-j}e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0\end{cases}$$

where $q(x)$ is a polynomial[1], and j is an integer[2]. So we've completed the inductive step.

Now we have successfully proven by induction on n that $f^{(n)}(x)$ exists and has a certain form. In particular, we've shown that it exists, so we're done. □

[1] Namely $p'(x)x^3 - kp(x)x^2 - 2p(x)$.
[2] Namely $k+3$.