Thursday Aug. 28

Solutions to in class problems:

A.1 Evaluate each expression without using a calculator:

(a) $(-3)^4 = (-3)\cdot(-3)\cdot(-3)\cdot(-3) =$

$9 \cdot (-3)\cdot(-3) =$

$(-27) \cdot (-3) =$

$\boxed{81}$

(b) -3^4 means $-(3^4)$

3^4 is $3\cdot3\cdot3\cdot3 = 9\cdot3\cdot3 = 27\cdot3 = 81$

So, -3^4 is $\boxed{-81}$

(c) $3^{-4} = \frac{1}{3^4} = \boxed{1/81}$

the negative exponent roughly means divide by three, four times, rather than multiply by three, four times.

So $(((1/3)/3)/3)/3 = \frac{1}{3\cdot3\cdot3\cdot3} = \frac{1}{81}$

$x^{-y} = 1/(x^y)$

(d) $5^{23}/5^{21} = 5^{23-21} = 5^2 = 5\times5 = \boxed{25}$

because of general rule: $x^y/x^z = x^{y-z}$

(e) $\left(\frac{2}{3}\right)^{-2} = \frac{1}{\left(\frac{2}{3}\right)^2} = \frac{1}{4/9} = \boxed{9/4}$

Or: $\left(\frac{2}{3}\right)^{-2} = \left(\frac{2}{3}\right)^{(-1)\cdot2} = \left(\left(\frac{2}{3}\right)^{-1}\right)^2 = \left(\frac{3}{2}\right)^2 = 9/4$

A.1. (f) $16^{-3/4} = \left((16)^{1/4}\right)^{-3}$

$16^{1/4}$ means $\sqrt[4]{16} = 2$. [Fourth roots can be gotten by taking square roots twice]

~~So we want~~

$$\sqrt[4]{16} = \sqrt{\sqrt{16}} = \sqrt{4} = 2$$

So we want $2^{-3} = \frac{1}{2^3} = \boxed{\frac{1}{8}}$

A5. Simplify each rational expression.

(a) $\dfrac{x^2+3x+2}{x^2-x-2}$

We simplify rational expressions the same way we simplify rational numbers (like: $2/6 = 1/3$) by looking for common factors in the numerator and denominator.

So we need to factor the numerator and the denominator:

Numerator: $x^2+3x+2 = (x+a)\cdot(x+b)$

$a+b=3$, $a\cdot b = 2$

$a=2$, $b=1$ works,

and indeed,

$x^2+3x+2 = (x+2)(x+1)$

Denominator: $x^2-x-2 = (x-2)(x+1)$?

$-2+1=-1$ ✓

$(-2)\cdot(1) = -2$ ✓ OK.

A.5. a. continued

So, $\dfrac{x^2+3x+2}{x^2-x-2} = \dfrac{(x+2)\cancel{(x+1)}}{(x-2)\cancel{(x+1)}} = \boxed{\dfrac{x+2}{x-2}}$

(b) $\dfrac{2x^2-x-1}{x^2-9} \cdot \dfrac{x+3}{2x+1}$

We'll need to factor the numerator and denominator, so it's best to not multiply out the polynomials

$(x^2-9)(2x+1) = 2x^3+x^2-18x-9 \leftarrow$ not useful

~~x^2-9~~ Instead, factor $2x^2-x-1$ and x^2-9

$2x^2-x-1 = ?$ $\qquad x^2-9 = (x-3)(x+3)$

[in general, $x^2-a^2 = (x-a)(x+a)$]

Perhaps $2x+1$ is a factor of $2x^2-x-1$, to make the problem easy?

$$\begin{array}{r|l} & x-1 \\ 2x+1 & 2x^2-x-1 \\ & -\ 2x^2+x \\ \hline & -2x-1 \\ & -\ (-2x-1) \\ \hline & 0 \end{array}$$

~~Ok, so 2x+1~~

Ok, so $(2x^2-x-1) = (2x+1)(x-1)$

So $\dfrac{2x^2-x-1}{x^2-9} \cdot \dfrac{x+3}{2x+1} = \dfrac{\cancel{(2x+1)}(x-1)\cancel{(x+3)}}{(x-3)\cancel{(x+3)}\cancel{(2x+1)}} = \boxed{\dfrac{x-1}{x-3}}$

A.5.(c) $\dfrac{x^2}{x^2-4} - \dfrac{x+1}{x+2}$.

We need the least common multiple of the denominators
x^2-4 and $x+2$
Well, x^2-4 is $(x-2)(x+2)$, so x^2-4 is the LCM.

$$\frac{x+1}{x+2} = \frac{(x+1)(x-2)}{(x+2)(x-2)} = \frac{x^2-x-2}{x^2-4}$$

So $$\frac{x^2}{x^2-4} - \frac{x+1}{x+2} = \frac{x^2}{x^2-4} - \frac{x^2-x-2}{x^2-4}$$

$$= \frac{x^2-(x^2-x-2)}{x^2-4} = \frac{x+2}{x^2-4}.$$

We can simplify this further, ~~sinc~~ factoring x^2-4

$$\frac{x+2}{x^2-4} = \frac{x+2}{(x+2)(x-2)} = \boxed{\frac{1}{x-2}}$$

(d) $\dfrac{y/x - x/y}{1/y - 1/x}$

What is $y/x - x/y$? The least common denominator is $\frac{1}{xy}$.

So, it's $\dfrac{y\cdot y}{x\cdot y} - \dfrac{x\cdot x}{y\cdot x} = \dfrac{y^2-x^2}{xy}$

Likewise, $\dfrac{1}{y} - \dfrac{1}{x} = \dfrac{x}{xy} - \dfrac{y}{xy} = \dfrac{x-y}{xy}$.

So now we have $\dfrac{(y^2-x^2)/(xy)}{(x-y)/(xy)} = \dfrac{y^2-x^2}{x-y}$.

But $y^2 - x^2$ can be factored as $(y-x)(y+x)$

So we really have $\dfrac{(y-x)(y+x)}{(x-y)}$.

As $(y-x) = -(x-y)$, this is $\boxed{-(y+x)}$

A.9 Solve inequalities; write answers using intervals.

(a) $-4 < 5 - 3x \leq 17$

We can multiply both sides of an inequality by a negative number, if we reverse the direction of all the signs.

So $-4 < 5 - 3x \leq 17$ is equivalent to

$4 > 3x - 5 \geq -17$

Add 5 to all sides:

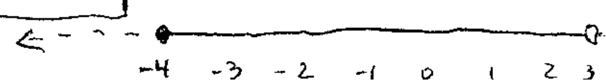
$5 + 4 > 3x \geq 5 - 17$

$9 > 3x \geq -12$

Divide by three (which is positive)

$3 > x \geq -4$

In interval notation, ~~the set~~ the set of x such that $3 > x \geq -4$ is denoted $\boxed{[-4, 3)}$



A, 9.(b) $x^2 < 2x + 8$

One method: move everything to one side

$$x^2 - 2x - 8 < 0$$

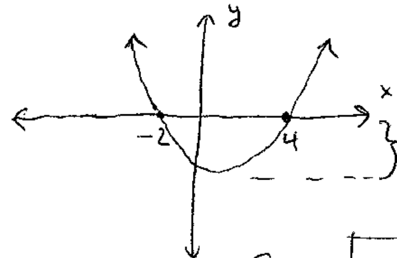
Factor:

$$(x-4)(x+2) < 0$$

The polynomial $(x-4)(x+2)$ has zeros at $4, -2$.

~~Cases:~~ ~~$x < -2$. Then x~~ It changes sign there

~~$x < -2$~~
~~$x = -2$~~



we want $y < 0$ this part.

So: $\boxed{(-2, 4)}$

Or, break into cases:

Case	$(x-4)$	$(x+2)$	$(x-4)\cdot(x+2)$	
$x < -2$	negative	negative	positive	NO
$x = -2$	negative	zero	zero	NO
$-2 < x < 4$	negative	positive	negative	YES
$x = 4$	zero	positive	zero	NO
$x > 4$	positive	positive	positive	NO

Another method: Complete the square . . .

$$x^2 < 2x + 8$$
$$x^2 - 2x < 8$$
$$x^2 - 2x + 1 < 8 + 1 = 9$$
$$(x-1)^2 < 9$$

Take square roots: $|x-1| < \sqrt{9} = 3$ ← $\sqrt{a^2} = |a|$.

So, x is within 3 of 1.

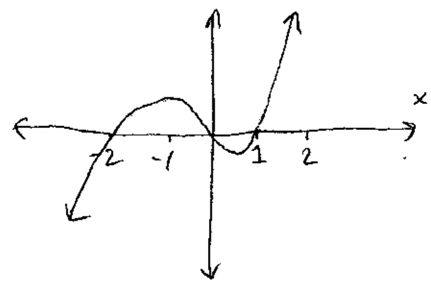
$$1 - 3 < x < 1 + 3$$
$$-2 < x < 4.$$

So: $\boxed{(-2, 4)}$

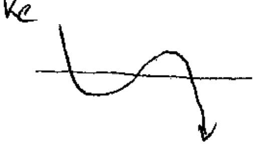
A.9. (c)

$$\underbrace{x\,(x-1)(x+2)}_{\text{this is a cubic polynomial,}} > 0$$

with zeros at 0, 1, and -2.
It changes sign at each zero.
The graph looks like



[It doesn't look like



because leading term is positive]

So it's positive
between -2 and ~~1~~0 (exclusive),
and greater than 1.

So, $\boxed{(-2,0) \cup (1,\infty)}$

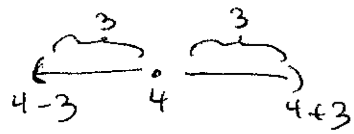
Or, break into cases:

Case	x	$(x-1)$	$(x+2)$	Product	OK?
$x < -2$	$-$	$-$	$-$	$-$	NO
$x = -2$	$-$	$-$	0	0	NO
$-2 < x < 0$	$-$	$-$	$+$	$+$	YES
$x = 0$	0	$-$	$+$	0	NO
$0 < x < 1$	$+$	$-$	$+$	$-$	NO
$x = 1$	$+$	0	$+$	0	NO
$x > 1$	$+$	$+$	$+$	$+$	YES

So, $-2 < x < 0$ OR $x > 1$.
In interval notation: $\boxed{(-2,0) \cup (1,\infty)}$

A.9. (d) $\quad |x-4| < 3$

This says that the distance between x and 4 is less than 3.

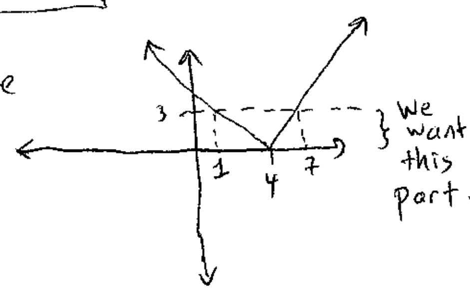


So, $\quad 4-3 < x < 4+3$

$\quad\quad 1 < x < 7$

This interval is $\boxed{(1, 7)}$

The graph of $|x-4|$ looks like



(e) $\quad \dfrac{2x-3}{x+1} \leq 1. \qquad (*)$

Ok, well, $x+1$ had better not be 0.

<u>If $x+1>0$,</u> $(*)$ is equivalent to

$$2x-3 \leq x+1$$
$$x-3 \leq 1$$
$$x \leq 4$$

And $x+1>0$ means $x>-1$.

So if $-1 < x \leq 4$, then x is okay

if $x>4$, x is not okay

<u>If $x+1<0$</u>, $(*)$ is equivalent to

$$2x-3 \geq x+1$$
$$x-3 \geq 1$$
$$x \geq 4$$

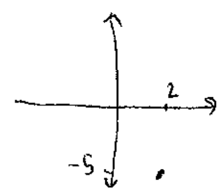
A, 9.(e) continued...

But $x+1<0$ means $x<-1$.
So we would need $x<-1$ and $x \geq 4$, impossible.
So the $x+1<0$ case gives no solutions.

So the only solutions are $-1 < x \leq 4$

$$\boxed{(-1, 4]}$$

B1 Find an equation for the line that passes thru the point $(2,-5)$

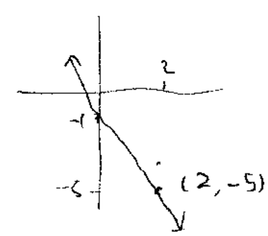


(a) ... and has slope -3.

OK, so, the equation is $y = -3x + b$, where b is some number — the y-intercept.

We want $(2,-5)$ to be on the line, so plug in $x=2$, $y=-5$.

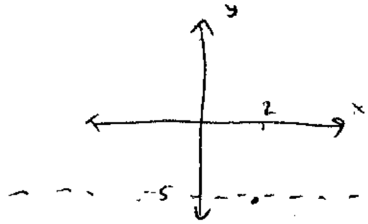
$$\begin{aligned} y &= -3\,x + b \\ -5 &= (-3)(2) + b \\ -5 &= -6 + b \\ 1 &= b. \end{aligned}$$



So: $\boxed{y = -3x + 1}$

B.1. (b) Line thru $(2,-5)$, parallel to the x-axis.

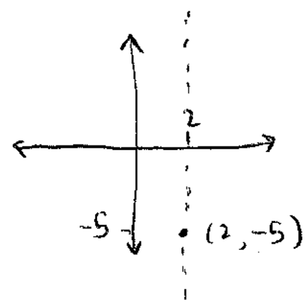
$$\boxed{y=-5}$$



Lines parallel to the x-axis have slope 0, are all of the form $y=$ (some number).

(c) Line thru $(2,-5)$ parallel to the y-axis.

$$\boxed{x=2}$$



The y-axis is given by the equation $x=0$,
Parallel lines have equations like: $x=$ (some number)

(d) Line thru $(2,-5)$, parallel to the line $2x-4y=3$.

Parallel lines have the same slope.
What's the slope of $2x-4y=3$?
Solve for y:

$$2x-4y=3$$
$$2x-3=4y$$
$$\tfrac{1}{2}x-\tfrac{3}{4}=y$$

The slope is $\frac{1}{2}$.
We want a line $y=\frac{1}{2}x+b$, having $(2,-5)$
Plug in $y=-5, x=2$:

$$-5=\tfrac{1}{2}\cdot 2+b \quad = 1+b$$

$b=-6$. So: $\boxed{y=\frac{1}{2}x-6}$

B3 : Find the center and radius of the circle with equation

$$x^2 + y^2 - 6x + 10y + 9 = 0$$

We do this by completing the squares

$$x^2 - 6x \rightsquigarrow x^2 - 6x + 9$$
$$y^2 + 10y \rightsquigarrow y^2 + 10y + 25$$

So... $x^2 + y^2 - 6x + 10y + 9 = 0$

becomes

$$(x^2 - 6x + 9) + (y^2 + 10y + 25) = 25$$

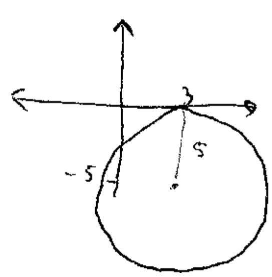
what we wanted, $\longrightarrow$ $(x-3)^2 + (y+5)^2 = 25 = 5^2$

because a circle with center (x_0, y_0) and radius r has equation

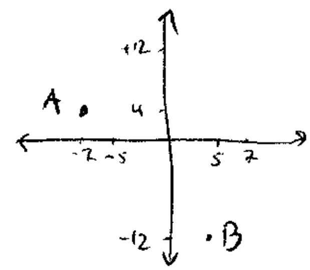
$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

* Here, $x_0 = 3$, $y_0 = -5$, $r = 5$.

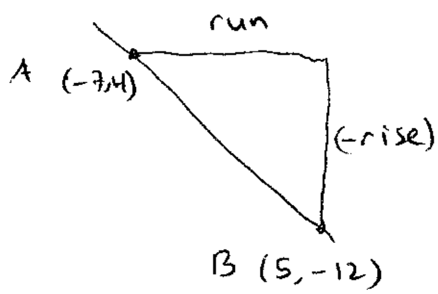
So $\boxed{\text{the center is } (3, -5), \text{ the radius is } 5}$



B4 Let $A = (-7, 4)$, $B = (5, -12)$ points in the plane



(a) Find the slope of the line containing A and B.



slope is $\frac{\text{rise}}{\text{run}}$

rise is $-12 - 4$
$= -16$

run is $5 - (-7) = 12$

Slope is $\cancel{-18/12} = \cancel{\boxed{-3/2}}$ oops $\frac{-16}{12} = \boxed{-4/3}$

Or, using the formula, it's $\frac{(-12) - (4)}{5 - (-7)} = \cancel{-18/12 = -3/2}$

$\frac{-16}{12} = \boxed{-4/3}$

(b) Find the equation for that line, and the intercepts.

$y = -4/3\, x + b$, where b is the y-intercept.
Plug in $(x, y) = (-7, 4)$, since A is on the line.

$4 = -4/3 \cdot (-7) + b.$

$4 = 28/3 + b$ $\qquad b = 12/3 - 28/3 = -16/3.$

$\boxed{y = -4/3\, x - 16/3, \quad \text{y-intercept is } -16/3}$

B.4. (b) continued.

For x-intercept, ~~solve~~ set $y=0$, solve for x

$$0 = -\tfrac{4}{3}x - \tfrac{16}{3}$$

$$16 = -4x$$

$$\boxed{x = -4}$$

B.4. (c) Find the midpoint of $\overline{AB}$ (the line segment)

$A = (-7,4) \qquad B = (5,-12)$

Coordinates of the midpoint are averages of the respective coordinates of A, B

$$\left(\frac{(-7+5)}{2}, \frac{4+(-12)}{2}\right) = \left(\frac{-2}{2}, \frac{-8}{2}\right) = \boxed{(-1,-4)}$$

B.4. (d) Find the length of that segment

Distance formula from (x_1, y_1) to (x_2, y_2) is $\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

Here: $\sqrt{(-7-5)^2 + (4-(-12))^2}$

$$= \sqrt{(-12)^2 + 16^2} = \sqrt{144+256} = \sqrt{400} = \boxed{20}$$

B.4. (e) Find equation of perpendicular bisector of the segment $\overline{AB}$,
$A = (-7,4)$, $B = (5,-12)$

The bisector goes through the midpoint $(-1,-4)$ and is perpendicular to $\overleftrightarrow{AB}$

Fact: Perpendicular lines have negative reciprocal slopes

$\overline{AB}$ had slope $-4/3$. So the bisector has slope $-1/(-4/3) = 3/4$.

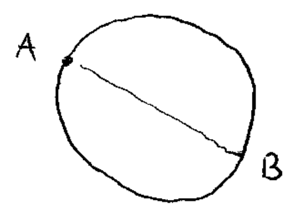
$y = 3/4\, x + b$. Plug in $(-1,-4)$

$-4 = 3/4 \cdot (-1) + b$ $\quad b = -4 + 3/4 = -16/4 + 3/4 = -13/4$

so, $\boxed{y = 3/4\, x - 13/4}$

B.4. (f) Find equation of circle with diameter $\overline{AB}$

Well, the center will be at the middle, and the radius will be half the length of $\overline{AB}$.



Center = midpoint of $\overline{AB}$ = $(-1,-4)$
Radius = $\frac{1}{2}$ diameter = $\frac{1}{2}$(length of $\overline{AB}$) = $\frac{1}{2} \cdot 20 = 10$

So: $\boxed{(x+1)^2 + (y+4)^2 = 10^2}$

In general, $(x-x_0)^2 + (y-y_0)^2 = r^2$ is the equation for the circle with center (x_0, y_0) and radius r.

C.3. ~~(a)~~ Find the domains of the functions...

(a) $f(x) = \dfrac{2x+1}{x^2+x-2}$. } this is well-defined as long as ~~x2xx~~ there's no division by zero.

(Division by zero is the only thing that could go wrong)

When is

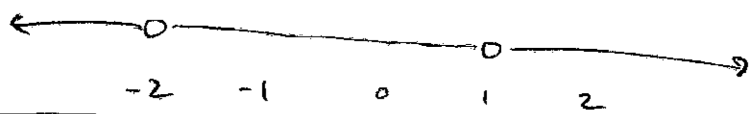
$$x^2 + x - 2 = 0 ?$$

Factor:

$$(x+2)(x-1) = 0.$$

$$x = -2, \text{ or } x = 1$$

So the domain is all x <u>except</u> $-2, 1$



It's $\boxed{(-\infty, -2) \cup (-2, 1) \cup (1, \infty)}$

(b) $g(x) = \dfrac{\sqrt[3]{x}}{x^2+1}$

The cube root function is always defined (unlike the square root function, which can't be done to negative #'s).

The only thing that could go wrong here is division-by-0.

When is $x^2 + 1 = 0$?

$x^2 = -1$?

Never.

Thi Nothing ever goes wrong. $g(x)$ makes sense for all x.

C.3.(b) continued

The domain of $g(x)$ is all numbers.

$\boxed{(-\infty, \infty)}$ also known as $\mathbb{R}$.

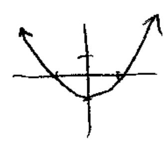
C.3.(c) $h(x) = \sqrt{4-x} + \sqrt{x^2-1}$, find the domain.

What could go wrong?
$4-x$ or x^2-1 could be negative.

The domain is those numbers x
for which $4-x \geq 0$ and $x^2-1 \geq 0$

$4 \geq x$ "

$x \leq 4$ " $x^2-1 \geq 0$

x^2-1 looks like  it's negative between -1 and 1

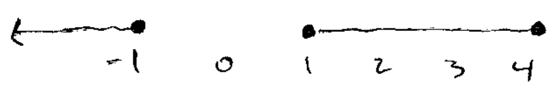
$x^2-1 \geq 0$
is equivalent to $x \geq 1$ or $x \leq -1$.

$(x \leq 4)$ AND $(x \geq 1$ OR $x \leq -1)$

[AND distributes over OR!]

$(x \leq 4$ AND $x \geq 1)$ OR $(x \leq 4$ AND $x \leq -1)$

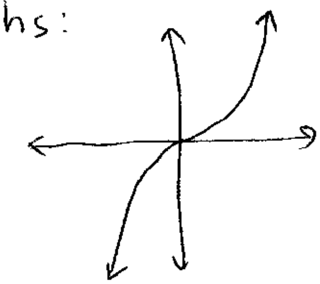
$1 \leq x \leq 4$ OR $x \leq -1$.



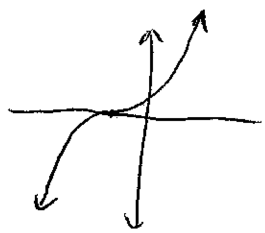
$\boxed{(-\infty, -1] \cup [1, 4]}$

C.5. Sketches of graphs:

(a) $y = x^3$

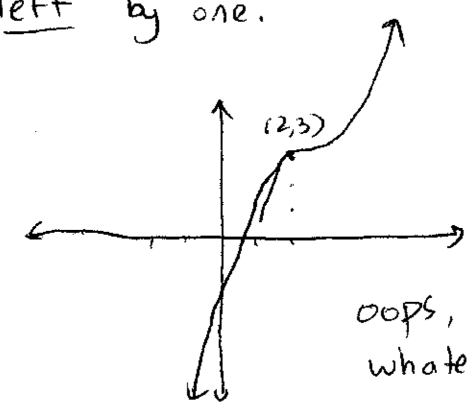


(b) $y = (x+1)^3$.



Same as $y = x^3$, but shifted left by one.

(c) $y = (x-2)^3 + 3$



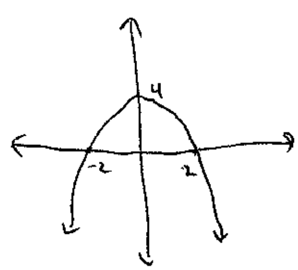
oops, whatever

Same as $y = x^3$, but shifted right by 2, up by 3

y-intercept: $(-2)^3 + 3 = -8 + 3 = -5$.

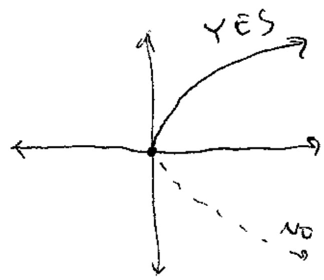
x-intercept: $2 - \sqrt[3]{3}$ or something horrible
cube root

(d) $y = 4 - x^2$



The parabola curves down because the "leading term" $-x^2$ has negative coefficient

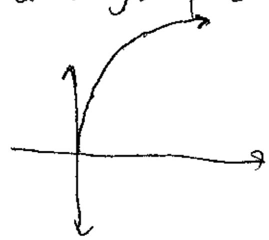
C.5.(e) $y = \sqrt{x}$



This is the top half of a parabola $y^2 = x$.

$\sqrt{4}$ is 2, so it can't also be -2, if we want our mathematical expressions to be meaningful.
By convention, $\sqrt{x}$ is always positive or zero.

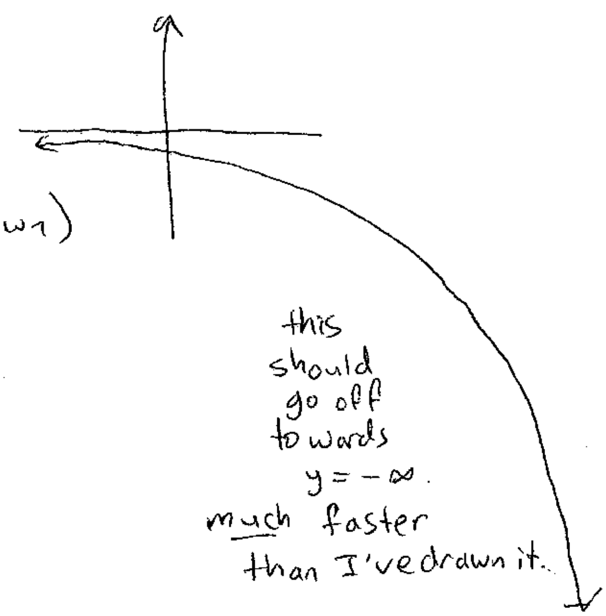
C.5.(f) $y = 2\sqrt{x}$



Same as $y = \sqrt{x}$, but vertically stretched by a factor of 2.

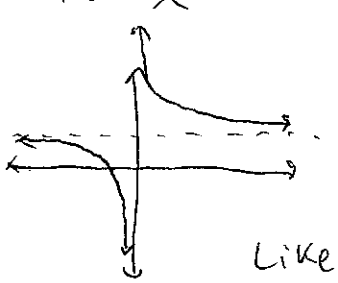
C.5.(g) $y = -2^x$

(like $y = 2^x$, but upsidedown)



this should go off towards $y = -\infty$ much faster than I've drawn it.

(h) $y = 1 + 1/x$



Like $y = 1/x$, but shifted up by 1.

D.6. If $\sin x = 1/3$ and $\sec y = 5/4$,

and $0 \le x \le \pi/2$, $0 \le y \le \pi/2$,

what is $\sin(x+y)$?

Well, trig identity

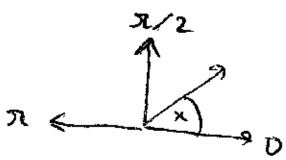
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

We have $\sin x = 1/3$. We also have $\cos y = \frac{1}{\sec y} = 4/5$

trig identity: $\sin^2 x + \cos^2 x = 1$

$$1/9 + \cos^2 x = 1$$

$$\cos^2 x = 1 - 1/9 = 8/9$$



$0 < x < \pi/2 = 90°$,

$\cos x$ is positive.

So $\cos x = \sqrt{8/9} = \frac{2}{3}\sqrt{2}$

Similarly:

$$\sin^2 y + \cos^2 y = 1$$

$$\sin^2 y = 1 - \cos^2 y = \cancel{1-\frac{16}{25}}\ 1 - (4/5)^2 = \frac{25-16}{25} = 9/25$$

$$\sin y = \sqrt{9/25} = 3/5.$$

<u>So</u>: $\sin x \cos y + \cos x \sin y =$

$$1/3 \cdot 4/5 + \left(\tfrac{2}{3}\sqrt{2}\right)\left(3/5\right)$$

$$= \frac{4}{15} + \frac{2\sqrt{2}}{5} = \frac{4 + 6\sqrt{2}}{15}$$

D7 Prove the identities:

(a) $\tan\theta \sin\theta + \cos\theta = \sec\theta.$ $(*)$

Express everything in terms of sines & cosines!

$(*)$ is $\dfrac{\sin\theta}{\cos\theta}\sin\theta + \cos\theta \overset{?}{\underset{?}{=}} \dfrac{1}{\cos\theta}$

Least common denominators on the left:

$$\frac{\sin\theta\,\sin\theta}{\cos\theta} + \frac{\cos\theta\,\cos\theta}{\cos\theta} \overset{?}{\underset{?}{=}} \frac{1}{\cos\theta}.$$

This is <u>true</u> – it's just the usual identity

$$\sin^2\theta + \cos^2\theta = 1,$$

with both sides divided by $\cos\theta$.

If we want to write the proof more directly, ~~we could write...~~ rather than backwards, we would say:

~~secθ~~

It is known that $\sin^2\theta + \cos^2\theta = 1$.

Divide both sides by $\cos\theta$:

$$\frac{\sin^2\theta}{\cos\theta} + \cos\theta = \frac{1}{\cos\theta}.$$

Now $\dfrac{\sin^2\theta}{\cos\theta} = \left(\dfrac{\sin\theta}{\cos\theta}\right)\overset{\sin}{\cancel{\cos}}\theta = \tan\theta\,\overset{\sin}{\cancel{\cos}}\theta,$ and $\dfrac{1}{\cos\theta} = \sec\theta,$

so $\tan\theta \sin\theta + \cos\theta = \sec\theta.$

D7 (b)

Prove the identity

$$\frac{2\tan x}{1+\tan^2 x} = \sin 2x \;.\,.$$

OK, first express everything in terms of $\sin x$, $\cos x$.

...

$\tan x = \sin x \;/\cos x$.

So $$\frac{2\tan x}{1+\tan^2 x} = \frac{2\sin x\;/\cos x}{1+\sin^2 x/\cos^2 x}$$

Multiply top and bottom by $\cos^2 x$.

hey, this is $\sin 2x$

$$\frac{2\tan x}{1+\tan^2 x} = \frac{2\sin x\,\cos x}{\cos^2 x+\sin^2 x}. = 2\sin x\cos x = \sin 2x.$$

hey, this is 1

Well,
we're done!