Let $f(x) = x$'s biological father, and $m(x) = x$'s biological mother. Let $g(x)$ be the piecewise function

$$g(x) = \begin{cases} x\text{'s spouse} & \text{if } x \text{ is married} \\ f(x) & \text{if } x \text{ is unmarried} \end{cases}$$

1. What are the domain and range of m?

2. Solve for x:

$$f(x) = \text{Charles, Prince of Wales}$$

3. What is $f \circ m$? What is $f \circ f$?

4. What is the domain of g?

5. Fill in the table

x	g(x)
Bill Clinton	Hillary Clinton
Hillary Clinton	
Kanye West	
North West	
Prince William	
Prince Harry	
Prince Charles	
42/3	

6. Can we graph g?

7. Solve for x:

   (a) $g(x) = \text{Prince Charles}$

   (b) $g(x) = \text{North West}$

   (c) $g(x) = \text{Kim Kardashian}$

8. Of Prince Charles, North West, and Kim Kardashian, who is in the range of g?

9. What is the range of g?

10. Solve for x:

$$g(x) \text{ is younger than } x$$

11. Write $g \circ g$ as a piecewise function

12. Solve for x:

$$g(g(x)) = x$$

1 Answers

For future reference, note that $g(x)$ is a piecewise function

$$g(x) = \begin{cases} x\text{'s spouse} & \text{if } x \text{ is married} \\ x\text{'s biological father} & \text{if } x \text{ is unmarried} \end{cases}$$

1. The domain of m is the set of x such that x has a mother. I was originally thinking of the set of all people, but in hindsight this wasn't clear, so perhaps the domain of m should really be the set of all things that have mothers. In what follows, though, I will assume that these are functions on the set of people. The range of m is the set of all mothers.

2. We want the set of x such that Prince Charles is x's biological father. According to Wikipedia, Prince Charles has two children, Prince Harry and Prince William. So the set of solutions would be

$$\{\text{Prince William, Prince Harry}\}$$

3. Your mother's father is your maternal grandfather, so $f \circ m$ is the function sending x to x's (biological) maternal grandfather. In more detail,

$$\begin{aligned}(f \circ m)(x) = f(m(x)) &= f(x\text{'s mother}) \\ &= (x\text{'s mother})\text{'s father} = x\text{'s maternal grandfather.}\end{aligned}$$

   Similarly, $(f \circ f)(x)$ is x's *paternal* grandfather.

4. The domain of g contains all people, because if x is any person, then $g(x)$ makes sense. Indeed, if x is married, then x has a spouse (by definition of "married") while if x is unmarried, then x has a biological father, because all people have biological fathers.

5.

x	g(x)
Bill Clinton	Hillary Clinton
Hillary Clinton	Bill Clinton
Kanye West	Kim Kardashian
North West	Kanye West
Prince William	Kate Middleton
Prince Harry	Prince Charles
Prince Charles	Camilla Parker-Bowles
42/3	*not defined*

   Note that $\frac{42}{3}$ is not in the domain of g.

6. No, since g isn't a function on numbers.

7. **(a)** Since g is a piecewise function, we break into two cases:

   - Case 1: x is married. Then $g(x) =$ Prince Charles if and only if x's spouse is Prince Charles. The solutions to this are exactly the people who are married to Prince Charles. There is only one: Camilla, Duchess of Cornwall.
   - Case 2: x is not married. Then we want x's biological father to be Prince Charles. As noted above, the solutions to this are Prince William and Prince Harry. Prince William is married, though, so the only solution in Case 2 is Prince Harry.

   So the set of solutions is

   $$\{\text{Camilla, Harry}\}$$

   (b) North West is neither a spouse nor a father, so she is certainly not in the range of g, and there are no solutions. The set of solutions is $\emptyset$.

   (c) As in (a), we break into two cases:

   - Case 1: x is married. Then we want x's spouse to be Kim Kardashian. There is a unique solution: Kanye West.
   - Case 2: x is unmarried. Then we want x's biological father to be Kim Kardashian. Kim is no-one's father, so this yields no solutions.

   Combining the two cases, we see that the unique solution is $x =$ Kanye West.

8. Prince Charles and Kim Kardashian are in the range, but North West is not. Indeed, we could solve $g(x) = y$ for $y =$ Prince Charles and $y =$ Kim Kardashian, but not for $y =$ North West.

9. Since g is piecewise, it is useful to write the range

   $$\{g(x) : x \text{ a person}\}$$

   as a union of the ranges of the two pieces

   $$\{g(x) : x \text{ married}\} \cup \{g(x) : x \text{ unmarried}\}.$$

   Now

   $$\{g(x) : x \text{ married}\} = \{x\text{'s spouse} : x \text{ married}\}$$

   i.e., the set of spouses of married people. Of course this is just the set of married people. And

   $$\{g(x) : x \text{ unmarried}\} = \{x\text{'s father} : x \text{ unmarried}\},$$

   i.e., the set of fathers of unmarried people, or equivalently, the set of men who have unmarried children.

   So, the range of g is the union of the following two sets:

- The set of married people
- The set of men with unmarried children.

10. We break into cases according to whether x is married or unmarried.

    - Case 1: x is married. Then $g(x)$ is x's spouse, so we get as solutions all married people who are older than their spouses.
    - Case 2: x is unmarried. Then $g(x)$ is x's biological father, so we get as solutions all unmarried people who are older than their biological fathers. Nobody is older than their biological father, so this yields no solutions.

    Taking the union of these two sets (the set of married people older than their spouses, and the empty set), we get the final set of solutions being exactly the following:

    The set of people married to someone younger than themselves.

11. We want to write $g(g(x))$ as a piecewise function in terms of x. Either x is married, or not. If x is married, then $g(x)$ is x's spouse. Of course x's spouse is also married, so

    $$g(g(x)) = g(x\text{'s spouse}) = (x\text{'s spouse})\text{'s spouse} = x.$$

    So if x is married, then $g(g(x)) = x$.

    Next suppose that x is unmarried. Then $g(x)$ is x's (biological) father. What is $g(g(x))$? Well, we have to break into cases again. Either x's father is married, or not. If x's father is married, then

    $$g(g(x)) = g(x\text{'s father}) = (x\text{'s father})\text{'s spouse},$$

    which would be x's (step)mother. If x is unmarried, then

    $$g(g(x)) = g(x\text{'s father}) = x\text{'s father's father}$$

    which is x's paternal grandfather.

    So, assembling all the cases together, we get

    $$(g \circ g)(x) = \begin{cases} x & \text{if } x \text{ is married} \\ x\text{'s (step)mother} & \text{if } x \text{ is unmarried but } x\text{'s father is married} \\ x\text{'s paternal grandfather} & \text{if } x \text{ and } x\text{'s father are unmarried} \end{cases}$$

12. Given our description of $g(g(x))$, there are three cases:

    - Case 1: x is married. If x is any married person, then $g(g(x)) = x$. So every married person is a solution.

- Case 2: x is unmarried, but x's father is married. Then $g(g(x))$ is the spouse of x's father. In particular, $g(g(x))$ is married, so does not equal x. So $g(g(x)) = x$ never holds in this case, and we get no solutions of this case.
- Case 3: x and x's father are both unmarried. Then $g(g(x))$ is x's paternal grandfather, so we want x to be x's own paternal grandfather. This is impossible, so we again get no solutions.

Combining all three cases, the set of solutions is

$$\{\text{married people}\} \cup \emptyset \cup \emptyset = \{\text{married people}\}.$$

So the solutions to the equation $g(g(x)) = x$ are exactly the people who are married.