Some of the problems don't make sense or can't be answered. You should be able to tell when this is the case.

1. If $f$ is any function, is $f^2$ even? (Recall that $(f \cdot f)(x) = f(x) \cdot f(x)$. So for example, if $f(x) = \sin(x)$, then $f^2(x) = \sin^2(x) = \sin(x) \cdot \sin(x)$.)

2. Prove that if $x$ is an integer, then $x(x-5)/2$ is also an integer. Hint: break into cases depending on whether $x$ is even or odd.

3. True or false: for any function $f$, if $x$ is in the domain of $f$ and $x = y$, then $f(x) = f(y)$.

4. Alan is John's boss. John is the mayor of Springfield. Who is the boss of the mayor of Springfield?

5. True or false: for any function $f$, if $x$ is in the domain of $f$ and $f(x) = f(y)$, then $x = y$.

6. True or false:
$$\left(\frac{1}{0}\right) \cdot 0 \neq 0$$

7. True or false: the King of France has two sisters.

8. Suppose $f(x) = 2x$ and $g(y) = 2y$ and $h(z) = 2(z+2)-4$. Which of the three functions $f, g, h$ are equal?

9. Which of the following functions are rational? Which are algebraic? Which are transcendental? Which are polynomial?
$$f(x) = e^x, \qquad g(x) = 0 \qquad h(x) = \frac{x+2}{x^2-7} \qquad d(x) = x + e^x - e^x \qquad s(x) = \frac{\sqrt{x}}{x^6+1}$$

10. Consider $f \circ g$, where $g(x) = x^2$ and $f(x)$ is the function on $\{1, 2, 3\}$ given by the following table:

| $x$ | $f(x)$ |
|---|---|
| 1 | Hydrogen |
| 2 | Helium |
| 3 | Lithium |

Write out all values of $f \circ g$ in a table. What is the domain of $f \circ g$?

11. A ship is 20 feet long and 7 feet wide. How old is the captain?

12. Consider the function $u(x)$ from people to people defined by
$$u(x) = x\text{'s mother-in-law}$$
What is the domain of $u$? Write $u$ as a composition of two functions (not using the identity function).

13. Let $f$ be a function from integers to integers. Suppose that $f(0) = f(1) = 1$, and that for every integer $n$, $f(n) + f(n+1) = f(n+2)$.
    - Prove that $f(3) = 3$. (OOPS, the original worksheet said $f(3) = 7$, which is wrong.)
    - Prove that for every integer $m$, $f(m-1) = f(m+1) - f(m)$.
    - Prove or disprove: for every integer $n$, $f(n)$ is odd.

14. Solve the following inequality for $x$:
$$x \leq x$$
Write the solution in terms of intervals.

15. Solve the following inequality for $x$:
$$x < x$$
Write the solution in terms of intervals.

16. Solve the following for $x$:
$$x^2 > 4 \text{ and } x > 0$$
That is, determine the set of numbers $x$ such that "$x^2 > 4$ and $x > 0$" is true. Write the solution in terms of intervals.

17. Solve the following for $x$:
$$x^2 > 4 \text{ or } x > 0$$
That is, determine the set of numbers $x$ such that "$x^2 > 4$ or $x > 0$" is true. Write the solution in terms of intervals.

18. If $f(x)$ is an arbitrary function, can the function $f(|x|)$ ever take negative values? That is, is there a function $f$ and a number $x$ such that $f(|x|) < 0$?

19. Let $g$ be the function from functions to numbers, sending $f$ to $f(0)$. So $g$ is just the function $g(f) = f(0)$. What is $g(\sin)$? What is $g(\cos)$? What is the domain of $g$?

20. Which of the following statements are true and which are false?
$$\forall x \exists y : x < y$$
(i.e., for every number $x$ there is a number $y$ such that $x < y$)
$$\exists x \forall y : x \leq y$$
(i.e., there is an $x$ such that for every $y$, $x \leq y$)
$$\exists x \forall y : x < y^2$$
(i.e., there is an $x$ such that for every $y$, $x < y^2$)
Hint(?): try plugging various values of $x$ into the statements $\exists y : x < y$, and $\forall y : x \leq y$, and $\forall y : x < y^2$.

# 1 Answers

1. No. For example, if $f(x) = x+1$, then $f^2(x) = (x+1)^2$ which is neither even nor odd.

2. It suffices to show that $x(x-5)$ is even. Recall that the product of an even integer with any integer is even. So if $x$ is even, then $x(x-5)$ is even. If $x$ is odd, then $(x-5)$ is even, so $(x-5)x = x(x-5)$ is even.

3. True. (For example, see the next question.)

4. Alan.

5. False, because if $f$ is the function $f(z) = z^2$, and $x = -1$, and $y = 1$, then

$$f(x) = (-1)^2 = 1 = (1)^2 = f(y),$$

but $x \neq y$. Not all functions are one-to-one.

6. Meaningless, since $1/0$ doesn't mean anything.

7. Meaningless, since "the King of France" doesn't mean anything.

8. All three functions are equal. They all have the same domain (namely $\mathbb{R}$), and if $w$ is any real number, then

$$f(w) = 2w$$
$$g(w) = 2w$$

and

$$h(w) = 2(w+2) - 4 = 2w,$$

so in particular

$$f(w) = g(w) = h(w).$$

Saying that this holds for all $w \in \mathbb{R}$ is what it means to say that $f = g = h$.

9. The functions $g, h, d$ are rational. The functions $g, h, d, s$ are algebraic. The function $f$ is transcendental. The functions $g$ and $d$ are polynomial.

10. For $f(g(x))$ to be defined, $x$ must be in the domain of $g$, which is $\mathbb{R}$, so $x$ must be real; AND $g(x) = x^2$ must be in the domain of $f$, which is $\{1, 2, 3\}$. The set of $x$ such that $x^2 \in \{1, 2, 3\}$ is just

$$\{-\sqrt{3}, -\sqrt{2}, -1, 1, \sqrt{2}, \sqrt{3}\}$$

so this is the domain of $f \circ g$. We can plug each value into $f \circ g$, and we get the following outputs:

| $x$ | $f(g(x))$ |
|---|---|
| $-\sqrt{3}$ | Lithium |
| $-\sqrt{2}$ | Helium |
| $-1$ | Hydrogen |
| $1$ | Hydrogen |
| $\sqrt{2}$ | Helium |
| $\sqrt{3}$ | Lithium |

11. Not enough information.

12. The domain of $u$ is people with mothers-in-law. If "mother" is interpreted sufficiently broadly (e.g., as biological mother, possibly deceased), this would just be the set of married people.

    A mother-in-law is the mother of a spouse, so the function $u$ would be the composition of

    $$x \mapsto x\text{'s mother}$$

    and

    $$x \mapsto x\text{'s spouse}$$

    So, if we let $m(x) = x$'s mother and $s(x) = x$'s spouse, then $u = m \circ s$.

13. 
    - Plugging $n = 0$ into the given equation, we see that

      $$f(0) + f(1) = f(2)$$

      On the other hand, $f(0) = f(1) = 1$, so

      $$1 + 1 = f(2),$$

      and $f(2) = 2$. Plugging $n = 1$ into the given equation, we also know that

      $$f(1) + f(2) = f(3).$$

      So

      $$f(3) = f(1) + f(2) = 1 + 2 = 3.$$

    - We know that for every $n$, $f(n) + f(n + 1) = f(n + 2)$. In particular, taking $n = m - 1$, we see that

      $$f(m - 1) + f(m) = f(m + 1).$$

      Rearranging gives the desired result $f(m - 1) = f(m + 1) - f(m)$.

    - We showed above that $f(2) = 2$, so the claim is false.

14. If $x$ is any real number, then $x \leq x$, so the set of solutions is the set of all real numbers. In terms of intervals, this would be

$$(-\infty, \infty).$$

15. If $x$ is any real number, then $x \not< x$, so this has no solutions. The set of solutions is $\emptyset$.

16. Note that

$$x^2 > 4 \iff x^2 - 4 > 0 \iff (x-2)(x+2) > 0$$

If $x$ equals 2 or $-2$, then $(x-2)(x+2)$ is zero, so we don't get a solution. If $x > 2$, then $x-2$ and $x+2$ are both positive, so their product $(x-2)(x+2)$ is positive, so $x$ is a solution. Similarly if $x < 2$, then $x-2$ and $x+2$ are both negative, so their product is positive and $x$ is a solution. Finally, if $-2 < x < 2$, then $(x-2)$ is negative and $(x+2)$ is positive, so $(x-2)(x+2)$ is negative, and $x$ is not a solution. To summarize, the solutions to $x^2 > 4$ are the set of $x$ which are less than $-2$ or greater than 2. In terms of intervals, this would be $(-\infty, -2) \cup (2, \infty)$.

Now, we want the set of $x$ which satisfy this *and* are positive, so we want the *intersection* of $(-\infty, -2) \cup (2, \infty)$ with the positive numbers. This is just $(2, \infty)$. So $(2, \infty)$ is the answer.

17. Here, we want the set of $x$ which are in $(\infty, -2) \cup (2, \infty)$, *or* are positive, so we want the union

$$(-\infty, -2) \cup (2, \infty) \cup (0, \infty) = (-\infty, -2) \cup (0, \infty).$$

(Note that $(2, \infty) \cup (0, \infty)$ is just $(0, \infty)$, since every element of $(2, \infty)$ is already in $(0, \infty)$.)

So the answer is $(-\infty, -2) \cup (0, \infty)$.

18. Yes. For example, $f(x) = -1$. Then no matter what $x$ is, $f(|x|)$ is $-1$, which is negative. (So, in particular, we could take $x = 42$.)

19. 

$$g(\sin) = \sin(0) = 0$$

$$g(\cos) = \cos(0) = 0$$

A function $f$ is in the domain of $g$ if and only if $f(0)$ makes sense, so in particular if and only if 0 is in the domain of $f$. So the domain of $g$ is

$$\{f : f \text{ is a function with 0 in its domain}\}.$$

That is, the domain of $g$ is the set of all functions with 0 in their domains. For example, $\sin, \cos, \tan$ are in the domain of $g$, but cot is not, and neither is the reciprocal function $r(x) = 1/x$.

20. The first statement

$$\forall x \exists y : x < y$$

is true: given $x$, we can take $y = x + 1$, and then $y > x$.

The second statement

$$\exists x \forall y : x \leq y$$

is false: if it were true, there would be some $x$ such that $x$ is less than or equal to any number $y$. But if we in particular took $y = x - 1$, then $x$ would not be less than or equal to $y$, a contradiction.

The third statement

$$\exists x \forall y : x < y^2$$

is true: we can take $x = -1$, and then it remains to show that for every $y$, we have $-1 < y^2$. This is clear, because $y^2$ is always zero or positive, so either way, it's bigger than $-1$.