1. True/False quiz. If it is true, explain why. If it is false, give a counterexample that disproves the statement.

   (a) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ don't exist, then $\lim_{x\to a}[f(x)+g(x)]$ does not exist.

   (b) $\frac{d^2y}{dx^2} = \left(\frac{dx}{dy}\right)^2$

   (c) If $f(x) = (x^6 - x^4)^5$, then $f^{(31)}(x) = 0$.

   (d) If f and g are increasing on an interval I, then fg is increasing on I.

   (e) The tangent line to the parabola $y = x^2$ at $(-2, 4)$ is $y - 4 = 2x(x + 2)$.

2. Differentiate

   - $x \ln x - x$

   - $(x^2 + 1)^{2x}$

3. Find the point on the line $y = 2x + 5$ closest to the origin (0,0).

4. Evaluate by realizing as a derivative

   $$\lim_{x\to e} \frac{\ln(x) \cdot \ln(x) - 1}{x - e} =$$

5. Write down an antiderivative of the absolute value function $f(x) = |x|$, using integrals.

   $$F(x) = \int$$

6. Among rectangles having perimeter 4, which has the greatest area?

7. Use L'Hôpital's rule to evaluate

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \underline{\hspace{3cm}} =$$

8. Graph $y = x\ln x$, finding intercepts, critical points, inflection points, and asymptotes, if they exist.

9. Evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2^{i/n}}{n} =$$

10. Apply Newton's method to solve $e^x = 0$. If $x_0 = 0$, what are...

$x_1 =$

$x_2 =$

$x_n =$

11. (a) Show that $2 = x + e^x$ has a unique solution.

    (b) Use Newton's method to solve for x. Start with $x_0 = 2$. What is x_1?

12. Evaluate the limit without FTC. What definite integral have we just calculated?

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2} = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} i =$$

13. Draw the region bounded by the curves $y = e^{-x^2}$ and $y = 1/e$. Rotate around the y-axis and find the volume.

14. Draw the region bounded by $y = |x|$ and $y = 1$. Rotate around $y = -1$, and find the volume. Use cylindrical shells if born in January through June, and washers if born in July through December.

1 Solutions

1. (a) False. For example, if

$$f(x) = \begin{cases} 1 & x < a \\ 0 & x \geq a \end{cases},$$

and $g(x) = 1 - f(x)$, then

$$\lim_{x \to a^+} f(x) = 0 \neq 1 = \lim_{x \to a^-} f(x),$$

so $\lim_{x \to a} f(x)$ does not exist. Also, $\lim_{x \to a} g(x)$ does not exist, because if it *did* exist, then by the limit laws,

$$\lim_{x \to a} f(x) = \lim_{x \to a} (1 - g(x)) = 1 - \lim_{x \to a} g(x) \qquad \text{would exist.}$$

On the other hand, $f(x) + g(x) = 1$, so

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} 1 = 1,$$

and in particular, the limit of $f(x) + g(x)$ *does* exist.

(b) False. $\frac{d^2y}{dx^2}$ denotes the second derivative of y with respect to x, while $\left(\frac{dy}{dx}\right)^2$ denotes the square of the derivative. If $y = x^3$, for example, then the first derivative is $3x^2$, and the second derivative is $6x$. And

$$\frac{d^2y}{dx^2} = 6x \neq 9x^4 = (3x^2)^2 = \left(\frac{dy}{dx}\right)^2.$$

(c) True. Note that $f(x) = (x^6 - x^4)^5$ is a degree 30 polynomial, because it is the fifth power of a degree 6 polynomial. Taking the derivative of a polynomial always decreases the degree, so the thirtieth derivative of f will be a degree zero polynomial, i.e., a constant. Therefore the 31^{st} derivative of f will be the derivative of a constant, hence will vanish.

(d) False. For example, $f(x) = x$ and $g(x) = x$ are both increasing on the interval $[-1, 1]$, but their product $fg = x^2$ is not increasing on this interval, as $-1 < 1$ but $(-1)^2 \not< 1^2$.

(e) False, since $y - 4 = 2x(x + 2)$ is not a line, so it can't be the tangent line.

2.

$$\frac{d}{dx}[x \ln x - x] = \left(\frac{d}{dx}x\right)\ln x + x\left(\frac{d}{dx}\ln x\right) - \frac{d}{dx}x = 1 \cdot \ln x + x \cdot \frac{1}{x} - 1 = \ln x.$$

For the second one, let $y = (x^2 + 1)^{2x}$. Then taking logarithms of both sides and differentiating...

$$\ln y = \ln\left((x^2 + 1)^{2x}\right) = 2x \ln(x^2 + 1).$$

$$\frac{y'}{y} = 2\ln(x^2+1) + 2x\frac{1}{x^2+1}2x = 2\ln(x^2+1) + \frac{4x^2}{x^2+1}.$$

So

$$y' = y\left(2\ln(x^2+1) + \frac{4x^2}{x^2+1}\right) = (x^2+1)^{2x}\left(2\ln(x^2+1) + \frac{4x^2}{x^2+1}\right).$$

3. A point on this line has the form $(x, 2x+5)$, for $x \in \mathbb{R}$, and its distance from the origin is $\sqrt{x^2 + (2x+5)^2}$, by the distance formula. We can call this expression $d(x)$. So

$$d(x) = \sqrt{x^2 + (2x+5)^2},$$

   and we want to find the value of x which minimizes $d(x)$. From geometry, we know that there *is* a unique point on the line which is closest to the origin.[1] By Fermat's theorem, this minimum must be a critical point of $d(x)$, so let's find all the critical points:

$$d'(x) = \frac{1}{2\sqrt{x^2+(2x+5)^2}}\left(2x + 2(2x+5)2\right) = \frac{10x+20}{2\sqrt{x^2+(2x+5)^2}}.$$

   This will vanish exactly when $10x+20$ vanishes. Solving $10x+20 = 0$ for x, we see that the unique critical point is at $x = -2$. So the global minimum of $d(x)$ is at $x = -2$. The corresponding point on the line is $(x, 2x+5) = (-2, -4+5) = (-2, 1)$.

   So, ultimately we conclude that the closest point to the origin is $(-2, 1)$.

4. Note that $\ln e = 1$, so

$$\lim_{x\to e}\frac{(\ln x)^2 - 1}{x - e} = \lim_{x\to e}\frac{(\ln x)^2 - (\ln e)^2}{x-e}.$$

   This last expression is just $f'(e)$, for the function $f(x) = (\ln x)^2$. We can determine what $f'(x)$ is by the chain rule:

$$f'(x) = 2(\ln x)\frac{1}{x},$$

   so

$$f'(e) = \frac{2\ln e}{e} = \frac{2}{e}.$$

   So the limit is $2/e$.

5.

$$F(x) = \int_0^x |t|\,dt$$

[1] The function $d(x)$ is continuous on its domain, but the Extreme Value Theorem doesn't apply because the domain is a closed interval, so something slightly fishy is going on here.

6. Let x and y be the side lengths of the rectangle. Then we're trying to maximize xy subject to the constraints that $x \geq 0$, $y \geq 0$, and $2x + 2y = 4$. This last equation lets us write y in terms of x, as $2 - x$. The area is then $x(2 - x)$. The constraint on y, that $y \geq 0$, then turns into the constraint that $2 - x \geq 0$, or equivalently, that $x \leq 2$. So equivalently, we're trying to maximize the expression $A(x) = x(2-x)$ where $0 \leq x \leq 2$.

   We know how to do this kind of optimization problem: we just need to evaluate $A(x)$ at the endpoints and critical points, and find the biggest value. To find the critical points, we take the derivative

$$A'(x) = 1 \cdot (2 - x) + x \cdot (-1) = 2 - x - x = 2 - 2x.$$

   Then $A'(x) = 0$ when $2 - 2x = 0$, which is when $x = 1$. So the critical point is at $x = 1$, and we also need to check the end points $x = 0$ and $x = 2$:

$$\begin{aligned} A(0) &= 0(2-0) = 0 \\ A(1) &= 1(2-1) = 1 \\ A(2) &= 2(2-2) = 0. \end{aligned}$$

   So the maximum is at $x = 1$. For this value of x, the other side length y is $y = 2 - x = 2 - 1 = 1$. So both side lengths are 1.

   So in conclusion, the rectangle with maximum area, among those having perimeter 4, is the 1×1 rectangle, i.e., the square.

7. L'Hôpital's rule applies because $\lim_{x\to 0^+} \ln x = -\infty$ and $\lim_{x\to 0^+} x^{-1} = +\infty$. Then

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} \overset{\text{L'H}}{=\!=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2},$$

   assuming the right hand limit exists. But

$$\frac{1/x}{-1/x^2} = -x,$$

   so

$$\lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0,$$

   and therefore

$$\lim_{x\to 0^+} x \ln x = 0.$$

8. Since $\ln x$ is only defined for $x > 0$, the domain of this function is $x > 0$. Therefore there are no y-intercepts. To find x-intercepts, you solve the equation

$$0 = x \ln x.$$

The only possible solutions are when $x = 0$ (but this doesn't work because then $x \ln x$ isn't defined), and when $\ln x = 0$, which is when $x = 1$. So the only x-intercept is at $x = 1$. Moreover, since $\ln x$ is negative for $x < 1$ and positive for $x > 1$, we see that $x \ln x$ is negative for $x < 1$ and positive for $x > 1$.

To find the local minima and maxima, we find the derivative:

$$y' = 1 \ln x + x\frac{1}{x} = 1 + \ln x.$$

This equals zero exactly when $\ln x = -1$, i.e., when $x = e^{-1} = 1/e$. The corresponding y-value is

$$x \ln x = e^{-1} \ln e^{-1} = -e^{-1} = 1/e.$$

So there is a critical point at $(1/e, -1/e)$.

To see whether it's a local maximum or local minimum or neither, we use the second derivative test:

$$y'' = \frac{d}{dx}(1 + \ln x) = \frac{1}{x}.$$

This is always positive, so the function is concave up, and the critical point is a local minimum.

Also, since the first derivative is $1+\ln x$, and $\ln x$ is increasing, we see that $y' = 1+\ln x$ is positive for $x > 1/e$, and negative for $x < 1/e$. This means that the original function $y = x \ln x$ is increasing on the interval $[1/e, +\infty)$, and decreasing on the interval $(0, 1/e]$.

The only possible place there could be a vertical asymptote is at $x = 0$. But as $x \to 0^+$, we know that y gets bigger (since the function is decreasing on the interval $(0, 1/e]$ and x is getting smaller), and also that y stays below 0, because $x \ln x$ is negative for $x < 1$. So y can't go to $-\infty$ (because it's getting bigger), and it can't go to $+\infty$ (because it's stuck below 0).

In fact, from the previous problem, we know that $\lim_{x\to 0^+} x \ln x = 0$, so there's definitely not a vertical asymptote.

Finally, we check for horizontal and slant asymptotes. Note that $\lim_{x\to+\infty} x \ln x = +\infty$, because x and $\ln x$ are both going to $+\infty$ in the limit. So there's no horizontal asymptote. In order for there to be a slant asymptote,

$$\lim_{x\to+\infty} \frac{x \ln x}{x}$$

would *need* to exist (though this wouldn't *suffice*). But

$$\lim_{x\to+\infty} \frac{x \ln x}{x} = \lim_{x\to+\infty} \ln x = +\infty,$$

so this limit does not exist. Therefore there is no slant asymptote.

So, we know the following facts about the graph:

- The domain is exactly the values of x that are positive.
- The function approaches 0 as x approaches 0 from above, and it approaches $+\infty$ as x approaches $+\infty$.
- There are no horizontal, vertical, or slant asymptotes.
- The function is concave up everywhere.
- The function crosses the x-axis at $(1, 0)$.
- The function is decreasing from 0 to $1/e$, and is increasing from $1/e$ to $+\infty$. In particular, the function has a global minimum at $1/e$.
- The function is continuous and differentiable on its domain, so there are no corners or jumps.

Using all this, you should be able to sketch the graph. See here for what the graph looks like.

9. If we evenly divide the interval $[0, 1]$ into n pieces $[x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{n-1}, x_n]$, each of length Δx, then $\Delta x = 1/n$, and $x_i = i/n$. Thus

$$\sum_{i=1}^{n} \frac{2^{i/n}}{n} = \sum_{i=1}^{n} 2^{x_i} \Delta x,$$

so the limit in question is

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2^{x_i} \Delta x.$$

This is a right-endpoints Riemann sum for the function 2^x on the interval $[0, 1]$. Since this function on that interval is integrable (because it's continuous), the limit exists and equals

$$\int_0^1 2^x \, dx = \left[\frac{2^x}{\ln 2}\right]_0^1 = \frac{2^1 - 2^0}{\ln 2} = \frac{1}{\ln 2}.$$

10. To solve $e^x = 0$, Newton's method gives us the formula

$$x_{new} = x_{old} - \frac{f(x_{old})}{f'(x_{old})} = x_{old} - \frac{e^{x_{old}}}{e^{x_{old}}} = x_{old} - 1.$$

So each iteration decreases x by 1, and thus

$$x_0 = 0, \quad x_1 = -1, \quad x_2 = -2, \quad x_3 = -3, \ldots$$

The pattern is clear: $x_n = -n$.

11. (a) Let $f(x)$ be the function $x+e^x$. This function is a sum of differentiable functions, so it is itself differentiable, and continuous. We need to show that there is exactly one value of x such that $f(x)=2$. We first show that there is at least one, using the Intermediate Value Theorem. Note that

$$f(100) = 100 + e^{100} > 100 > 2,$$

and

$$f(-100) = -100 + e^{-100} < -100 + 1 = -99 < 2,$$

where we have used the fact that $e^{-100} < 1$. So 2 is a number between $f(-100)$ and $f(100)$. As f is continuous on $[-100, 100]$, it follows that $f(x) = 2$ for some x between -100 and 100. So there is at least one value of x such that $f(x) = 2$.

Finally we show that there is at most one value of x such that $f(x) = 2$. Suppose for the sake of contradiction that there were two values a and b such that $f(a) = f(b) = 2$. Since f is continuous and differentiable everywhere, it follows by Rolle's theorem that $f'(c) = 0$ for some c between a and b. But

$$f'(c) = 1 + e^c,$$

which is always positive, contradicting $f'(c) = 0$.

(b) To solve the equation $e^x + x - 2 = 0$, Newton's method yields the iteration

$$x_{new} = x_{old} - \frac{e^{x_{old}} + x_{old} - 2}{e^{x_{old}} + 1}.$$

In particular, if we start with $x_{old} = 2$, then the next value of x is

$$x_{new} = 2 - \frac{e^2 + 2 - 2}{e^2 + 1} = 2 - \frac{e^2}{e^2 + 1}.$$

If we like, we can "simplify" this answer as

$$\frac{2e^2 + 2}{e^2 + 1} - \frac{e^2}{e^2 + 1} = \frac{e^2 + 2}{e^2 + 1}.$$

12. First, we evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2} = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} = \lim_{n\to\infty} \frac{1}{n^2} \frac{n(n+1)}{2} = \lim_{n\to\infty} \frac{n^2 + n}{2n^2} = \lim_{n\to\infty} \frac{1 + 1/n}{2} = \frac{1}{2}.$$

Next, we interpret it as an integral, as in problem 9. If one subdivides the interval $[0, 1]$ into n pieces $[x_0, x_1] \cup [x_1, x_2] \cup \cdots [x_{n-1}, x_n]$ of equal length Δx, then

$$x_i = i/n$$

and

$$\Delta x = 1/n.$$

So

$$\sum_{i=1}^{n} \frac{i}{n^2} = \sum_{i=1}^{n} x_i \Delta x.$$

So this is a right-endpoints approximation to $\int_0^1 x\,dx$. So, we've just shown that $\int_0^1 x\,dx = 1/2$.