Practice with logic

November 29, 2014

Which of the following proofs are invalid, and why?

1. Suppose f and g are differentiable everywhere, and $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Then there is some c between 0 and 1 such that $f'(c) = g'(c)$.

   Proof. The mean value theorem applies to both f and g, so there is some c such that

   $$f'(c) = \frac{f(1) - f(0)}{1 - 0} = \frac{1 - 0}{1 - 0} = 1$$

   and

   $$g'(c) = \frac{g(1) - g(0)}{1 - 0} = \frac{1 - 0}{1 - 0} = 1.$$

   Then

   $$f'(c) = 1 = g'(c),$$

   and c is between 0 and 1. □

2. For all x, $x^2 - 2x + 1 \geq 0$.

   Proof. We need to show that the minimum value of the function $f(x) = x^2 - 2x + 1$ is at least 0. To find the minimum, we take the derivative

   $$f'(x) = 2x - 2.$$

   The only critical point is when $2x - 2 = 0$, i.e., when $x = 1$. There, $f(x) = f(1) = 1^2 - 2 \cdot 1 + 1 = 0$. So the minimum value of f is 0, and for any other value of x, $f(x)$ is at least 0. □

3. Suppose f is differentiable everywhere, and $f'(x)$ is never 0. Then f is increasing or decreasing.

Proof. Suppose $x_1 < x_2$. We need to show that $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$. By the mean value theorem applied to f on the interval $[x_1, x_2]$, there is some $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}.$$

Now $f'(c) \neq 0$, so $f'(c) > 0$ or $f'(c) < 0$. If $f'(c) > 0$, then

$$f(x_1) - f(x_2) = (x_1 - x_2)f'(c) < 0,$$

so $f(x_1) - f(x_2) < 0$, i.e., $f(x_1) < f(x_2)$, and f is increasing. On the other hand, if $f'(c) < 0$, then

$$f(x_1) - f(x_2) = (x_1 - x_2)f'(c) > 0,$$

so $f(x_1) - f(x_2) > 0$, i.e., $f(x_1) > f(x_2)$, and f is decreasing. □

4. The function $t(x) = x^3 - e^{-x}$ has at most one zero.

   Proof. Note that t is continuous, and

$$\lim_{x \to +\infty} t(x) = +\infty - 0 = +\infty,$$

while

$$\lim_{x \to -\infty} t(x) = -\infty - \infty = -\infty.$$

So if we choose N large enough, then $t(N) > 0$ and $t(-N) < 0$. By the intermediate value theorem, there is exactly one zero between $-N$ and N. □

1 Solutions

All the proofs are invalid.

1. By the mean value theorem, there is some c such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = \frac{1 - 0}{1 - 0} = 1.$$

Likewise, by the mean value theorem, there is some c such that

$$g'(c) = \frac{g(1) - g(0)}{1 - 0} = \frac{1 - 0}{1 - 0} = 1.$$

But it might be a different c in each case! So really, we should call the two constants c_1 and c_2. All we know is that there are constants c_1 and c_2 such that

$$f'(c_1) = \frac{f(1) - f(0)}{1 - 0} = \frac{1 - 0}{1 - 0} = 1$$

and

$$g'(c_2) = \frac{g(1) - g(0)}{1 - 0} = \frac{1 - 0}{1 - 0} = 1.$$

But we don't know that $c_1 = c_2$, and in fact in general this won't be the case. (The correct way to do this proof is to apply the Mean Value Theorem to $f - g$.)

2. We need to use the first or second derivative test to check that the critical point at $x = 1$ is a *minimum*, rather than a maximum, or a point of inflection or something. The invalid prove never uses the fact that the function is $x^2 - 2x + 1$ rather than $-(x^2 - 2x + 1)$ or $(x - 1)^3$, both of which don't have minima at $x = 1$.

3. The given proof establishes the statement

   For every $x_1 < x_2$, (either $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$.)

   but what we really need is the stronger statement that

   Either (for every $x_1 < x_2$, it's true that $f(x_1) < f(x_2)$) or (for every $x_1 < x_2$, it's true that $f(x_1) > f(x_2)$.)

   In the first statement, whether $f(x_1)$ is bigger than or smaller than $f(x_2)$ might *depend* on the choice of x_1, x_2. In the second statement, the ordering of $f(x_1)$ and $f(x_2)$ has to be the same for every choice of x_1 less than x_2.

   The second statement is what we want to prove, since "f is increasing" exactly means "for every $x_1 < x_2$, it's true that $f(x_1) < f(x_2)$," and "f is decreasing" exactly means "for every $x_1 < x_2$, it's true that $f(x_1) > f(x_2)$" (see page 19 of the textbook, section 1.1).

   All the manipulations in the bogus proof are valid, the problem is just with the statement "We need to show that $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$." This isn't all we need to prove—it's not sufficient.

   (The statement being proven here is actually true, but the proof is not so easy. The given proof would work if we knew that either f' was always positive, or f' was always negative. In other words, we'd need to know that f' never changes sign. This can be established, using Darboux's Theorem.)

4. The given proof establishes that there is *at least* one zero. To prove that there's *at most* one zero, something completely different must be used. Really, Rolle's theorem should be used... since $t' = 3x^2 + e^{-x} > 0$, Rolle's theorem (or its contrapositive[1]) says that t can't take the same value twice.

[1] Rolle's Theorem says that if a nice function like t takes the same value twice, its derivative has a zero somewhere. So the contrapositive would say that if t' is never 0, then t doesn't take the same value twice.