Practice with Proofs

October 6, 2014

Recall the following

Definition 0.1. *A function f is* increasing *if for every x, y in the domain of f,*

$$x < y \implies f(x) < f(y)$$

1. Prove that $h(x) = x^3$ is increasing, using algebra.
2. Prove that $g(x) = x^3 - x$ is not increasing.
3. Suppose that f is an increasing function with domain $\mathbb{R}$. Prove that f is not even.
4. Suppose that f is an increasing function. Show that for every x, y in the domain of f,

$$x < y \iff f(x) < f(y)$$

5. Write down an increasing function whose domain is $\mathbb{R}$, but whose range is strictly smaller than $\mathbb{R}$.
6. Prove that increasing functions are one-to-one.
7. Suppose that f is an increasing function with domain and range $\mathbb{R}$.
   (a) What are the domain and range of f^{-1}?
   (b) Show that f^{-1} is increasing.
   (c) Show that for every x, y, we have equivalences

$$y < f(x) \iff f^{-1}(y) < x$$

   and

$$y > f(x) \iff f^{-1}(y) > x.$$

   (d) Show that f is continuous. Hint: you'll need to solve

$$f(a) - \epsilon < f(x) < f(a) + \epsilon$$

   for x, using f^{-1}.
8. Write down an increasing non-continuous function f whose domain is all of $\mathbb{R}$.

1 Solutions

1. Prove that $h(x) = x^3$ is increasing, using algebra.

   Proof. Suppose that $x < y$. We need to show that $x^3 < y^3$. We break into cases according to whether x and y are positive, negative, or zero.

   - *Case 1:* $0 < x < y$. Then x^2, xy, and y^2 are positive, so we can multiply the inequality $x < y$ by these numbers, getting

     $$x^3 = x^2 \cdot x < x^2 \cdot y$$

     $$x^2 y = xy \cdot x < xy \cdot y = xy^2$$
     $$xy^2 = y^2 \cdot x < y^2 \cdot y = y^3$$

     So $x^3 < x^2 y < xy^2 < y^3$, proving that $x^3 < y^3$.
   - *Case 2:* $0 = x < y$. Then y is positive, so its cube y^3 is positive. Thus

     $$x^3 = 0 < y^3$$

   - *Case 3:* $x < 0 < y$. Then x is negative, so its cube x^3 is negative. Likewise, y is positive, so its cube y^3 is positive. Then

     $$x^3 < 0 < y^3,$$

   - *Case 4:* $x < y = 0$. Then x is negative, so its cube x^3 is negative. Thus

     $$x^3 < 0 = y^3.$$

   - *Case 5:* $x < y < 0$. Then $0 < (-y) < (-x)$, so by Case 1 (applied to $-y$ and $-x$), we know
     $$(-y)^3 < (-x)^3$$
     Equivalently
     $$-y^3 < -x^3$$
     Multiplying both sides by -1 yields

     $$y^3 > x^3.$$

   So in every case, we have $x^3 < y^3$, completing the proof. (Note: in Hald's class, you can probably take for granted the fact that $f(x) = x^3$ is increasing, and you probably don't need to prove it!) □

2. Prove that $g(x) = x^3 - x$ is not increasing.

Proof. Note that $0 < 1$ but $g(0) = 0 \not< g(1) = 1$. □

3. Suppose that f is an increasing function with domain $\mathbb{R}$. Prove that f is not even.

   Proof. Since f is increasing, $f(-1) < f(1)$. In particular, $f(-1) \neq f(1)$, so f is not even. □

4. Suppose that f is an increasing function. Show that for every x, y in the domain of f,

$$x < y \iff f(x) < f(y)$$

   Proof. The $\Rightarrow$ direction is by definition of increasing function, so we need to show the converse, that if $f(x) < f(y)$, then $x < y$. Equivalently, we need to show the contrapositive: if $x \geq y$, then $f(x) \geq f(y)$. Suppose $x \geq y$. Then either $x = y$ or $y < x$. In the first case, x and y are the same thing, so $f(x) = f(y)$. In the other case, $y < x$, so $f(y) < f(x)$, because f is increasing. Either way, we have $f(x) \geq f(y)$. This proves the contrapositive, and we're done. □

5. Write down an increasing function whose domain is $\mathbb{R}$, but whose range is strictly smaller than $\mathbb{R}$.

   Answer. The exponential function $f(x) = e^x$, for example. □

6. Prove that increasing functions are one-to-one.

   Proof. Suppose that f is increasing and $f(x) = f(y)$. We need to show that $x = y$. *If this weren't true,* then $x < y$ or $y < x$. By definition of increasing, it would follows that $f(x) < f(y)$ or $f(y) < f(x)$. Either way, $f(x) \neq f(y)$, a contradiction. So $f(x) = f(y)$. □

7. Suppose that f is an increasing function with domain and range $\mathbb{R}$.

   (a) What are the domain and range of f^{-1}?

   Answer. The domain and range of f^{-1} are the range and domain of f, which are both $\mathbb{R}$. □

   (b) Show that f^{-1} is increasing.

   Proof. Suppose that $x < y$. Substituting $f^{-1}(x)$ and $f^{-1}(y)$ for x and y in problem 4, we see that

$$f^{-1}(x) < f^{-1}(y) \iff f(f^{-1}(x)) < f(f^{-1}(y))$$

   But $f(f^{-1}(x)) = x$ and $f(f^{-1}(y)) = y$, so the right hand side is true. Therefore the left hand side is true, i.e., $f^{-1}(x) < f^{-1}(y)$. So f^{-1} is increasing. □

(c) Show that for every x, y, we have equivalences

$$y < f(x) \iff f^{-1}(y) < x$$

and

$$y > f(x) \iff f^{-1}(y) > x.$$

Proof. Since the range of f is all of $\mathbb{R}$, we can write y as $f(z)$ for some z. Then $f^{-1}(y) = z$, so the first statement says

$$f(z) < f(x) \iff z < x$$

and the second says

$$f(z) > f(x) \iff z > x$$

These follow from problem 4. □

(d) Show that f is continuous. Hint: you'll need to solve

$$f(a) - \epsilon < f(x) < f(a) + \epsilon$$

for x, using f^{-1}.

Preliminary work to find δ. We need to show that

$$\lim_{x \to a} f(x) = f(a)$$

for arbitrary a. Given $\epsilon > 0$, we need to find $\delta > 0$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$$

Now

$$|f(x) - f(a)| < \epsilon \iff f(a) - \epsilon < f(x) < f(a) + \epsilon.$$

By the previous problem,

$$f(a) - \epsilon < f(x) < f(a) + \epsilon \iff f^{-1}(f(a) - \epsilon) < x < f^{-1}(f(a) + \epsilon).$$

So, we need to find δ such that

$$a - \delta < x < a + \delta \implies f^{-1}(f(a) - \epsilon) < x < f^{-1}(f(a) + \epsilon),$$

i.e., such that the open interval $(a - \delta, a + \delta)$ is a subset of the open interval

$$(f^{-1}(f(a) - \epsilon), f^{-1}(f(a) + \epsilon)).$$

As usual, we take δ to be the smaller of the two distances from a to the endpoints of this interval:

$$\delta = \min \left\{ f^{-1}(f(a) + \epsilon) - a, a - f^{-1}(f(a) - \epsilon) \right\}$$

One has to be careful that a actually sits inside this interval, so that δ is actually positive! This uses the fact that f^{-1} is increasing, so that

$$f(a) - \epsilon < f(a) < f(a) + \epsilon$$

implies

$$f^{-1}(f(a) - \epsilon) < f^{-1}(f(a)) = a < f^{-1}(f(a) + \epsilon)$$

$\square$

Now here's the actual proof:

Proof. Fix a. We need to show continuity at a, i.e., that

$$\lim_{x \to a} f(x) = f(a).$$

So, we need to show that for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$$

Given ϵ, let

$$\delta = \min\left\{f^{-1}(f(a) + \epsilon) - a, a - f^{-1}(f(a) - \epsilon)\right\}$$

We first check that $\delta > 0$, which amounts to checking that the two numbers it is the minimum of, are both positive. Since $\epsilon > 0$, we have

$$f(a) - \epsilon < f(a) < f(a) + \epsilon.$$

By the previous problem

$$f^{-1}(f(a) - \epsilon) < a < f^{-1}(f(a) + \epsilon).$$

Consequently both $f^{-1}(f(a) + \epsilon) - a$ and $a - f^{-1}(f(a) - \epsilon)$ are positive, so their minimum δ is positive. So $\delta > 0$.
Next, suppose that $|x - a| < \delta$. Then

$$x < a + \delta \leq a + \big(f^{-1}(f(a) + \epsilon) - a\big) = f^{-1}(f(a) + \epsilon).$$

So $x < f^{-1}(f(a) + \epsilon)$. By the previous problem

$$f(x) < f(a) + \epsilon. \tag{1}$$

Similarly,

$$x > a - \delta \geq a - \big(a - f^{-1}(f(a) - \epsilon)\big) = f^{-1}(f(a) - \epsilon).$$

so $x > f^{-1}(f(a) - \epsilon)$. By the previous problem

$$f(x) > f(a) - \epsilon. \tag{2}$$

Combining (1) and (2), we see that

$$f(a) - \epsilon < f(x) < f(a) + \epsilon,$$

or equivalently that $|f(x) - f(a)| < \epsilon$, which is what we wanted to show. So we've shown that $\lim_{x \to a} f(x) = f(a)$. As a was arbitrary, f is continuous. □

8. Write down an increasing non-continuous function f whose domain is all of $\mathbb{R}$.

Answer. For example,

$$f(x) = \begin{cases} x & \text{when } x < 0 \\ x + 1 & \text{when } x \geq 0 \end{cases}$$

□