Solving epsilon-delta problems

Math 1A, 313,315 DIS

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There will probably be at least one epsilon-delta problem on the midterm and the final. These kind of problems ask you to show¹ that

$$\lim_{x \to a} f(x) = L$$

for some particular f and particular L, using the actual definition of limits in terms of ϵ 's and δ 's rather than the limit laws. For example, there might be a question asking you to show that

$$\lim_{x \to a} 7x + 3 = 7a + 3 \tag{1}$$

or

$$\lim_{x \to 5} x^2 - x - 1 = 19, \tag{2}$$

using the definition of a limit.

1 The rules of the game

Normally, the answer to this kind of question will be of the following form:

Given $\epsilon > 0$, let $\delta = [$ something positive, usually depending on ϵ and a]. If $0 < |x - a| < \delta$ then [some series of steps goes here], so $|f(x) - L| < \epsilon$.

Some examples of this are Examples 2-4 of section 2.4. Note that "[some series of steps goes here]" should consist of a proof that $|f(x) - L| < \epsilon$, from the assumptions that

- $\epsilon > 0$
- δ is whatever we said it was, and
- $0 < |x a| < \delta$.

 1 i.e., prove

In these kind of problems, much of the work goes into figuring out what δ should be. None of this work is shown in the actual answer. To clarify: in examples 2-4 of section 2.4, each "solution" consists of two parts. Part 2 ("showing that this δ works") is the actual answer–what you would turn in if asked this question on a homework or an exam. Part 1 ("guessing a value for δ ") is the bulk of the work done to produce this answer.

So there's a sense in which you don't have to show your work in this kind of problem; it suffices to just write down the final answer. This is a little strange because for most math problems it *is* necessary to show your work. For example, if there was a problem asking you to evaluate

$$\lim_{x \to 1} \frac{x^4 - 1}{x - 1},$$

it would not be acceptable to just write down "4." This would be unacceptable because there's no way for the person reading your answer to see why the limit should be 4. But if the answer to a question is a proof, rather than a number or an expression, then the reader can see directly whether or not the answer is correct, because the correctness of a proof is self-evident. In problems where the answer is a number or an expression, when we say "show your work" we really mean "show that the answer is correct." For example, a more correct answer to $\lim_{x\to 1} (x^4 - 1)/(x - 1)$ would be

$$\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim_{x \to 1} \frac{(x^3 + x^2 + x + 1)(x - 1)}{x - 1} = \lim_{x \to 1} (x^3 + x^2 + x + 1)$$
$$= 1^3 + 1^2 + 1 + 1 = 4.$$

The first step is just rewriting the thing whose limit is being taken. The second step is using the fact that $\lim_{x\to 1}$ only looks at values of x that aren't 1, for which we can cancel out the factors of (x-1). The third step is the direct substitution principle for polynomials, and the last step is basic arithmetic.

2 Common mistakes

From looking through people's homework, I got the impression that the following mistakes were common:

- Dividing by zero, or treating ∞ as if it were an actual number.
- Writing things like

$$\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = x^3 + x^2 + x + 1 = 4.$$

In $\lim_{x\to 1} \frac{x^4-1}{x-1}$, the variable x is a bound variable. To paraphrase Wikipedia, "there is nothing called x on which $\lim_{x\to 1} (x^4-1)/(x-1)$ could depend." It doesn't make sense to say that the limit is equal to $x^3 + x^2 + x + 1$, because what is x?

• Not specifying what you chose δ to be! If you don't do this, it's really unclear what you're ultimately trying to prove.

- Confusing the preliminary analysis to figure out δ , with the actual answer (the proof), or flat out omitting the actual answer.
- Making δ depend on x. Perhaps you're trying to show that

$$\lim_{x \to 0} \frac{x}{x^2 + 1} = 0$$

and so you need to show that for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x| < \delta$ implies $|x/(x^2 + 1)| < \epsilon$. You note

$$\left|\frac{x}{x^2+1}\right| < \epsilon \iff \frac{|x|}{|x^2+1|} < \epsilon \iff |x| < |x^2+1|\epsilon,$$

so you would like to take δ to be $|x^2 + 1|\epsilon$.

But you can't, since the rules of the ϵ - δ game say that you have to specify δ before being told what x is. In this case, you need to find a δ which will be guaranteed to be less than $|x^2 + 1|\epsilon$. Since $|x^2 + 1|$ is always at least 1, you could take $\delta = \epsilon/2$ or something similar.

3 Strategies for finding delta

One general strategy is to try solving $|f(x) - L| < \epsilon$ for x. Once you know what values of x will work, you choose δ so that the interval $(a - \delta, a + \delta)$ sits inside the set of solutions.

For example, suppose you're trying to prove that $\lim_{x\to 8} \sqrt[3]{x} = 2$. Given $\epsilon > 0$, you need to find $\delta > 0$ such that

$$0 < |x - 8| < \delta \implies |\sqrt[3]{x} - 2| < \epsilon.$$

One approach is to just solve the inequality $|\sqrt[3]{x} - 2| < \epsilon$ for x, as follows:

$$|\sqrt[3]{x} - 2| < \epsilon \iff 2 - \epsilon < \sqrt[3]{x} < 2 + \epsilon$$
$$\iff (2 - \epsilon)^3 < x < (2 + \epsilon)^3$$

In order for $(8 - \delta, 8 + \delta)$ to sit inside the interval from $(2 - \epsilon)^3$ to $(2 + \epsilon)^3$, one needs

$$(2-\epsilon)^3 \le 8-\delta$$
 and $8+\delta \le (2+\epsilon)^3$,

or equivalently

$$\delta \le 8 - (2 - \epsilon)^3$$
 and $\delta \le (2 + \epsilon)^3 - 8$

So the biggest value of δ that would work is

$$\delta = \min\{8 - (2 - \epsilon)^3, (2 + \epsilon)^3 - 8\}.$$

If f(x) is a polynomial or a nice enough rational function, so that L = f(a), then another approach is to look at

$$\frac{f(x) - f(a)}{x - a}$$

If you can find some constant C and guarantee that

$$\left|\frac{f(x) - f(a)}{x - a}\right| \le C,$$

then it's safe to take $\delta = \epsilon/C$, because then

$$|x-a| < \delta \implies |f(x)-L| = |f(x)-f(a)| = |x-a| \cdot \left|\frac{f(x)-f(a)}{x-a}\right| < \delta \cdot C = \epsilon.$$

In practice, one usually can't find such a C without assuming that x is bounded. But this is okay, because we can always take δ to be the smaller of two numbers. If C only works when x is within 1/2 of a, we just take δ to be the minimum of 1/2 and ϵ/C .

For example, suppose you're trying to show that $\lim_{x\to 1} x^3 - 2x = -1$. Look at

$$\frac{x^3 - 2x + 1}{x - 1}.$$

Factoring the numerator, this is

$$\frac{(x-1)(x^2+x-1)}{x-1}$$

which is the same thing as $x^2 + x - 1$, since x is not 1. Now $x^2 + x - 1$ could be pretty big. But if we decide that x will be within 1/2 of 1, then |x| is at most 3/2. So

$$|x^2 + x - 1| \le |x|^2 + |x| + |1| \le 9/4 + 3/2 + 1 = 19/4 < 5$$

So it turns out that we can take δ to be $\min(1/2, \epsilon/5)$.

This kind of approach *always* works for polynomials, and often works for rational functions.

For taking limits of rational functions, it helps to remove any discontinuities that exist. For example, the first step in analyzing

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

is to replace it with the equivalent expression

$$\lim_{x \to 1} x + 1.$$

Another way of thinking about these problems is to keep track of what things can be made small (because they have limit 0), and what things can be bounded (because they have some finite limit, or at least don't have limit infinity).

For example, if you're trying to prove using ϵ - δ that $\lim_{x\to 0} x(\cos x)(x^2+1) = 0$, then the goal is to make $x(\cos x)(x^2+1)$ be really small. This is a product of three things. The first, x, can be made arbitrarily small, because $\lim_{x\to 0} x = 0$. On the other hand $\lim_{x\to 0} \cos x$ and $\lim_{x\to 0} (x^2+1)$ are nonzero, so we shouldn't expect to make those small. But they do approach finite limits, so we can at least make them be bounded: by choosing δ small enough, we can ensure that $\cos x$ will be at most 2 (duh), and that $x^2 + 1$ will be at most 2, because $2 > \lim_{x\to 0} (x^2 + 1)$.

So of the three factors in $x \cdot (\cos x) \cdot (x^2 + 1)$, we can make the first one as small as we like, and the second and third be as small as 2. We want the product to be smaller than ϵ , so we should make the first one be as small as $\epsilon/4$.

So now we just need to choose δ to ensure that $|x| < \epsilon/4$, that $|\cos x| < 2$, and that $|x^2 + 1| < 2$. The first condition is ensured by $\delta \le \epsilon/4$. The second is ensured by anything; it's always true. The third is ensured by, I guess, taking $\delta \le 1/2$. We need to take the smallest of these three values of δ , so we take $\delta = \min(\epsilon/4, 1/2)$.