A MARKOV CHAIN ON THE SYMMETRIC GROUP WHICH IS SCHUBERT POSITIVE?

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Abstract. We study a multivariate Markov chain on the symmetric group with remarkable enumerative properties. We conjecture that the stationary distribution of this Markov chain can be expressed in terms of positive sums of Schubert polynomials. This Markov chain is a multivariate generalization of a Markov chain introduced by the first author in the study of random affine Weyl group elements.

1. A Markov chain on the symmetric group

A recent trend in algebraic combinatorics is the study of Markov chains whose stationary distributions have a combinatorial description. For example, the Razumov-Stroganov (ex)-conjecture concerns a Markov chain on the set of link patterns on $2n$ points around a circle; Razumov and Stroganov conjectured that each component of the stationary distribution could be described as a sum over alternating sign matrices [RS]. After intensive study (see [DZZ] and references therein), this conjecture was recently proved by Cantini and Sportiello [CS]. To give another example, the asymmetric exclusion process (ASEP) is a Markov chain on the 2^n words of length n in two letters. Duchi and Schaeffer [DS], and Corteel and the second author [CW1, CW2] studied different variants of the ASEP, and described the stationary distributions in terms of Dyck paths, permutations, and staircase tableaux.

In this paper, we study a Markov chain on the symmetric group, whose stationary distribution appears to have remarkable combinatorial properties. Let S_n, $n \geq 3$ denote the symmetric group on n letters and let (i,j) denote the transposition which swaps i and j. We use conventions so that left multiplication acts on values and right multiplcation acts on positions.

Define a matrix $P = (p_{w,v})_{w,v \in S_n}$

$$p_{w,v} = \begin{cases} x_{w^{-1}(i+1)} & \text{if } v = (i,i+1)w < w. \\ x_{w^{-1}(1)} & \text{if } v = (1,n)w > w. \\ * & \text{if } w = v. \\ 0 & \text{otherwise.} \end{cases}$$

where $*$ is chosen so that $\sum_{v \in S_n} p_{w,v} = 1$ for each $w \in S_n$. If the x_i's are non-negative real numbers summing to at most 1, then we can think of P as defining a Markov chain on S_n.

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When we set $x_i = 1/n$, we obtain the Markov chain defined in [Lam, Section 3]. This specialized Markov chain $P|_{1/n}$ was introduced to study the asymptotic behavior of random elements in the affine symmetric group, or equivalently, random walks in the affine braid arrangement. The stationary distribution of $P|_{1/n}$ was shown to control the asymptotic "shapes" of random affine symmetric group elements. In particular, a detailed understanding of the Markov chain P, would have applications to the problem studied in [Lam] – our conjectures here imply [Lam, Conjecture 1].

Proposition 1. *The matrix $P^T - I$ has a one-dimensional nullspace for generic values of x. In particular, when the x_i's are nonnegative real numbers summing to at most 1, the Markov chain defined by P has a unique stationary distribution.*

Proof. When all x_i are positive and sum to at most 1, then it follows from [Lam, Proposition 1] that we have an irreducible and aperiodic Markov chain on S_n, and thus we have a unique invariant distribution. If we treat $x_1, \ldots, x_{n-2}, x_{n-1}$ as variables, then a basis of the nullspace of $P^T - I$ can be written as a rational function in the x_i. This nullspace must be one-dimensional. □

Let $\{\zeta(w) \in \mathbb{Q}(x_1, x_2, \ldots, x_{n-1}) \mid w \in W\}$ denote a vector spanning the nullspace of Proposition 1, which we normalize by setting

$$\zeta(w_0) = x_1^{1+2+\cdots+n-2} x_2^{1+2+\cdots+n-3} \cdots x_{n-2}.$$

Suppose $w = w_1 w_2 \cdots w_n \in S_n$. Let $\chi(w) = (w_1+1)(w_2+1)\cdots(w_n+1) \in S_n$ be the cyclic shift of w, where the letters of $\chi(w)$ are interpreted modulo n. The following follows immediately from the definitions.

Proposition 2. *For each $w \in W$, we have $\zeta(\chi(w)) = \zeta(w)$.*

2. Schubert polynomials

We fix notations concerning Schubert polynomials. Let ∂_i denote the divided difference operator on polynomials in $x_1, x_2, \ldots$, defined by

$$\partial_i f(x_1, x_2, \ldots) = \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots) - f(x_1, \ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}.$$

For the longest permutation $w_0 \in S_n$, we first define

$$\mathfrak{S}_{w_0}(x_1, x_2, \ldots) := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

Next for $w \in S_n$, we let $w^{-1} w_0 = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ be a reduced expression. Then

$$\mathfrak{S}_w := \partial_{i_1} \partial_{i_2} \cdots \partial_{i_\ell} \mathfrak{S}_{w_0}.$$

The polynomial $\mathfrak{S}_w$ does not depend on the choice of reduced expression. Furthermore, $\mathfrak{S}_w$ does not depend on which symmetric group w is considered an element of.

3. Conjectures

Our main conjecture is

Conjecture 1. *In increasing strength:*

(1) *Each $\zeta(w)$ is a polynomial.*
(2) *Each $\zeta(w)$ is a polynomial with nonnegative integer coefficients.*
(3) *Each $\zeta(w)$ is a nonnegative integral sum of Schubert polynomials $\mathfrak{S}_u(x_1, x_2, \ldots)$.*

Let $\eta(w)$ denote the largest monomial that can be factored out of $\zeta(w)$. By Proposition 2, $\eta(w) = \eta(\chi(w))$. Write $[m]$ to denote $\{0, 1, 2, \ldots, m\}$.

Conjecture 2 (Monomial factor)**.** *Assume Conjecture 1(1). The map $w \mapsto \eta(w)$ is an n-to-1 map from S_n to*

$$\left\{x_1^{a_1+a_2+\cdots+a_{n-2}} x_2^{a_2+\cdots+a_{n-2}} \cdots x_{n-2}^{a_{n-2}} \mid (a_1, a_2, \ldots, a_{n-2}) \in [n-2] \times [n-3] \times \cdots \times [1]\right\}.$$

Moreover, $\eta(w) = x_1^{a_1+a_2+\cdots+a_{n-2}} x_2^{a_2+\cdots+a_{n-2}} \cdots x_{n-2}^{a_{n-2}}$ is given by

$$a_i = \#\{k \in [i+2, n] \mid w_k \in [w_i, w_{i+1}]\},$$

where $[w_i, w_{i+1}]$ denotes a cyclic subinterval of $[n]$.

Conjecture 3 (Special value)**.**

$$\zeta(\mathrm{id}) = \mathfrak{S}_{123\cdots n}\mathfrak{S}_{1n23\cdots(n-1)}\mathfrak{S}_{1(n-1)n23\cdots(n-2)}\cdots\mathfrak{S}_{134\cdots n2}$$

Conjecture 4 (Special Schubert factors)**.** *Consider the letters of $w \in S_n$ in (cyclic) order. If there is an adjacent string of letters $1, 2$, then $\zeta(w)$ is a multiple of the Schubert polynomial $\mathfrak{S}_{1345\ldots n2}$. More generally, if there is an adjacent string of letters $1, 2, 3, \ldots, k$, then $\zeta(w)$ is a multiple of the Schubert polynomial $\mathfrak{S}_{1(k+1)(k+2)\ldots n23\ldots k}$.*

4. Data

We provide experimental data supporting these conjectures.

4.1. $n = 3$**.** See Figure 1.

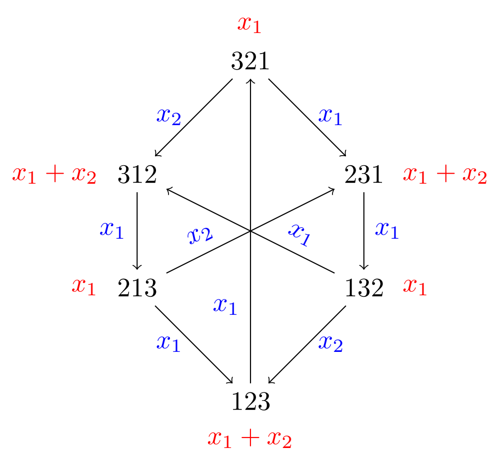

FIGURE 1. The transition matrix on S_3 (in blue) with the transitions from a vertex to itself removed and the normalized stationary distribution ζ (in red).

4.2. $n = 4$. Using Proposition 2, we need only provide data for permutations w where $w_1 = n$. In the following we use $a = x_1$, $b = x_2$, and $c = x_3$. We also write the answers as products of Schubert polynomials. Since a product of Schubert polynomials is also a nonnegative linear combination of Schubert polynomials this supports Conjecture 1(3).

w	$\zeta(w)$	
4123	$(a^2+ab+b^2)(ab+ac+bc)$	$\mathfrak{S}_{1423}\mathfrak{S}_{1342}$
4132	$(a^2+ab+b^2)ab$	$\mathfrak{S}_{1423}\mathfrak{S}_{231}$
4213	$(a+b+c)a^2b$	$\mathfrak{S}_{1243}\mathfrak{S}_{321}$
4231	$(a^2b+a^2c+ab^2+abc+b^2c)a$	$\mathfrak{S}_{1432}\mathfrak{S}_{21}$
4312	$(ab+ac+bc)a^2$	$\mathfrak{S}_{1342}\mathfrak{S}_{312}$
4321	a^3b	$\mathfrak{S}_{4213}$

Note that $a^2b+a^2c+ab^2+abc+b^2c$ is the only non-trivial factor which is not a symmetric polynomial.

4.3. $n = 5$. For $n = 5$ we write our answers as products and sums of Schubert polynomials, multiplied by the monomial factor $\eta(w)$.

w	$\zeta(w)$
51234	$\mathfrak{S}_{15234}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$
51243	$\mathfrak{S}_{15234}\mathfrak{S}_{14523}abc$
51324	$\mathfrak{S}_{15234}\mathfrak{S}_{12453}a^2b^2c$
51342	$\mathfrak{S}_{15234}\mathfrak{S}_{14532}ab$
51423	$\mathfrak{S}_{15234}\mathfrak{S}_{13452}a^2b^2$
51432	$\mathfrak{S}_{15234}a^3b^3c$
52134	$\mathfrak{S}_{12534}\mathfrak{S}_{13452}a^3b^2$
52143	$\mathfrak{S}_{12534}a^4b^3c$
52314	$(\mathfrak{S}_{15432}+\mathfrak{S}_{164235})a^2bc$
52341	$(\mathfrak{S}_{1753246}+\mathfrak{S}_{265314}+\mathfrak{S}_{2743156}+\mathfrak{S}_{356214}+\mathfrak{S}_{364215}+\mathfrak{S}_{365124})a$
52413	$(\mathfrak{S}_{164325}+\mathfrak{S}_{25431})a^2b$
52431	$\mathfrak{S}_{15243}a^3b^2c$
53124	$(\mathfrak{S}_{146325}+\mathfrak{S}_{24531})a^3b$
53142	$\mathfrak{S}_{12543}a^4b^2c$
53214	$\mathfrak{S}_{12354}a^5b^3c$
53241	$\mathfrak{S}_{13542}a^4b^2$
53412	$\mathfrak{S}_{15423}\mathfrak{S}_{13452}a^2$
53421	$\mathfrak{S}_{15423}a^3bc$
54123	$\mathfrak{S}_{14523}\mathfrak{S}_{13452}a^3$
54132	$\mathfrak{S}_{14523}a^4bc$
54213	$\mathfrak{S}_{12453}a^5b^2c$
54231	$\mathfrak{S}_{14532}a^4b$
54312	$\mathfrak{S}_{13452}a^5b^2$
54321	a^6b^3c

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