POSITROIDS AND NON-CROSSING PARTITIONS

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Abstract. We investigate the role that non-crossing partitions play in the study of positroids, a class of matroids introduced by Postnikov. We prove that every positroid can be constructed uniquely by choosing a non-crossing partition on the ground set, and then freely placing the structure of a connected positroid on each of the blocks of the partition. This structural result yields several combinatorial facts about positroids. We show that the face poset of a positroid polytope embeds in a poset of weighted non-crossing partitions. We enumerate connected positroids, and show how they arise naturally in free probability. Finally, we prove that the probability that a positroid on $[n]$ is connected equals $1/e^2$ asymptotically.

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1. Introduction

A *positroid* is a matroid on an ordered set which can be represented by the columns a full rank $d \times n$ real matrix such that all maximal minors are non-negative. Such matroids were first considered by Postnikov [Pos] in his study of the *totally non-negative part of the Grassmannian.* In particular, Postnikov showed that positroids are in bijection with several interesting classes of combinatorial objects, including Grassmann necklaces, decorated permutations, Ⅎ-diagrams, and equivalence classes of plabic graphs.

Positroids have many nice matroidal properties. They are closed under restriction, contraction, and duality, as well as a cyclic shift of the ground set. Positroid polytopes also have nice properties. A general matroid polytope for a matroid on the ground set $[n]$ can be described by using 2^n inequalities; in contrast, as we describe in Section 5, a positroid polytope for a rank d positroid on $[n]$ can be described using $dn + n + 2$ inequalities.

The main structural result of this paper shows the connection between positroids and non-crossing partitions. In Theorem 7.6 we show that the connected components of a positroid form a non-crossing partition. Conversely, each positroid on $[n]$ can be uniquely constructed by choosing a non-crossing partition $(S_1, \dots, S_t)$ of $[n]$, and then putting the structure of a connected positroid on each block S_i. We remark that the first statement was recently discovered independently by Nicolas Ford in his upcoming preprint [For13]. We also give an alternative description of this non-crossing partition in terms of Kreweras complementation.

Our structural result allows us to enumerate connected positroids, as described in Theorem 10.8. Along the way, we show in Theorem 10.7 that the connected positroids on $[n]$ are in bijection with the stabilized-interval-free permutations on $[n]$; that is, the permutations π such that $\pi(I) \neq I$ for all intervals $I \subsetneq [n]$. We then show in Theorem 10.9 that the proportion of positroids on $[n]$ which are connected is equal to $1/e^2$ asymptotically. This result is somewhat surprising in light of the conjecture [MNWW11] that "most matroids are connected"; more specifically, that as n goes to infinity, the ratio of connected matroids on $[n]$ to matroids on $[n]$ tends to 1.

Our enumerative results on positroids also allow us to make a connection to free probability. Concretely, we show that if Y is the random variable $1 + Exp(1)$, then the nth moment $m_n(Y)$ equals the number of positroids on $[n]$, and the nth free cumulant $k_n(Y)$ equals the number of connected positroids on $[n]$.

We also obtain some results on the matroid polytope of a positroid. In Proposition 5.6 we state and prove an inequality description for positroid polytopes, which we learned from Alex Postnikov. More strongly, we show in Theorem 9.3 that the face poset of a positroid polytope naturally embeds in a poset of weighted non-crossing partitions.

The structure of this paper is as follows. In Section 2 we review the notion of a matroid, as well as the operations of restriction, contraction, and duality. In Section 3 we show that positroids are closed under these operations as well as a cyclic shift of the ground set. We also show that if $\{S_1, \dots, S_t\}$ is a non-crossing partition of $[n]$, and M_i is a positroid on S_i, then the direct sum of the M_i's is a positroid. In Section 4 we review Postnikov's notion of Grassmann necklaces, decorated permutations, Ⅎ-diagrams, and plabic graphs, all of which are combinatorial objects parameterizing positroids. We review some of the bijections between them. In Section 5 we turn our attention to positroid polytopes, and provide a simple inequality description of them due to Postnikov. We also show that each face of a positroid polytope is a positroid polytope. In Section 6 we explain how to read off the bases and basis exchanges of a positroid from a corresponding plabic graph. In Section 7 we prove our main structural result on positroids, that the connected components of a positroid comprise a non-crossing partition of the ground set. We also prove a converse to this result. The proofs of these results use plabic graphs as well as positroid polytopes. In Section 8 we give an alternative description of the non-crossing partition of a positroid, relating the Kreweras complement of the partition to the positroid polytope. In Section 9 we define the poset of *weighted non-crossing partitions*, and show that the face poset of a positroid polytope is embedded in it. In Section 10 we give our enumerative results for positroids, and in Section 11 we make the connection to free probability.

2. Matroids

A **matroid** is a combinatorial object which unifies several notions of independence. Among the many equivalent ways of defining a matroid we will adopt the point of view of bases, which is one of the most convenient for the study of positroids and matroid polytopes. We refer the reader to [Oxl92] for a more in-depth introduction to matroid theory.

Definition 2.1. A **matroid** M is a pair $(E, \mathcal{B})$ consisting of a finite set E and a nonempty collection of subsets $\mathcal{B} = \mathcal{B}(M)$ of E, called the **bases** of M, which satisfy the **basis exchange axiom**:

If $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 - B_2$, then there exists $b_2 \in B_2 - B_1$ such that $B_1 - \{b_1\} \cup \{b_2\} \in \mathcal{B}$.

A subset $F \subseteq E$ is called **independent** if it is contained in some basis. All the maximal independent sets contained in a given set $A \subseteq E$ have the same size, which is called the **rank** $r_M(A) = r(A)$ of A. In particular, all the bases of M have the same size, which is called the **rank** $r(M)$ of M.

Example 2.2. Let A be a $d \times n$ matrix of rank d with entries in a field K, and denote its columns by $\mathbf{a_1}, \mathbf{a_2}, \dots, \mathbf{a_n} \in K^d$. The subsets $B \subseteq [n]$ for which the columns $\{\mathbf{a_i} \mid i \in B\}$ form a linear basis for K^d are the bases of a matroid $M(A)$ on the set $[n]$. Matroids arising in this way are called **representable**, and motivate much of the theory of matroids. $\diamondsuit$

There are several natural operations on matroids.

Definition 2.3. Let M be a matroid on E and N a matroid on F. The *direct sum* of matroids M and N is the matroid $M \oplus N$ whose underlying set is the disjoint union of E and F, and whose bases are the disjoint unions of a basis of M with a basis of N.

Definition 2.4. Given a matroid $M = (E, \mathcal{B})$, the *orthogonal* or *dual matroid* $M^* = (E, \mathcal{B}^*)$ is the matroid on E defined by $\mathcal{B}^* = \{E - B \mid B \in \mathcal{B}\}$.

Definition 2.5. Given a matroid $M = (E, \mathcal{B})$, and a subset S of E, the *restriction* of M to S, written $M|S$, is the matroid on the ground set S whose independent sets are all independent sets of M which are contained in S. Equivalently, the set of bases of $M|S$ is

$$\mathcal{B}(M|S) = \{B \cap S \ \mid \ B \in \mathcal{B}, \text{ and } |B \cap S| \text{ is maximal among all } B \in \mathcal{B}\}.$$

The dual operation of restriction is contraction.

Definition 2.6. Given a matroid $M = (E, \mathcal{B})$ and a subset T of E, the *contraction* of M by T, written M/T, is the matroid on the ground set $E - T$ whose bases are the following:

$$\mathcal{B}(M/T) = \{B - T \ \mid \ B \in \mathcal{B}, \text{ and } |B \cap T| \text{ is maximal among all } B \in \mathcal{B}\}.$$

Proposition 2.7. [Oxl92, Chapter 3.1, Exercise 1] *If M is a matroid on E and $S \subseteq E$, then*

$$(M/S)^* = M^*|(E - S).$$

3. Positroids

In this paper we study a special class of representable matroids introduced by Postnikov in [Pos]. We begin by collecting several foundational results on positroids, most of which are known [Oh11, Pos].

Definition 3.1. Suppose A is a $d \times n$ matrix of rank d with real entries such that all its maximal minors are nonnegative. Such a matrix A is called *totally nonnegative*, and the representable matroid $M(A)$ associated to A is called a **positroid**.

Remark 3.2. We will often identify the ground set of a positroid with the set $[n]$, but more generally, the ground set of a positroid may be any finite set $E = \{e_1, \ldots, e_n\}$, endowed with a specified total order $e_1 < \cdots < e_n$. Note that the fact that a given matroid is a positroid is strongly dependent on the total order of its ground set; in particular, being a positroid is not invariant under matroid isomorphism.

If A is as in Definition 3.1 and $I \in \binom{[n]}{d}$ is a d-element subset of $[n]$, then we let $\Delta_I(A)$ denote the $d \times d$ minor of A indexed by the column set I. These minors are called the *Plücker coordinates* of A.

In our study of positroids, we will repeatedly make use of the following notation. Given $k, \ell \in [n]$, we define the **(cyclic) interval** $[k, \ell]$ to be the set

$$[k,\ell] := \begin{cases} \{k, k+1, \dots, \ell\} & \text{if } k \leq \ell, \\ \{k, k+1, \dots, n, 1, \dots, \ell\} & \text{if } \ell < k. \end{cases}$$

We also refer to a cyclic interval as a *cyclically consecutive* subset of $[n]$. We will often put a total order on a cyclic interval: in the first case above we use the total order $k < k+1 < \cdots < \ell$, and in the second case, we use the total order $k < k+1 < \cdots < n < 1 < \cdots < \ell$.

Positroids are closed under several key operations:

Lemma 3.3. *Let M be a positroid on the ground set $E = \{1 < \cdots < n\}$. Then for any $1 \leq a \leq n$, M is also a positroid on the ordered ground set $\{a < a+1 < \cdots < n < 1 < \cdots < a-1\}$.*

Proof. Let $M = M(A)$ for some totally nonnegative full rank $d \times n$ matrix A. Write $A = (v_1, \dots, v_n)$ as a concatenation of its column vectors $v_i \in \mathbb{R}^d$. Then as noted in [Pos, Remark 3.3], the matrix $A' = (v_2, \dots, v_n, (-1)^{d-1} v_1)$ obtained by cyclically shifting the columns of A and multiplying the last column by $(-1)^{d-1}$ is also totally nonnegative. Moreover, $\Delta_I(A) = \Delta_{I'}(A')$, where I' is the cyclic shift of the subset I. Therefore $M(A')$ is a positroid, which coincides with M after cyclically shifting the ground set. It follows that M is a positroid on $\{2 < 3 < \cdots < n < 1\}$, and by iterating this construction, the lemma follows. □

Proposition 3.4. *Suppose we have a decomposition of $[n]$ into two cyclic intervals $[\ell+1, m]$ and $[m+1, \ell]$. Let M be a positroid on the ordered ground set $[\ell+1, m]$ and let M' be a positroid on the ordered ground set $[m+1, \ell]$. Then $M \oplus M'$ is a positroid on the ordered ground set $[n] = \{1 < \cdots < n\}$.*

Proof. First assume $\ell = n$. Let M be a positroid on the ground set $[m] = \{1 < \cdots < m\}$ and M' be a positroid on the ground set $\{m+1 < \cdots < n\}$. Then $M = M(A)$ and $M' = M(B)$, where A and B are full rank $d \times m$ and $d' \times (n-m)$ matrices whose maximal minors are nonnegative. Use A and B to form the $(d+d') \times n$ block matrix of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Clearly this matrix has all maximal minors nonnegative and represents the direct sum $M(A) \oplus M(B)$ of the matroids $M(A)$ and $M(B)$. It follows that $M \oplus M'$ is a positroid. Now the proposition follows from Lemma 3.3. □

The following proposition says that positroids are closed under duality, restriction, and contraction.

Proposition 3.5. *Let M be a positroid on $[n]$. Then M^* is also a positroid on $[n]$. Furthermore, for any subset S of $[n]$, the restriction $M|S$ is a positroid on S, and the contraction M/S is a positroid on $[n]-S$. Here the total orders on S and $[n]-S$ are the ones inherited from $[n]$.*

Proof. Suppose that $A = (a_{ij})$ is a full rank $d \times n$ real matrix such that $M = M(A)$ and all maximal minors of A are nonnegative. By performing row operations on A and multipliying rows by -1 when necessary, we may assume without loss of generality that A is in reduced row-echelon form. In particular, A contains the identity matrix in columns $i_1, i_2, \ldots, i_d$ for some $i_1 < \cdots < i_d$. Let us label the rows of A by $i_1, \ldots, i_d$. Now construct an $(n-d) \times n$ matrix $A' = (a'_{ij})$, with rows labeled by $[n] - \{i_1, \ldots, i_d\}$, as follows. First we place the identity matrix in columns $[n] - \{i_1, \ldots, i_d\}$. For the remaining entries we define $a'_{ij} = a_{ji}$, see Example 3.6. It is not hard to check that for each $I \in \binom{[n]}{d}$, we have that $\Delta_I(A) = \Delta_{[n]-I}(A')$. It follows that $M(A')$ is the dual M^* of M and is also a positroid, as we wanted.

We will now prove that the contraction M/S is a positroid on $[n]-S$. If $S \cap S' = \emptyset$ then $(M/S)/S' = M/(S \cup S')$, so by induction it is enough to prove that M/S is a positroid for S a subset of size 1. Moreover, in view of Lemma 3.3, we can assume without loss of generality that $S = \{1\}$. Again, suppose that $A = (a_{ij})$ is a full rank $d \times n$ real matrix in reduced row-echelon form such that $M = M(A)$ and all maximal minors of A are nonnegative. If $\{1\}$ is a dependent subset in M then the first column of A contains only zeros, and M/S is the rank d positroid on S represented by the submatrix of A obtained by eliminating its first column. If $\{1\}$ is an independent subset in M then the first column of A is the vector $e_1 \in \mathbb{R}^d$. The matroid M/S is then represented by the submatrix A' of A obtained by eliminating its first column and its first row, which also has nonnegative maximal minors since $\Delta_I(A') = \Delta_{\{1\} \cup I}(A)$.

Finally, since positroids are closed under duality and contraction, by Proposition 2.7 they are also closed under restriction. $\square$

Example 3.6. Let

$$A = \begin{pmatrix} a_{21} & 1 & a_{23} & 0 & a_{25} & a_{26} \\ a_{41} & 0 & a_{43} & 1 & a_{45} & a_{46} \end{pmatrix}$$

represent a matroid $M(A)$ on $[6]$. Then the matrix A' as defined in the proof of Proposition 3.5 is given by

$$A' = \begin{pmatrix} 1 & a_{21} & 0 & a_{41} & 0 & 0 \\ 0 & a_{23} & 1 & a_{43} & 0 & 0 \\ 0 & a_{25} & 0 & a_{45} & 1 & 0 \\ 0 & a_{26} & 0 & a_{46} & 0 & 1 \end{pmatrix},$$

and $M(A') = M(A)^*$. Moreover, for each $I \in \binom{[6]}{2}$, $\Delta_I(A) = \Delta_{[6]-I}(A')$. $\diamondsuit$

4. Combinatorial objects parameterizing positroids

In [Pos], Postnikov gave several families of combinatorial objects in bijection with positroids. In this section we will start by defining his notion of *Grassmann necklace*, and explain how each one naturally labels a positroid. We will then define *decorated permutations*, *Ⅎ-diagrams*, and equivalence classes of *reduced plabic graphs*, and give (compatible) bijections among all these objects. This will give us a canonical way to label each positroid by a Grassmann necklace, a decorated permutation, a Ⅎ-diagram, and a plabic graph.

4.1. **Grassmann necklaces.**

Definition 4.1. Let $d \leq n$ be positive integers. A **Grassmann necklace** of type (d, n) is a sequence $(I_1, I_2, \ldots, I_n)$ of subsets $I_k \in \binom{[n]}{d}$ such that for any $i \in [n]$

- if $i \in I_i$ then $I_{i+1} = I_i - \{i\} \cup \{j\}$ for some $j \in [n]$,
- if $i \notin I_i$ then $I_{i+1} = I_i$,

where $I_{n+1} = I_1$.

The ***i*-order** $<_i$ on the set $[n]$ is the total order

$$i <_i i+1 <_i \cdots <_i n <_i 1 <_i \cdots <_i i-2 <_i i-1.$$

For any rank d matroid $M = ([n], \mathcal{B})$, let I_k be the lexicographically minimal basis of M with respect to the order $<_k$, and denote

$$\mathcal{I}(M) = (I_1, I_2, \ldots, I_n).$$

Proposition 4.2 ([Pos, Lemma 16.3])**.** *For any rank d matroid $M = ([n], \mathcal{B})$, the sequence $\mathcal{I}(M)$ is a Grassmann necklace of type (d, n).*

In the case where the matroid M is a positroid we can actually recover M from its Grassmann necklace, as described below.

Let $i \in [n]$. The **Gale order** on $\binom{[n]}{d}$ (with respect to $<_i$) is the partial order $\leq_i$ defined as follows: for any two d-subsets $S = \{s_1 <_i \cdots <_i s_d\} \subseteq [n]$ and $T = \{t_1 <_i \cdots <_i t_d\} \subseteq [n]$, we have $S \leq_i T$ if and only if $s_j \leq_i t_j$ for all $j \in [d]$.

Theorem 4.3 ([Pos, Oh11])**.** *Let $\mathcal{I} = (I_1, I_2, \ldots, I_n)$ be a Grassmann necklace of type (d, n). Then the collection*

$$\mathcal{B}(\mathcal{I}) := \left\{ B \in \binom{[n]}{d} \;\middle|\; B \geq_j I_j \text{ for all } j \in [n] \right\}$$

is the collection of bases of a rank d positroid $\mathcal{M}(\mathcal{I}) := ([n], \mathcal{B}(\mathcal{I}))$. Moreover, for any positroid M we have $\mathcal{M}(\mathcal{I}(M)) = M$.

Corollary 4.4 follows directly from the definitions.

Corollary 4.4. *Let M be a matroid. Then every basis of M is also a basis of $\mathcal{M}(\mathcal{I}(M))$.*

Theorem 4.3 shows that $\mathcal{M}$ and $\mathcal{I}$ are inverse bijections between the set of Grassmann necklaces of type (d, n) and the set of rank d positroids on the set $[n]$. Moreover, it follows that for any matroid M the positroid $\mathcal{M}(\mathcal{I}(M))$ is the smallest positroid containing M, in the sense that any positroid containing all bases of M must also contain all bases of $\mathcal{M}(\mathcal{I}(M))$.

4.2. **Decorated permutations.** The information contained in a Grassmann necklace can be encoded in a more compact way, as follows.

Definition 4.5. A **decorated permutation** of the set $[n]$ is a bijection $\pi : [n] \to [n]$ whose fixed points are colored either "clockwise" or "counterclockwise." We denote a clockwise fixed point by $\pi(j) = \overline{j}$ and a counterclockwise fixed point by $\pi(j) = \underline{j}$. A **weak i-excedance** of the decorated permutation π is an element $j \in [n]$ such that either $j <_i \pi(j)$ or $\pi(j) = \underline{j}$ is a "counterclockwise" fixed point. The number of weak i-excedances of π is the same for any $i \in [n]$, and we will simply call it the number of **weak excedances** of π.

Given a Grassmann necklace $\mathcal{I} = (I_1, I_2, \ldots, I_n)$ we can construct a decorated permutation $\pi_{\mathcal{I}}$ of the set $[n]$ in the following way.

- If $I_{i+1} = I_i - \{i\} \cup \{j\}$ for $i \neq j$ then $\pi_{\mathcal{I}}(j) := i$.
- If $I_{i+1} = I_i$ and $i \notin I_i$ then $\pi_{\mathcal{I}}(i) := \overline{i}$.
- If $I_{i+1} = I_i$ and $i \in I_i$ then $\pi_{\mathcal{I}}(i) := \underline{i}$.

Conversely, given a decorated permutation π of $[n]$ we can construct a Grassmann necklace $\mathcal{I}_\pi = (I_1, I_2, \ldots, I_n)$ by letting I_k be the set of weak k-excedances of π. It is straightforward to verify the following.

Proposition 4.6. *The maps $\mathcal{I} \mapsto \pi_{\mathcal{I}}$ and $\pi \mapsto \mathcal{I}_\pi$ are inverse bijections between the set of Grassmann necklaces of type (d, n) and the set of decorated permutations of $[n]$ having d weak excedances.*

4.3. **Le-diagrams.**

Definition 4.7. Fix d and n. Let Y_λ denote the Young diagram of the partition λ. A Ꞁ-*diagram* (or Le-diagram) D of shape λ and type (d, n) is a Young diagram Y_λ contained in a $d \times (n - d)$ rectangle whose boxes are filled with 0's and $+$'s, in such a way that the Ꞁ-*property* is satisfied: there is no 0 which has a $+$ above it in the same column and a $+$ to its left in the same row. See Figure 1 for an example of a Ꞁ-diagram.

Lemma 4.8. *The following algorithm is a bijection between Ꞁ-diagrams of type (d, n) and decorated permutations on n letters with d weak excedances.*

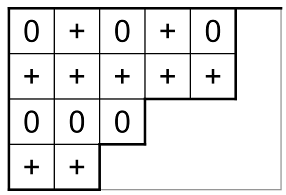


FIGURE 1. A Le diagram with $\lambda = 5532, d = 4$, and $n = 10$.

(1) *Replace each* $+$ *in the* ⅃*-diagram* D *with an elbow joint* , *and each* 0 *in* D *with a cross* .
(2) *Note that the south and east border of* Y_λ *gives rise to a length-*n *path from the north-east corner to the south-east corner of the* $d \times (n-d)$ *rectangle. Label the edges of this path with the numbers* 1 *through* n.
(3) *Now label the edges of the north and west border of* Y_λ *so that opposite horizontal edges and opposite vertical edges have the same label.*
(4) *View the resulting "pipe dream" as a permutation* $\pi \in S_n$, *by following the "pipes" from the northwest border to the southeast border of the Young diagram. If the pipe originating at label* i *ends at the label* j, *we define* $\pi(i) = j$.
(5) *If* $\pi(j) = j$ *and* j *labels two horizontal (respectively, vertical) edges of* Y_λ, *then* $\pi(j) := \overline{j}$ *(respectively,* $\pi(j) := \underline{j}$*).*

Figure 2 illustrates this procedure for the ⅃-diagram of Figure 1, giving rise to the decorated permutation $\overline{1}, 7, 9, 3, 2, \underline{6}, 5, 10, 4, 8$.

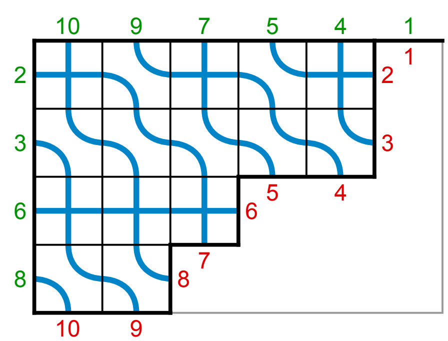


FIGURE 2. A "pipe dream"

4.4. **Plabic graphs.**

Definition 4.9. A *plabic graph*[1] is an undirected graph G drawn inside a disk (considered modulo homotopy) with n *boundary vertices* on the boundary of the disk, labeled $b_1, \ldots, b_n$ in clockwise order, as well as some colored

[1] "Plabic" stands for "planar bicolored."

internal vertices. These internal vertices are strictly inside the disk and are colored in black and white such that each boundary vertex b_i in G is incident to a single edge.

A *perfect orientation* $\mathcal{O}$ of a plabic graph G is a choice of orientation of each of its edges such that each black internal vertex u is incident to exactly one edge directed away from u; and each white internal vertex v is incident to exactly one edge directed towards v. A plabic graph is called *perfectly orientable* if it admits a perfect orientation. Let $G_{\mathcal{O}}$ denote the directed graph associated with a perfect orientation $\mathcal{O}$ of G. The *source set* $I_{\mathcal{O}} \subseteq [n]$ of a perfect orientation $\mathcal{O}$ is the set of i for which b_i is a source of the directed graph $G_{\mathcal{O}}$. Similarly, if $j \in \overline{I}_{\mathcal{O}} := [n] - I_{\mathcal{O}}$, then b_j is a sink of $\mathcal{O}$.

Figure 5a shows a plabic graph with a perfect orientation. In that example, $I_{\mathcal{O}} = \{2, 3, 6, 8\}$

All perfect orientations of a fixed G have source sets of the same size d where $d-(n-d) = \sum \operatorname{color}(v)\,(\deg(v)-2)$. Here the sum is over all internal vertices v, $\operatorname{color}(v) = 1$ for a black vertex v, and $\operatorname{color}(v) = -1$ for a white vertex; see [Pos]. In this case we say that G is of *type* (d, n).

The following construction, which comes from [Pos, Section 20], associates a plabic graph to a Ⅎ-diagram.

Definition 4.10. Let D be a Ⅎ-diagram. Delete the 0's, and replace each $+$ with a vertex. From each vertex we construct a hook which goes east and south, to the border of the Young diagram. The resulting diagram is called the "hook diagram" $H(D)$. After replacing the edges along the southeast border of the Young diagram with boundary vertices labeled by $1, 2, \ldots, n$, we obtain a graph with n boundary vertices and one internal vertex for each $+$ from D. Then we replace the local region around each internal vertex as in Figure 3, and embed the resulting bicolored graph in a disk. Finally, for each clockwise (respectively, counterclockwise) fixed point, we add a black (respectively, white) boundary leaf at the corresponding boundary vertex. This gives rise to a plabic graph which we refer to as $G(D)$.

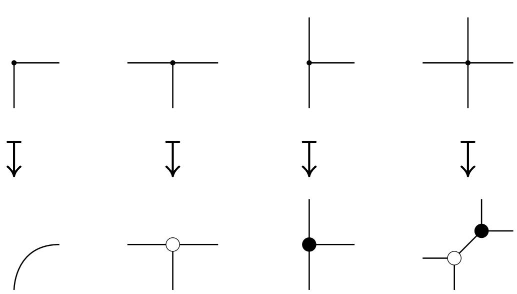
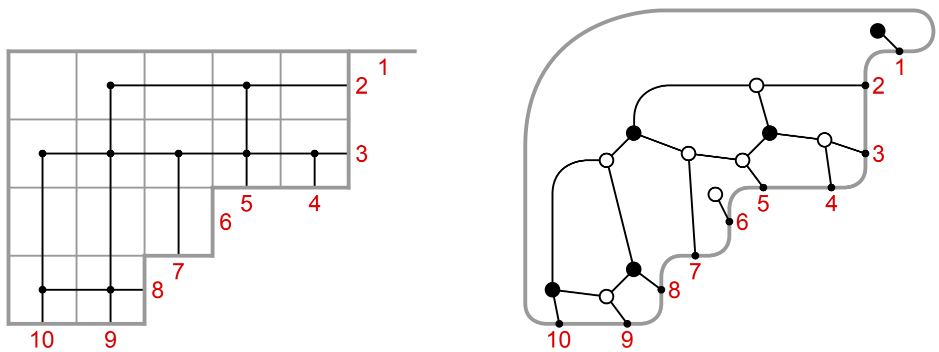


FIGURE 3. Local substitutions for getting the plabic graph $G(D)$ from the hook diagram $H(D)$.

Figure 4a depicts the hook diagram corresponding to the ⅃-diagram given in Figure 1, and Figure 4b shows its corresponding plabic graph.



(A) A hook diagram (B) A plabic graph

Figure 4

More generally each ⅃-diagram D is associated with a family of *reduced plabic graphs* consisting of $G(D)$ together with other plabic graphs which can be obtained from $G(D)$ by certain *moves*, see [Pos, Section 12].

From the plabic graph constructed in Definition 4.10 (and more generally from any leafless reduced plabic graph G without isolated components), one may read off the corresponding decorated permutation π_G as follows.

Definition 4.11. Let G be a reduced plabic graph as above with boundary vertices $b_1, \ldots, b_n$. The *trip* from b_i is the path obtained by starting from b_i and traveling along edges of G according to the rule that each time we reach an internal white vertex we turn right, and each time we reach an internal black vertex we turn left. This trip ends at some boundary vertex $b_{\pi(i)}$. If the boundary vertex b_j is attached to a black (respectively, white) boundary leaf, then we set $\pi(j) = \overline{j}$ (respectively, $\pi(j) = \underline{j}$). In this way we associate a decorated permutation $\pi_G = (\pi(1), \ldots, \pi(n))$ to each reduced plabic graph G, which is called the *decorated trip permutation* of G.

We invite the to verify that when we apply these rules to Figure 4b we obtain the trip permutation $\overline{1}, 7, 9, 3, 2, \underline{6}, 5, 10, 4, 8$.

Remark 4.12. All bijections that we have defined in this section are compatible. This gives us a canonical way to label each positroid of rank d on $[n]$ by: a Grassmann necklace, a decorated permutation, a ⅃-diagram, and an equivalence class of plabic graphs.

5. Positroid polytopes

The following geometric representation of a matroid will be useful in our study of positroids.

Definition 5.1. Given a matroid $M = ([n], \mathcal{B})$, the (basis) **matroid polytope** Γ_M of M is the convex hull of the indicator vectors of the bases of M:

$$\Gamma_M := \text{convex}\{e_B \mid B \in \mathcal{B}\} \subset \mathbb{R}^n,$$

where $e_B := \sum_{i \in B} e_i$, and $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

When we speak of "a matroid polytope," we refer to the polytope of a specific matroid in its specific position in $\mathbb{R}^n$.

The following elegant characterization of matroid polytopes is due to Gelfand, Goresky, MacPherson, and Serganova.

Theorem 5.2 ([GGMS87])**.** *Let $\mathcal{B}$ be a collection of subsets of $[n]$ and let $\Gamma_{\mathcal{B}} := \text{convex}\{e_B \mid B \in \mathcal{B}\} \subset \mathbb{R}^n$. Then $\mathcal{B}$ is the collection of bases of a matroid if and only if every edge of $\Gamma_{\mathcal{B}}$ is a parallel translate of $e_i - e_j$ for some $i, j \in [n]$.*

When the conditions of Theorem 5.2 are satisfied, the edges of $\Gamma_{\mathcal{B}}$ correspond exactly to the basis exchanges; that is, to the pairs of distinct bases B_1, B_2 such that $B_2 = B_1 - \{i\} \cup \{j\}$ for some $i, j \in [n]$. Two such bases are called **adjacent bases**.

The following result is a restatement of the greedy algorithm for matroids.

Proposition 5.3. [AK06, Prop. 2] *Let M be a matroid on $[n]$. Then any face of the matroid polytope Γ_M is itself a matroid polytope. More specifically, let $w : \mathbb{R}^n \to \mathbb{R}$ be a linear functional. Let $w_i = w(e_i)$; note that by linearity, these values determine w. Consider the flag of sets $\emptyset = A_0 \subsetneq A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_k = [n]$ such that $w_a = w_b$ for $a, b \in A_i - A_{i-1}$, and $w_a < w_b$ for $a \in A_i - A_{i-1}$ and $b \in A_{i+1} - A_i$. Then the face of Γ_M minimizing the linear functional w is the matroid polytope of the matroid*

$$\bigoplus_{i=1}^{k} (M|A_i)/A_{i-1}.$$

Corollary 5.4. *Every face of a positroid polytope is a positroid polytope.*

Proof. This follows from Propositions 3.5 and 5.3. $\square$

Next we will give an inequality description of positroid polytopes.

Proposition 5.5 ([Wel76])**.** *Let $M = ([n], \mathcal{B})$ be any matroid of rank d, and let $r_M : 2^{[n]} \to \mathbb{Z}_{\geq 0}$ be its rank function. Then the matroid polytope Γ_M can be described as*

$$\Gamma_M = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum_{i \in [n]} x_i = d, \sum_{i \in A} x_i \leq r_M(A) \;\textit{ for all } A \subseteq [n] \right\}.$$

Proposition 5.5 describes a general matroid polytope using the 2^n inequalities arising from the rank of all subsets of its ground set. For positroid polytopes, however, there is a much shorter description, which we learned from Alex Postnikov [Pos12].

Proposition 5.6. *Let $\mathcal{I} = (I_1, I_2, \ldots, I_n)$ be a Grassmann necklace of type (d, n), and let $M = (\mathcal{I})$ be its corresponding positroid. For any $j \in [n]$, suppose the elements of I_j are $a_1^j <_j a_2^j <_j \cdots <_j a_d^j$. Then the matroid polytope Γ_M can be described by the inequalities*

$$x_1 + x_2 + \cdots + x_n = d, \tag{1}$$

$$x_j \geq 0 \qquad \textit{for all } j \in [n], \tag{2}$$

$$x_j + x_{j+1} + \cdots + x_{a_k^j - 1} \leq k - 1 \qquad \textit{for all } j \in [n] \textit{ and } k \in [d], \tag{3}$$

where all the subindices are taken modulo n.

In Proposition 5.6, when we refer to taking some number i modulo n, we mean taking its representative modulo n in the set $\{1, \ldots, n\}$.

Proof. Let P be the polytope described by (1), (2), and (3). First we claim that the vertices of P are $0/1$ vectors. To see this, rewrite the polytope in terms of the "y-coordinates" given by $y_i = x_1 + \cdots + x_i$ for $1 \leq i \leq n-1$. The inequalities of P are of the form $y_i - y_j \leq a_{ij}$ for integers a_{ij}. Since the matrix whose row vectors are $e_i - e_j$ is totally unimodular [Sch86], the vertices of P have integer y-coordinates, and hence also integer x-coordinates. The inequalities (2) and (3) (for $k = 2$) imply that the x-coordinates of any vertex are all equal to 0 or 1.

Since P and Γ_M are $0/1$ polytopes, it suffices to show they have the same vertices. But for a $0/1$ vector e_B satisfying (1), the inequalities (3) are equivalent to $B \geq_j I_j$ for all j, *i.e.*, to $B \in \mathcal{B}(\mathcal{I})$, as desired. $\square$

Proposition 5.7. *A matroid M is a positroid if and only if all the facets of its matroid polytope Γ_M correspond to cyclic intervals, that is, they are given by equations of the form*

$$\sum_{\ell \in [i,j]} x_\ell = r_M([i, j]) \textit{ for some } i, j \in [n].$$

Proof. Proposition 5.6 shows that all the facets of a positroid polytope have the desired form. To prove the converse, assume M is a rank d matroid on the set $[n]$ such that all the facets of Γ_M correspond to cyclic intervals. Let $r_{ij} = r_M([i, j])$ be the rank in M of the cyclic interval $[i, j]$. The polytope Γ_M can then be described by the inequalities

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= d, \\ x_i + x_{i+1} + \cdots + x_j &\leq r_{ij} \qquad \text{for all } i, j \in [n]. \end{aligned}$$

Let $\mathcal{I} := \mathcal{I}(M) = (I_1, I_2, \ldots, I_n)$ be the associated Grassmann necklace, and let $M' := \mathcal{M}(\mathcal{I}(M))$ be its corresponding positroid. Recall from Section

4.1 that this is the smallest positroid containing M. We will now show that $M = M'$.

By Corollary 4.4, every basis of M is also a basis of M'. Now, suppose B is basis of M', and consider any cyclic interval $[i, j]$. Denote $I_i = \{a_1 <_i a_2 <_i \cdots <_i a_d\}$, and let $k = |I_i \cap [i, j]|$. Then $i \leq j \leq a_{k+1} - 1$ in cyclic order. Combining this with Proposition 5.6, we see that the vertex e_B of $\Gamma_{M'}$ satisfies the inequality

$$x_i + x_{i+1} + \cdots + x_j \leq x_i + x_{i+1} + \cdots + x_{a_{k+1}-1} \leq k.$$

(with the convention that $a_{d+1} = i$.) Moreover, the definitions of $\mathcal{I}_i$ and k imply that $k = r_{ij}$, showing that e_B satisfies all the inequalities that describe Γ_M. It follows that B is also a basis of M, as desired. $\square$

6. Matroidal properties of positroids from plabic graphs

As shown in [Pos, Section 11], every perfectly orientable plabic graph gives rise to a positroid as follows.

Proposition 6.1. *Let G be a plabic graph of type (d, n). Then we have a positroid M_G on $[n]$ whose bases are precisely*

$$\{I_{\mathcal{O}} \mid \mathcal{O} \text{ is a perfect orientation of } G\}.$$

where $I_{\mathcal{O}}$ is the set of sources of $\mathcal{O}$.

Moreover, every positroid can be realized in this way, using the construction of Definition 4.10. (One perfect orientation of $G(D)$ may be obtained by orienting each horizontal edge in Figure 3 west, each vertical edge south, and each diagonal edge southwest.)

There is another way to read off from G the bases of M_G, which follows from Proposition 6.1 and [Tal08, Theorem 1.1]. To state this result, we need to define the notion of a *flow*. For $|J| = |I_{\mathcal{O}}|$, a *flow from $I_{\mathcal{O}}$ to J* is a collection of self-avoiding walks and self-avoiding cycles, all pairwise vertex-disjoint, such that the sources of these walks are $I_{\mathcal{O}} - (I_{\mathcal{O}} \cap J)$ and the destinations are $J - (I_{\mathcal{O}} \cap J)$.

Proposition 6.2. *Let G be a plabic graph of type (d, n). Choose a perfect orientation $\mathcal{O}$ of G. Then the bases of the positroid M_G are precisely*

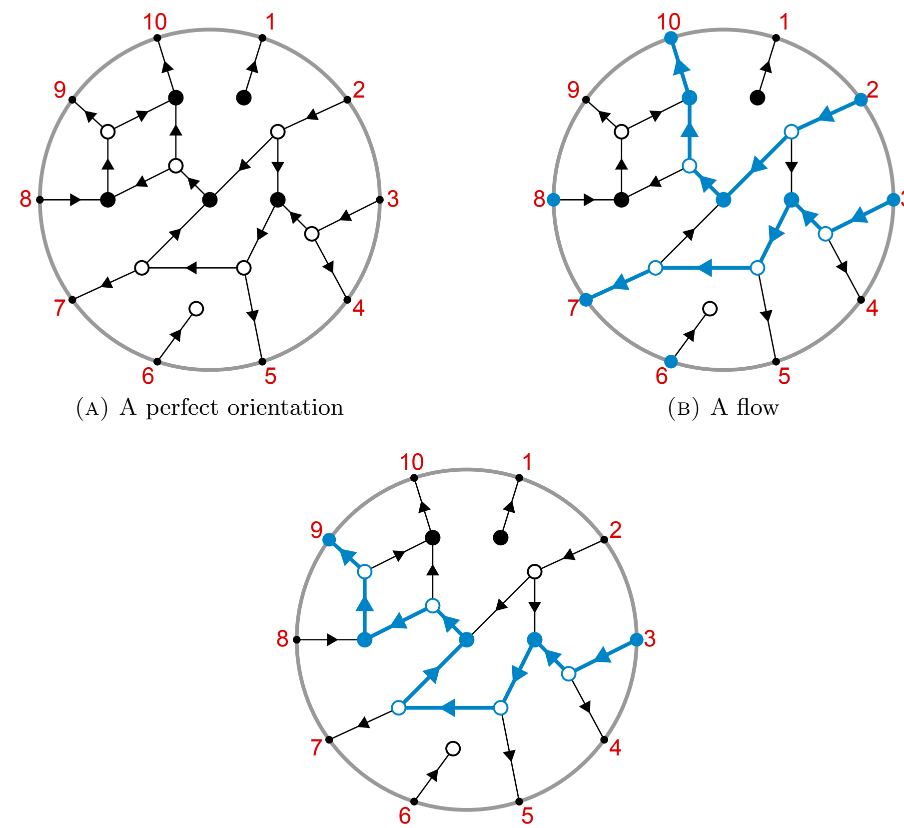
$$\{I \mid \text{there exists a flow from } I_{\mathcal{O}} \text{ to } I\}.$$

Not only can we read off bases from plabic graph, we can also read off basis exchanges. The backwards direction of Proposition 6.3 below was observed in [PSW09, Section 5].

Proposition 6.3. *Consider a positroid M which is encoded by the perfectly orientable plabic graph G. Consider a perfect orientation $\mathcal{O}$ of G and let $I = I_{\mathcal{O}}$. Then there is a basis exchange between I and $J = I - \{i\} \cup \{j\}$ if and only if there is a directed path P in $\mathcal{O}$ from the boundary vertex i to the boundary vertex j.*

Proof. Suppose that P is a directed path in $\mathcal{O}$ from i to j. Then if we modify $\mathcal{O}$ by reversing all edges along P, we obtain another perfect orientation $\mathcal{O}'$, whose source set is $J = I - \{i\} \cup \{j\}$. It follows from Proposition 6.1 that J is also a basis of M and hence there is a basis exchange between I and J that swaps i and j.

Conversely, suppose that there is a basis exchange between I and $J = I - \{i\} \cup \{j\}$. Then by Proposition 6.1 there is a perfect orientation $\mathcal{O}'$ of G such that $I_{\mathcal{O}'} = J$. Comparing $\mathcal{O}'$ to $\mathcal{O}$, it is clear that the set of edges where the perfect orientations differ is a subgraph H' of G such that all vertices have degree 2 (except possibly at the boundary), see e.g. [PSW09, Lemma 4.5]. More specifically, H is a disjoint union of some closed cycles $C_1, \ldots, C_l$ together with a path P between vertices i and j, and $\mathcal{O}'$ is obtained from $\mathcal{O}$ by reversing all edges in H. It follows from the definition of perfect orientation that P must be a directed path in both $\mathcal{O}$ and $\mathcal{O}'$. □



(A) A perfect orientation

(B) A flow

(C) A directed path

FIGURE 5. A perfect orientation, a flow, and a directed path in a plabic graph.

Example 6.4. Figure 5a shows the plabic graph G given in Figure 4b (rearranged without changing the combinatorial type), together with a perfect orientation $\mathcal{O}$ of its edges. The corresponding source set $I_{\mathcal{O}} = \{2, 3, 6, 8\}$ is then a basis of the corresponding positroid $M = \mathcal{M}(G)$. Figure 5b depicts a flow from $I_{\mathcal{O}}$ to the set $I = \{6, 7, 8, 10\}$, which implies that I is a basis of M. The directed path in $\mathcal{O}$ from 3 to 9 highlighted in Figure 5c shows that the set $I_{\mathcal{O}} - \{3\} \cup \{9\}$ is a basis of M. Finally, since there is no directed path in $\mathcal{O}$ from 2 to 4, the set $I_{\mathcal{O}} - \{2\} \cup \{4\}$ is not a basis of M. $\diamondsuit$

7. Positroids, connected positroids, and non-crossing partitions

In this section we begin to illustrate the role that non-crossing partitions play in the theory of positroids. More specificaly, Theorem 7.6 shows that the connected components of a positroid form a non-crossing partition. Conversely, it also says that positroids on $[n]$ can be built out of connected positroids by first choosing a non-crossing partition on $[n]$, and then putting the structure of a connected positroid on each of the blocks of the non-crossing partition.

Definition 7.1. A matroid which cannot be written as the direct sum of two nonempty matroids is called *connected*.

Proposition 7.2. [Oxl92]. *Let M be a matroid on E. For two elements $a, b \in E$, we set $a \sim b$ whenever there are bases B_1, B_2 of M such that $B_2 = B_1 - \{a\} \cup \{b\}$. The relation $\sim$ is an equivalence relation, and the equivalence classes are precisely the connected components of M.*

Proof. It is more customary to define $a \sim b$ if $a, b \in C$ for some circuit C. It is known that this is an equivalence relation whose equivalence classes are the connected components of M [Oxl92, Chapter 4.1]. We now verify that these two definitions are equivalent:

If B_1 and B_2 are bases and $B_2 = B_1 - \{a\} \cup \{b\}$ then there is a unique circuit $C \subseteq B_1 \cup \{b\}$, called the *fundamental circuit of B with respect to b*. It has the property that $a, b \in C$. Conversely, if a, b are contained in a circuit C, let D be a basis of M/C. Then $B_2 := D \cup C - \{a\}$ and $B_1 := D \cup C - \{b\}$ are bases of M such that $B_2 = B_1 - \{a\} \cup \{b\}$. $\square$

Lemma 7.3. *Let M be a positroid on E, and write it as a direct sum of connected matroids $M = M_1 \oplus \cdots \oplus M_l$. Then each M_i is a positroid.*

Proof. This follows from the fact that each M_i is the restriction of M to some subset of E, and restrictions of positroids are positroids. $\square$

Proposition 7.4. *Suppose that M is a positroid on $[n]$ which is the direct sum $M_1 \oplus M_2$ of two connected positroids M_1 and M_2 on ground sets E_1 and E_2. Then E_1 and E_2 are cyclic intervals of $[n]$.*

Proof. We propose two arguments: one in terms of perfect orientations, and one in terms of matroid polytopes.

1. (Perfect orientations) Suppose that E_1 and E_2 are not cyclic intervals. Then there exist positive integers $1 \le i < j < k < l \le n$ such that $i, k \in E_1$ and $j, l \in E_2$. By Proposition 7.2, there exist bases B_1 and B_1' of M_1 such that $i \in B_1$ and $B_1' = B_1 - \{i\} \cup \{k\}$, and there exist bases B_2 and B_2' of M_2 such that $j \in B_2$ and $B_2' = B_2 - \{j\} \cup \{l\}$. Then $B = B_1 \cup B_2$ is basis of M which contains i and j. Moreover, B admits a basis exchange which replaces i with k, and a basis exchange which replaces j with l.

Therefore by Proposition 6.1 there exists a perfect orientation $\mathcal{O}$ of a plabic graph for M whose set I of sources contains i and j. And by Proposition 6.3, $\mathcal{O}$ has a directed path P_1 from i to k, and a directed path P_2 from j to l. Because $i < j < k < l$ these directed paths must intersect at some internal vertex v. But now it is clear that $\mathcal{O}$ also contains a directed path P_3 from i to l (and from j to k): P_3 is obtained by following P_1 from i to v, and then following P_2 from v to l. Therefore M has a basis exchange which switches i and l, contradicting our assumption that i and l lie in different connected components of M.

2. (Matroid polytopes) The matroid polytope Γ_M satisfies the equality

$$\sum_{e \in E_1} x_e = r_M(E_1). \tag{4}$$

Since the polytope is cut out by the "cyclic" equalities and inequalities of Proposition 5.6, (4) must be a linear combination of cyclic equalities satisfied by Γ_M; *i.e.*, equalities of the form $\sum_{e \in I} x_e = r_M(I)$ for cyclic intervals I.

If E_1 is not a cyclic interval, then we need at least two cyclic equalities different from $\sum_{e \in [n]} x_i = r(M)$ to obtain (4). Therefore Γ_M satisfies at least three linearly independent equations, and $\dim \Gamma_M \le n-3$. This contradicts the fact [BGW03] that $\dim \Gamma_M = n - c$ where c is the number of connected components of M. □

Definition 7.5. Let S be a partition $[n] = S_1 \sqcup \cdots \sqcup S_t$ of $[n]$ into pairwise disjoint, non-empty subsets. We say that S is a *non-crossing partition* if there are no a, b, c, d in cyclic order such that $a, c \in S_i$ and $b, d \in S_j$ for some $i \ne j$.

Equivalently, place the numbers $1, 2, \ldots, n$ on n vertices around a circle in clockwise order, and then for each S_i, draw a polygon on the corresponding vertices. If no two of these polygons intersect, then S is a non-crossing partition of $[n]$.

Let NC_n be the set of non-crossing partitions of $[n]$.

Theorem 7.6. *Let M be a positroid on $[n]$ and let $S_1, S_2, \ldots, S_t$ be the ground sets of the connected components of M. Then $\Pi_M = \{S_1, \ldots, S_t\}$ is a non-crossing partition of $[n]$, called the* non-crossing partition *of M.*

Conversely, if S_1, S_2, …, S_t form a non-crossing partition of $[n]$ and if M_1, M_2, …, M_t are connected positroids on S_1, S_2, …, S_t, respectively, then $M_1 \oplus \cdots \oplus M_t$ is a positroid.

Proof. To prove the first statement of the theorem, let us suppose that S_1, S_2, …, S_t do not form a non-crossing partition of $[n]$. Then we can find two parts S_a and S_b and $1 \leq i < j < k < l \leq n$ such that $i, k \in S_a$ and $j, l \in S_b$. But then the restriction of M to $S_a \cup S_b$ is the direct sum of two connected positroids where S_a and S_b are not cyclic intervals. This contradicts Proposition 7.4.

We prove the second statement of the theorem by induction on t, the number of parts in the non-crossing partition. Since S_1, …, S_t is a non-crossing partition, we can assume that one of the parts, say S_t, is a cyclic interval in $[n]$. Then S_1, …, S_{t-1} is a non-crossing partition on $[n] - S_t$. By the inductive hypothesis, $M' = M_1 \oplus \cdots \oplus M_{t-1}$ is a positroid on $[n] - S_t$. But now M' and M_t are positroids on $[n] - S_t$ and S_t, which are cyclic intervals of $[n]$. Therefore by Proposition 3.4, $M = M' \oplus M_t$ is a positroid. □

As remarked earlier, the first half of Theorem 7.6 was obtained independently by Nicolas Ford, and will appear in his preprint [For13].

8. A complementary view on positroids and non-crossing partitions

We now give a complementary description of the non-crossing partition of a positroid, as defined by Theorem 7.6. To do that, we need the notion of Kreweras complementation.

Definition 8.1. Let Π be a non-crossing partition of $[n]$. Consider nodes $1, 1', 2, 2', \ldots, n, n'$ in that order around a circle, and draw the partition Π on the labels $1, 2, \ldots, n$. The *Kreweras complement* $K(\Pi)$ is the coarsest (non-crossing) partition of $[n]$ such that when we regard it as a partition $K(\Pi)'$of $1', 2', \ldots, n'$, the partition $\Pi \cup K(\Pi)'$ of $1, 1', 2, 2', \ldots, n, n'$ is non-crossing.

Figure 6 shows an example of Kreweras complementation.

Let M be a rank d positroid on $[n]$ and consider its matroid polytope Γ_M in $\mathbb{R}^n$. Instead of the usual coordinates $x_1, \ldots, x_n$, we use the system of coordinates $y_1, \ldots, y_n$ given by

$$y_i = x_1 + \cdots + x_i \qquad (1 \leq i \leq n).$$

Recall that by Proposition 5.7, the inequality description of Γ_M has the form

$$y_n = d, \qquad y_j - y_i \leq r_{ij} \text{ for } i \neq j$$

where $r_{ij} = r_M([i, j])$. For $i, j \in [n]$, define

$$i \sim^* j \text{ if and only if } y_j - y_i \text{ is constant for } y \in \Gamma_M.$$

This clearly defines an equivalence relation $\sim^*$ on $[n]$. Let Π^*_M be the partition of $[n]$ into equivalence classes of $\sim^*$

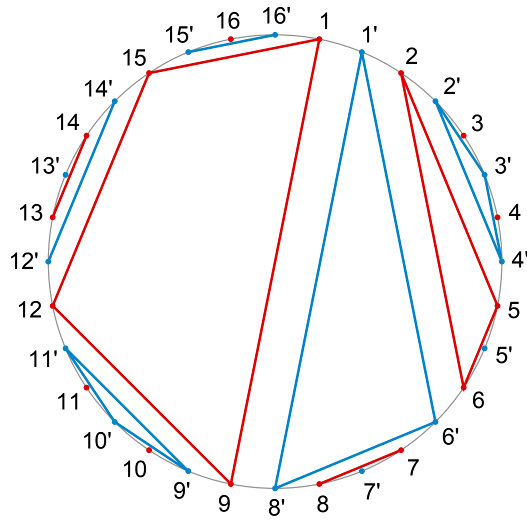


FIGURE 6. The Kreweras complement of the (blue) partition $\{\{1, 9, 12, 15\}, \{2, 5, 6\}, \{3\}, \{4\}, \{7, 8\}, \{10\}, \{11\}, \{13, 14\}, \{16\}\}$ is $\{\{1, 6, 8\}, \{2, 3, 4\}, \{5\}, \{7\}, \{9, 10, 11\}, \{12, 14\}, \{13\}, \{15, 16\}\}$, as shown in red.

Theorem 8.2. *The partition Π_M^* is the Kreweras complement of the non-crossing partition Π_M of M. Consequently, it is also non-crossing.*

Proof. First we prove that $K(\Pi_M)$ is a refinement of Π_M^*. Consider a block S of $K(\Pi_M)$ and two cyclically consecutive elements $i < j$ in S. Since $\Pi_M \cup K(\Pi_M)'$ is non-crossing in $1, 1', \dots, n, n'$, the cyclic interval $[i+1, j]$ of $[n]$ is a disjoint union of blocks $S_1, \dots, S_s$ of Π_M, which are themselves connected components of M. If $1 \leq i < j$ in cyclic order, we have that

$$y_j - y_i = \sum_{a \in [i+1,j]} x_a = \sum_{r=1}^{s} \sum_{a \in S_r} x_a = \sum_{r=1}^{s} r(S_r)$$

is constant in Γ_M, and therefore $i \sim^* j$. A similar computation holds if $i < 1 \leq j$. It follows that $K(\Pi_M)$ is a refinement of Π_M^*.

Now assume that $i \sim^* j$ but i and j are not in the same block of $K(\Pi_M)$. Looking at the non-crossing partition $\Pi_M \cup K(\Pi_M)'$, this means that the edge $i'j'$ (which is not in $K(\Pi_M)$) must cross an edge kl of Π_M. Assume $k \in [i+1, j]$ and $l \notin [i+1, j]$. Now, since $k \sim l$, we can find bases B and B' of M with $B' = B - \{k\} \cup \{l\}$. But then $\sum_{a \in [i+1,j]} x_a$ is not constant on Γ_M: more specifically, the value it takes on the vertex e_B is 1 more than the value it takes on the vertex $e_{B'}$. This contradicts the fact that $i \sim^* j$. $\square$

9. Positroid polytopes and non-crossing partitions

Having explained the role that non-crossing partitions play in the connectivity of positroids, we use that knowledge to show that the face poset of a positroid polytope lives inside the poset of *weighted non-crossing partitions.*

Definition 9.1. A *weighted non-crossing partition* S^w of $[n]$ is a non-crossing partition S of $[n]$, say $[n] = S_1 \sqcup \cdots \sqcup S_t$, together with a weight vector $w = (w_1, \ldots, w_t) \in \mathbb{N}^t$ of integer weights $w_1 = w(S_1), \ldots, w_t = w(S_t)$ with $0 \leq w_i \leq |S_i|$ for $i = 1, \ldots, t$. The *weight* of the partition S^w is $w_1 + \cdots + w_t$.

The set NC_n of non-crossing partitions of $[n]$ is partially ordered by refinement; that poset has many interesting properties and connections to several fields of mathematics. We extend that order to the context of weighted non-crossing partitions.

Definition 9.2. Let NC_n^d be the poset of non-crossing partitions of $[n]$ of weight d, where the cover relation is given by $S^w \lessdot T^v$ if

- $T = \{T_1, \ldots, T_t\}$ and $S = \{T_1, \ldots, T_{h-1}, A, T_h \setminus A, T_{h+1}, \ldots, T_t\}$ for some index $1 \leq h \leq t$ and some proper subset $\emptyset \subsetneq A \subsetneq T_h$, and
- $v(T_h) = w(A) + w(T_h \setminus A)$ and $v(T_j) = w(T_j)$ for all $j \neq h$.

Let $NC_n^d \cup \widehat{0}$ be this poset with an additional minimum element $\widehat{0}$.

The poset NC_n^d is ranked of height $n+1$. It has a unique maximal element $\widehat{1}$ corresponding to the trivial partition of $[n]$ into one part of weight d.

Readers familiar with the *poset Π_n^w of weighted partitions* defined by Dotsenko and Khorsohkin [DK07] and further studied by González and Wachs [DW13] may notice the relationship between these two posets. The subposet of Π_n^w consisting of the non-crossing partitions of weight d is almost equal to NC_n^d; the difference is that $0 \leq w(S_i) \leq |S_i| - 1$ in Π_n^w and $0 \leq w(S_i) \leq |S_i|$ in NC_n^d. In fact for our purposes we only need to allow $w(S_i) = |S_i|$ for $|S_i| = 1$, but this small distinction is important; see Remark 9.4.

Theorem 9.3. *If M is a rank d positroid on $[n]$, then the face poset of the matroid polytope Γ_M is an induced subposet of $NC_n^d \cup \widehat{0}$.*

Proof. By Corollary 5.4, any non-empty face F of the positroid polytope Γ_M is itself a positroid polytope, say $F = \Gamma_N$. Write $N = N_1 \oplus \cdots \oplus N_c$ as a direct sum of its connected components. By Theorem 7.6, the partition $\Pi_F = \{N_1, \ldots, N_c\}$ of $[n]$ is non-crossing. Assign weights $w(N_i) = r_N(N_i)$ for $1 \leq i \leq c$ to the blocks of this partition. Let Π_F^w be the resulting weighted non-crossing partition. Since $r_N(N_1) + \cdots + r_N(N_c) = r(N) = r(M) = d$ we have that $\Pi_F^w \in NC_n^d$. (If F is the empty face let $\Pi_F^w = \widehat{0}$.) We claim that

$$F \mapsto \Pi_F^w$$

is the desired embedding.

Figure 7 shows the positroid polytope Γ_M for the positroid M whose bases are $\{12, 13, 14, 23, 24\}$. It is a square pyramid. It also shows the face poset of Γ_M, with each face labeled with the corresponding weighted non-crossing partition of $[4]$.

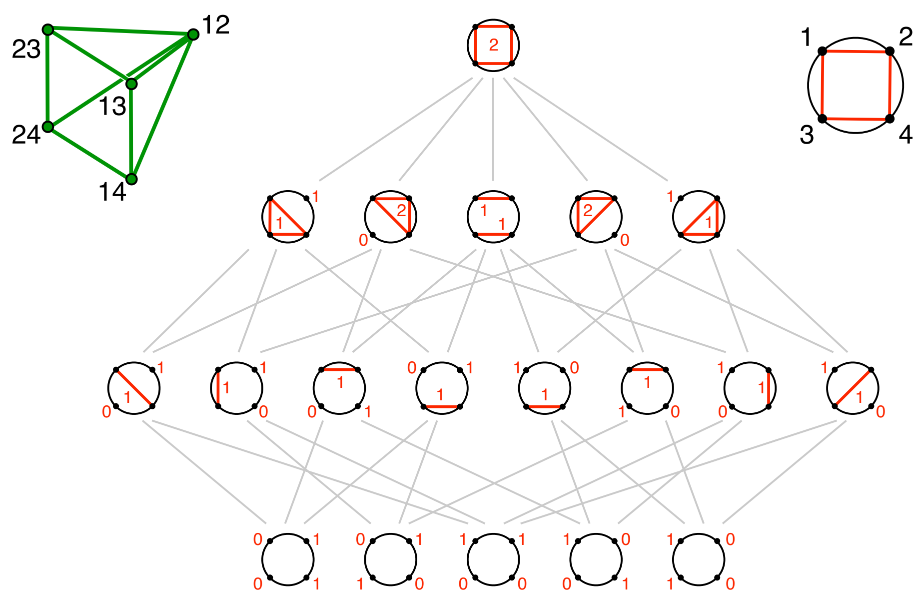


FIGURE 7. The face poset of the square pyramid inside NC_4^2.

First we show that this mapping is one-to-one. Suppose we know Π_F^w and we wish to recover F. Since F is a face of Γ_M, it satisfies the same inequalities as Γ_M, and some additional equalities. If $F = \Gamma_N$, the equalities that it satisfies are $\sum_{i \in N_j} x_i = r_N(N_j)$ for $j = 1, \dots, c$ and their linear combinations. But we know the N_js and the $r_N(N_j)$s from Π_F^w, so we can recover F as the intersection of Γ_M with these c hyperplanes.

Now we show that the mapping is order-preserving. Assume that $F \gtrdot G$ are faces of Γ_M; say $F = \Gamma_K$ and $G = \Gamma_L$ for positroids K and L. Let $K = K_1 \oplus \cdots \oplus K_c$ be the decomposition of K into connected components. Then $\dim F = n - c$ implies $\dim G = \dim F - 1 = n - c - 1$. By Proposition 5.3, the decomposition of L into connected components must then be of the form $L = K_1 \oplus \cdots \oplus K_{h-1} \oplus (K_h|A) \oplus (K_h/A) \oplus K_{h+1} \oplus \cdots \oplus K_c$ for some $1 \leq h \leq c$ and some proper subset $A \subset K_h$. Therefore $\Pi_G \lessdot \Pi_F$ in NC_n. Furthermore, since K_j has the same weight in Π_F and Π_G for all $j \neq i$ and $r(K) = r(L) = d$, the weight $r_K(K_h)$ in Π_F must equal the sum $r_L(K_h|A) + r_L(K_h/A)$ of weights in Π_G. Therefore $\Pi_G^w \lessdot \Pi_F^w$ in NC_n^d.

Finally, to show that the face poset of Γ_M is embedded as an **induced** subposet of NC_n^d, assume that $\Pi_G^w \lessdot \Pi_F^w$ for some faces F and G of Γ_M. We need to show that $G \lessdot F$. Again let $F = \Gamma_K$ and $G = \Gamma_L$, and let $K_1, \dots, K_c$ be the components of K. The components of L must be $K_1, \ldots, K_{h-1}, A, K_h \setminus A, K_{h+1}, \ldots, K_c$ for some $1 \leq h \leq c$ and some proper subset $A \subset K_h$, and we must have $r_L(A) + r_L(K_h \setminus A) = r_K(K_h)$. Now, the equalities that determine the face F as a subset of Γ_M are $\sum_{i \in K_j} x_i = r_K(K_j)$ for $j = 1, \ldots, c$. The face G is cut out of Γ_M by the same equalities, together with the two new equalities $\sum_{i \in A} x_i = r(A)$ and $\sum_{i \in K_h \setminus A} x_i = r(K_h \setminus A)$. In principle we should have removed the old equality $\sum_{i \in K_h} x_i = r(K_h)$ from the defining equalities for G, but we may keep it since it is a consequence of the two new equalities. So in total, G satisfies one additional equality which is linearly independent from those of F. It follows that $F \gtrdot G$. $\square$

Remark 9.4. In the correspondence above, the weight of a block N_i in a non-crossing partition Π_F^w is $w(N_i) = r_N(N_i)$. If we had $r_N(N_i) = |N_i|$, then N_i would consist solely of coloops. Since N_i is connected, we must have $|N_i| \in \{0, 1\}$. However, singletons may have weight equal to 0 or 1. This is the only reason why, in Π_n^d, we need to allow a block of size k to have weight k, instead of following [DK07, DW13].

As mentioned earlier, the poset $NC_n^d \cup \widehat{0}$ is ranked of height $n+1$. The face poset of any connected positroid polytope Γ_M of rank d on $[n]$ is also ranked of height $n+1$.

For each such positroid M, the order complex $\Delta(\Gamma_M \setminus \{\widehat{0}, \widehat{1}\})$ can be identified with the barycentric subdivision of the polytope Γ_M, so it is homeomorphic to an $(n-2)$-sphere. The interaction of these different $(n-2)$-spheres inside the order complex $\Delta(NC_n^d \setminus \{\widehat{1}\})$ is the subject of an upcoming project.

10. Enumeration of connected positroids

In this section we use Theorem 7.6, together with a result of the third author [Wil05], to enumerate connected positroids.

Many combinatorial objects (such as graphs or matroids) on a set $[n]$ decompose uniquely into connected components $S_1, \ldots, S_k$, where the partition $[n] = S_1 \sqcup \cdots \sqcup S_k$ has no additional structure. In that case, the Exponential Formula [Sta99, Theorem 5.1.3] tells us that the exponential generating functions $E_t(x)$ and $E_c(x)$ for the total number of objects and the total number of connected objects are related by the formula $E_c(x) = \log E_t(x)$.

In our situation, where the connected components of a positroid form a non-crossing partition, we need the following "non-crossing" analog of the Exponential Formula:

Theorem 10.1. [Spe94] *Let K be a field. Given a function $f : \mathbb{Z}_{>0} \to K$ define a new function $h : \mathbb{Z}_{>0} \to K$ by*

$$h(n) = \sum_{\{S_1, \ldots, S_k\} \in NC_n} f(\#S_1) f(\#S_2) \cdots f(\#S_k), \tag{5}$$

where the sum is over all non-crossing partitions of $[n]$*. Let* $F(x) = 1 + \sum_{n\geq 1} f(n)x^n$ *and* $H(x) = 1 + \sum_{n\geq 1} h(n)x^n$*. Then*

$$xH(x) = \left(\frac{x}{F(x)}\right)^{\langle -1\rangle},$$

where $G(x)^{\langle -1\rangle}$ *denotes the compositional inverse of* $G(x)$*.*

Definition 10.2. Let $p(n)$ be the number of positroids on $[n]$ and $p_c(n)$ be the number of connected positroids on $[n]$. Let

$$P(x) = 1 + \sum_{n\geq 1} p(n)x^n \quad \text{and} \quad P_c(x) = 1 + \sum_{n\geq 1} p_c(n)x^n.$$

Corollary 10.3. *The generating functions for positroids and connected positroids satisfy:*

$$xP(x) = \left(\frac{x}{P_c(x)}\right)^{\langle -1\rangle}.$$

Proof. Theorem 7.6 implies that

$$p(n) = \sum_{\{S_1,\dots,S_k\}\in NC_n} p_c(\#S_1)p_c(\#S_2)\cdots p_c(\#S_k)$$

and Theorem 10.1 then gives the desired result. □

Theorem 10.4. *We have*

$$P(x) = \sum_{k\geq 0} k!\frac{x^k}{(1-x)^{k+1}}, \qquad p(n) = \sum_{k=0}^{n} \frac{n!}{k!}, \qquad \lim_{n\to\infty} \frac{p(n)}{n!} = e.$$

Proof. In [Wil05], Williams gave a finer enumeration of positroids in terms of the size of the ground set, the rank, and the dimension of the positroid cell. The first equality follows from [Wil05, Prop. 5.11] by setting $q = y = 1$. This easily implies the second equality, which implies the third. □

The following formula also follows easily from the above:

Proposition 10.5. [Pos, Prop. 23.2] *The exponential generating function for* $p(n)$ *is*

$$1 + \sum_{n\geq 1} p(n)\frac{x^n}{n!} = \frac{e^x}{1-x}.$$

The sequence $\{p(n)\}_{n\geq 1}$ is entry A000522 in Sloane's Encyclopedia of Integer Sequences [Slo94]. The first few terms are $2, 5, 16, 65, 326, 1957, 13700, \dots$.

Definition 10.6. [Cal04] A *stabilized-interval-free* (SIF) permutation π of $[n]$ is a permutation which does not stabilize any proper interval of $[n]$; that is, $\pi(I) \neq I$ for all intervals $I \subsetneq [n]$.

Theorem 10.7. *For* $n \geq 2$*, the number of connected positroids on* $[n]$ *equals the number of SIF permutations on* $[n]$*.*

Proof. To prove Theorem 10.7, we will use plabic graphs to show that our bijection between positroids and decorated permutations restricts to a bijection from connected positroids to SIF permutations.

First assume that the positroid M is disconnected. By Proposition 3.4 and Proposition 3.5, M can be decomposed into a direct sum of two positroids M_1 and M_2. By Lemma 3.3 and Proposition 7.4, we may assume without loss of generality that the ground set of M_1 is $[1, \ell]$ and the ground set of M_2 is $[\ell+1, n]$ for some $1 \leq \ell \leq n-1$.

By Definition 4.10 there exists a reduced plabic graph G_1 encoding M_1 (whose boundary vertices are $b_1, \dots, b_\ell$) and also a reduced plabic graph G_2 encoding M_2 (whose boundary vertices are $b_{\ell+1}, \dots, b_n$). The trip permutation π_{G_1} is a decorated permutation on the letters $[1, \ell]$ and the trip permutation π_{G_2} is a decorated permutation on the letters $[\ell+1, n]$.

Now let G be the plabic graph with boundary vertices $b_1, \dots, b_n$ which is obtained by combining G_1 and G_2 in the obvious way. In particular G is disconnected: there is no path between any vertex in $\{b_1, \dots, b_\ell\}$ and any vertex in $\{b_{\ell+1}, \dots, b_n\}$. Moreover if we delete the vertices $b_1, \dots, b_\ell$ and all vertices connected to them, we obtain G_2, and if we delete the vertices $b_{\ell+1}, \dots, b_n$, we obtain G_1, so G naturally has two subgraphs isomorphic to G_1 and G_2. It follows immediately from any characterization of reduced plabic graph (see [Pos, Section 12]) that G is also reduced. Additionally G is perfectly orientable, and its perfect orientations are obtained by choosing a perfect orientation for G_1 together with a perfect orientation for G_2. We have $M_G = M$; this follows easily from the definition of direct sum of matroids and Proposition 6.1.

Let $\pi := \pi_{G_1} \sqcup \pi_{G_2}$ be the decorated permutation on $[n]$ whose restriction to $[1, \ell]$ and $[\ell+1, n]$ is π_{G_1} and π_{G_2}, respectively. Clearly π is not SIF. We claim that π is the decorated permutation associated to M. This follows from Definition 4.11, the fact that G is reduced, and the fact that $M_G = M$.

We have thus shown if M is a disconnected positroid, then its associated decorated permutation is not SIF.

To prove the converse, we can reverse all of the steps above. Namely suppose π is not SIF. We can assume that it fixes two intervals $[1, \ell]$ and $[\ell+1, n]$. Let π_1 and π_2 be the restrictions of π to these two intervals. Then we construct reduced plabic graphs G_1 and G_2 whose trip permutations are π_1 and π_2 respectively. We combine G_1 and G_2 into a reduced plabic graph G in the obvious way. The trip permutation π_G equals π. It is then clear that $M_G = M_{G_1} \oplus M_{G_2}$, and the decorated permutation associated to M_G is π. Therefore if we start with a decorated permutation π which is not SIF, the corresponding positroid is disconnected. $\square$

Theorem 10.8. *The number $p_c(n)$ of connected positroids on $[n]$ satisfies*

$$\begin{aligned} p_c(n) &= \frac{[x^n]P(x)^{1-n}}{1-n}, \\ p_c(n) &= (n-1)p_c(n-1) + \sum_{j=2}^{n-2}(j-1)p_c(j)p_c(n-j) \text{ for } n \geq 2, \text{ and} \\ \lim_{n\to\infty} \frac{p_c(n)}{n!} &= \frac{1}{e}. \end{aligned}$$

Proof. The first statement follows by applying the Lagrange inversion formula [Sta99, Theorem 5.4.2] to $F(x) = x/P_c(x)$ and $F^{\langle -1 \rangle}(x) = xP(x)$, which says:

$$\begin{aligned} m[x^m]\left(F^{\langle -1\rangle}(x)\right)^k &= k[x^{-k}]F(x)^{-m} \\ m[x^m]\,(xP(x))^k &= k[x^{-k}]\left(\frac{P_c(x)}{x}\right)^m \\ m[x^{m-k}]P(x)^k &= k[x^{m-k}]P_c(x)^m. \end{aligned}$$

It remains to set $m = 1$ and $k = 1 - n$.

In view of Theorem 10.7, the second statement is derived in [Cal04] and the third is a consequence of [ST09, Cor. 11]. □

The sequence $\{p_c(n)\}_{n\geq 1}$ is, except for the first term, equal to entry A075834 in Sloane's Encyclopedia of Integer Sequences [Slo94]. The first few terms are $2, 1, 2, 7, 34, 206, 1476, \ldots$

We conclude the following.

Theorem 10.9. *If $p(n)$ is the number of positroids on $[n]$ and $p_c(n)$ is the number of connected positroids on $[n]$, then*

$$\lim_{n\to\infty} \frac{p_c(n)}{p(n)} = \frac{1}{e^2} \approx 0.1353.$$

Proof. This is an immediate consequence of Theorems 10.4 and 10.8. □

This result is somewhat surprising in view of the conjecture that most matroids are connected:

Conjecture 10.10. *(Mayhew, Newman, Welsh, Whittle, [MNWW11]) If $m(n)$ is the number of matroids on $[n]$ and $m_c(n)$ is the number of connected matroids on $[n]$, then*

$$\lim_{n\to\infty} \frac{m_c(n)}{m(n)} = 1.$$

Theorem 10.9 should **not** be seen as evidence against Conjecture 10.10. Positroids possess strong structural properties that are quite specific to them. Furthermore, they are a relatively small family of matroids: compare the estimate $\log p(n) \approx n!e$ to the estimate $\log\log m(n) = n - \frac{3}{2}\log n + O(1)$ due to Knuth [Knu74], and Bansal, Pendavingh, and van der Pol [BPvdP12].

11. Positroids and free probability

The results of the previous section have an interesting connection with Voiculescu's theory of *free probability*. We give a very brief overview of the aspects of the theory which are relevant to our discussion; for a more thorough introduction, we recommend Speicher's excellent survey [Spe94].

The concept of *freeness* can be thought of as a "non-commutative analogue" to the classical notion of independence in probability. The role played by independence, moments, cumulants, and partitions in classical probability is now played by freeness, moments, free cumulants, and non-crossing partitions in free probability, as we now explain.

Given a real-valued random variable X with probability distribution $\mu(x)$, the moments of X are the expected values of the powers of X: the nth moment is $m_n(X) = E(X^n)$ for $n \geq 1$. (We assume for the rest of this discussion that all moments exist.) The *moment generating function*

$$M_X(t) = E(e^{tX}) = \sum_{n\geq 0} m_n(X)\frac{t^n}{n!}$$

is essentially the same as the Fourier transform of μ. The *cumulants* of X are the coefficients of the generating function:

$$\log M_X(t) = \sum_{n\geq 0} c_n(X)\frac{t^n}{n!}.$$

The independence of random variables X and Y translates into a linear relation of cumulants. Since expectation is multiplicative on independent variables, we have that $M_{X+Y}(t) = M_X(t)M_Y(t)$ when X and Y are independent, so

$$X, Y \text{ independent} \Rightarrow c_n(X+Y) = c_n(X) + c_n(Y) \text{ for all } n \geq 1.$$

In the non-commutative setting, our "random variables" are simply elements of a unital algebra $\mathcal{A}$ which is not necessarily commutative. Our "expectation" E is just a linear function $E : \mathcal{A} \to \mathbb{C}$ with $E(1) = 1$. We say that random variables X and Y are *free* if, for any polynomials $p_1, q_1, \ldots, p_k, q_k$,

$$E(p_i(X)) = E(q_j(Y)) = 0 \text{ for all } i, j \;\Rightarrow\; E(p_1(X)q_1(Y)\cdots p_k(X)q_k(Y)) = 0.$$

Again, the freeness of X and Y manifests linearly in terms of the *free cumulants*, which are the numbers $k_1, k_2, \ldots$ such that

$$m_n = \sum_{\{S_1,\ldots,S_k\}\in NC_n} k_{\#S_1}k_{\#S_2}\cdots k_{\#S_k} \tag{6}$$

for all n. While the formula for the moments of $X + Y$ is quite intricate, the free cumulants are related beautifully by:

$$X, Y \text{ free } \Rightarrow k_n(X+Y) = k_n(X) + k_n(Y) \text{ for all } n \geq 1.$$

There is also a remarkable formula for the free cumulants of $X \cdot Y$ [Spe94].

With all the necessary background in place, we can now establish a simple connection between free probability and positroids. Let $\mathrm{Exp}(\lambda)$ be an exponential random variable with rate parameter λ.

Theorem 11.1. *The moments of the random variable $Y \sim 1 + Exp(1)$ are*

$$m_n(Y) = \#\ \textit{positroids on } [n],$$

and its free cumulants are

$$k_n(Y) = \#\ \textit{connected positroids on } [n].$$

Proof. Using the fact that $M_{A+B}(t) = M_A(t)M_B(t)$ for independent random variables A and B, and that $M_{\mathrm{Exp}(\lambda)} = 1/(1-\frac{t}{\lambda})$, it follows that the moment generating function of Y is

$$M_Y(t) = M_{1+\mathrm{Exp}(1)}(t) = M_1(t)M_{\mathrm{Exp}(1)}(t) = e^t \cdot \frac{1}{1-t}.$$

Comparing with Proposition 10.5 gives the first formula. The second follows by combining Corollary 10.3 with the relation (6) between the moments and the free cumulants. □

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