Combinatorial Aspects of Total Positivity

by

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Abstract

In this thesis I study combinatorial aspects of an emerging field known as total positivity. The classical theory of total positivity concerns matrices in which all minors are nonnegative. While this theory was pioneered by Gantmacher, Krein, and Schoenberg in the 1930s, the past decade has seen a flurry of research in this area initiated by Lusztig. Motivated by surprising positivity properties of his canonical bases for quantum groups, Lusztig extended the theory of total positivity to arbitrary reductive groups and real flag varieties. In the first part of my thesis I study the totally nonnegative part of the Grassmannian and prove an enumeration theorem for a natural cell decomposition of it. This result leads to a new q-analog of the Eulerian numbers, which interpolates between the binomial coefficients, the Eulerian numbers, and the Narayana numbers. In the second part of my thesis I introduce the totally positive part of a tropical variety, and study this object in the case of the Grassmannian. I conjecture a tight relation between positive tropical varieties and the cluster algebras of Fomin and Zelevinsky, proving the conjecture in the case of the Grassmannian. The third and fourth parts of my thesis explore a notion of total positivity for oriented matroids. Namely, I introduce the positive Bergman complex of an oriented matroid, which is a matroidal analogue of a positive tropical variety. I prove that this object is homeomorphic to a ball, and relate it to the Las Vergnas face lattice of an oriented matroid. When the matroid is the matroid of a Coxeter arrangement, I relate the positive Bergman complex and the Bergman complex to the corresponding graph associahedron and the nested set complex.

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Chapter 1

Introduction

In this thesis we study combinatorial aspects of an emerging field known as total positivity, as well as its relations to tropical geometry and cluster algebras. The classical theory of total positivity concerns matrices in which all minors are nonnegative. While this theory was pioneered by Gantmacher, Krein, and Schoenberg in the 1930s, the past decade has seen a flurry of research in this area initiated by Lusztig [30, 29, 31]. Motivated by surprising positivity properties of his canonical bases for quantum groups, Lusztig extended the theory of total positivity by introducing the totally nonnegative variety $G_{\geq 0}$ in an arbitrary reductive group G and the totally nonnegative part $B_{\geq 0}$ of a real flag variety B, which he refers to as a "miraculous polyhedral subspace" [29]. This thesis concerns combinatorial aspects of the theory of total positivity, as well as its relations to tropical geometry and cluster algebras.

Tropical algebraic geometry is the geometry of the tropical semiring $(\mathbb{R}, \min, +)$. Its objects are polyhedral cell complexes which behave like complex algebraic varieties. Although this is a young field in which many basic questions have not yet been addressed [35], tropical geometry has already been shown to have applications to enumerative geometry, and connections to representation theory.

Cluster algebras are commutative algebras endowed with a certain combinatorial structure, which were introduced by Fomin and Zelevinsky in [21]. Though they were introduced a mere five years ago, it is already clear that cluster algebras have connections to total positivity, canonical bases, hyperbolic geometry, and quiver representations. Remarkably, the classification of the cluster algebras of "finite type" turns out to be identical to the *Cartan-Killing classification* of semisimple Lie algebras and finite root systems [22].

This thesis is divided into four chapters, which are based on the papers [48, 41, 3, 4]. We have included a more detailed introduction at the beginning of each chapter to outline some of the background material and outline the goals of the chapter.

The first project we are concerned with is the study of the poset of cells of Postnikov's [34] cell decomposition of the totally nonnegative part of the Grassmannian Gr_{kn}^+. This poset is very interesting because it has many different combinatorial descriptions, for example, in terms of certain tableaux, in terms of certain permutations, and in terms of the MacPhersonian. See Figures A-1 and A-2 for depictions of the poset of cells of Gr_{24}^+ in terms of these tableaux and permutations. Our first main result is an explicit formula for the rank generating function for the poset of cells of Gr_{kn}^+. One corollary of this theorem is a new proof that the Euler characteristic of Gr_{kn}^+ is 1. Additionally, this theorem leads to a new q-analog of the Eulerian numbers $E_{k,n}(q)$, which specializes to the binomial coefficients, Narayana numbers, and the Eulerian numbers.

Chapter 3 explores a link between totally positivity and tropical geometry. Specifically, we introduce the *totally positive part of the tropicalization of an arbitrary affine variety*, an object which has the structure of a polyhedral fan. We then investigate the case of the Grassmannian, denoting the resulting fan $\mathrm{Trop}^+ Gr_{k,n}$. We show that $\mathrm{Trop}^+ Gr_{2,n}$ is combinatorially the fan dual to the (type A_n) associahedron, and that $\mathrm{Trop}^+ Gr_{3,6}$ and $\mathrm{Trop}^+ Gr_{3,7}$ are essentially the fans dual to the types D_4 and E_6 associahedra. These results are strikingly reminiscent of the fact that the Grassmannian's cluster algebra structure is of types A_{n-3}, D_4, and E_6 for $Gr_{2,n}$, $Gr_{3,6}$, and $Gr_{3,7}$. Finally, we conjecture a tight relation between the combinatorial structure of a cluster algebra $\mathcal{A}$ and the combinatorial structure of $\mathrm{Trop}^+(\mathrm{Spec}\,\mathcal{A})$. This chapter is joint work with David Speyer [41].

Chapter 4 introduces a notion of total positivity for oriented matroids. Specifically, the *Bergman complex of a matroid* is a polyhedral complex which generalizes to matroids the notion of a tropical variety. Sturmfels introduced the Bergman complex $\mathcal{B}(M)$ of an arbitrary matroid M [46], and Ardila and Klivans [2] described the geometry of $\mathcal{B}(M)$: they showed that, appropriately subdivided, the Bergman complex of a matroid M is the order complex of the proper part of the lattice of flats L_M of the matroid; this implies that $\mathcal{B}(M)$ is homotopy equivalent to a wedge of spheres. In this chapter we define the *positive Bergman complex* $\mathcal{B}^+(M)$ of an oriented matroid M, in order to generalize to oriented matroids the notion of the totally positive part of a tropical variety. We also prove that, appropriately subdivided, $\mathcal{B}^+(M)$ is the order complex of the proper part of the Las Vergnas face lattice of M; it follows that $\mathcal{B}^+(M)$ is homeomorphic to a sphere. We conclude by showing that if M is the matroid of the complete graph, then $\mathcal{B}^+(M)$ is dual to the face poset of the associahedron. This chapter is joint work with Federico Ardila and Carly Klivans [3].

Chapter 5 is a continuation of the work begun in Chapter 4. In this chapter we relate the positive Bergman complex and Bergman complex of (the oriented matroid of) a Coxeter arrangement to *graph associahedra* and *nested set complexes*. Graph associahedra are polytopes generalizing the associahedron that were independently discovered in the past year by Carr and Devadoss [10] and Postnikov [33]; these polytopes have connections to the real moduli space of n-punctured Riemann spheres. The nested set complex of an arrangement encodes the combinatorics of its De Concini-Procesi wonderful model, as well as the combinatorics of resolutions of singularities in toric varieties. In our work we prove that the Bergman complex of a Coxeter arrangement $\mathcal{A}$ of type Φ is equal to the nested set complex of type Φ, and the positive Bergman complex of $\mathcal{A}$ is dual to the graph associahedron of type Φ. This chapter is joint work with Federico Ardila and Victor Reiner [4].

Chapter 2

Enumeration of totally positive Grassmann cells

2.1 Introduction

The theory of total positivity dates back to the 1930s, when Gantmacher, Krein, and Schoenberg studied matrices in which all minors are nonnegative. However, the last decade has seen a great deal of developments in this area initiated by Lusztig [30, 29, 31]. Motivated by surprising connections he discovered between his theory of canonical bases for quantum groups and the theory of total positivity, Lusztig extended this subject by introducing the totally nonnegative variety $G_{\geq 0}$ in an arbitrary reductive group G and the totally nonnegative part $B_{\geq 0}$ of a real flag variety B. A few years later, Fomin and Zelevinsky [19] advanced the understanding of $G_{\geq 0}$ by studying the decomposition of G into double Bruhat cells, and Rietsch [36] proved Lusztig's conjectural cell decomposition of $B_{\geq 0}$. Most recently, Postnikov [34] investigated the combinatorics of the totally nonnegative part of a Grassmannian $Gr_{k,n}^+$: he established a relationship between $Gr_{k,n}^+$ and planar oriented networks, producing a combinatorially explicit cell decomposition of $Gr_{k,n}^+$. In this chapter we continue Postnikov's study of the combinatorics of $Gr_{k,n}^+$: in particular, we enumerate the cells in the cell decomposition of $Gr_{k,n}^+$ according to their dimension.

The *totally nonnegative part* of the Grassmannian of k-dimensional subspaces in

$\mathbb{R}^n$ is defined to be the quotient $Gr^+_{k,n} = \mathrm{GL}_k^+ \setminus \mathrm{Mat}^+(k,n)$, where $\mathrm{Mat}^+(k,n)$ is the space of real $k \times n$-matrices of rank k with nonnegative maximal minors and GL_k^+ is the group of real matrices with positive determinant. If we specify which maximal minors are strictly positive and which are equal to zero, we obtain a cellular decomposition of $Gr^+_{k,n}$, as shown in [34]. We refer to the cells in this decomposition as *totally positive cells*. The set of totally positive cells naturally has the structure of a graded poset: we say that one cell covers another if the closure of the first cell contains the second, and the rank function is the dimension of each cell.

Lusztig [30] has proved that the totally nonnegative part of the (full) flag variety is contractible, which implies the same result for any partial flag variety. (We thank K. Rietsch for pointing this out to us.) The topology of the individual cells is not well understood, however. Postnikov [34] has conjectured that the closure of each cell in $Gr^+_{k,n}$ is homeomorphic to a closed ball.

In [34], Postnikov constructed many different combinatorial objects which are in one-to-one correspondence with the totally positive Grassmann cells (these objects thereby inherit the structure of a graded poset). Some of these objects include decorated permutations, Ⅎ-diagrams, positive oriented matroids, and move-equivalence classes of planar oriented networks. Because it is simple to compute the rank of a particular Ⅎ-diagram or decorated permutation, we will restrict our attention to these two classes of objects.

The main result of this chapter is an explicit formula for the *rank generating function* $A_{k,n}(q)$ of $Gr^+_{k,n}$. Specifically, $A_{k,n}(q)$ is defined to be the polynomial in q whose q^r coefficient is the number of totally positive cells in $Gr^+_{k,n}$ which have dimension r. As a corollary of our main result, we give a new proof that the Euler characteristic of $Gr^+_{k,n}$ is 1.

Additionally, using our result and exploiting the connection between totally positive cells and permutations, we find a simple expression for a polynomial $\hat{E}_{k,n}(q)$ which enumerates (regular) permutations according to weak excedences and alignments. This polynomial $\hat{E}_{k,n}(q)$ is a new q-analog of the Eulerian numbers which has many interesting combinatorial properties. For example, when we evaluate $\hat{E}_{k,n}(q)$

at $q = -1, 0$, and 1, we obtain the binomial coefficients, the Narayana numbers, and the Eulerian numbers. Recent work of S. Corteel [13] has shown that $\hat{E}_{k,n}(q)$ has yet another interpretation: it enumerates permutations according to descents and occurrences of the generalized pattern $13-2$. (This result was conjectured by the author and E. Steingrimsson.) Finally, the connection with the Narayana numbers suggests a way of incorporating noncrossing partitions into a larger family of "crossing" partitions.

Let us fix some notation. Throughout this chapter we use $[i]$ to denote the q-analog of i, that is, $[i] = 1 + q + \cdots + q^{i-1}$. (We will sometimes use $[n]$ to refer to the set $\{1, \ldots, n\}$, but the context should make our meaning clear.) Additionally, $[i]! := \prod_{k=1}^{i} [k]$ and $\begin{bmatrix} i \\ j \end{bmatrix} := \frac{[i]!}{[j]![i-j]!}$ are the q-analogs of $i!$ and $\binom{i}{j}$, respectively.

2.2 ⅃-Diagrams

A *partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a weakly decreasing sequence of nonnegative numbers. For a partition λ, where $\sum \lambda_i = n$, the *Young diagram* Y_λ of shape λ is a left-justified diagram of n boxes, with λ_i boxes in the ith row. Figure 2-1 shows a Young diagram of shape $(4, 2, 1)$.

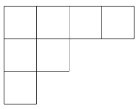


Figure 2-1: A Young diagram of shape $(4, 2, 1)$

Fix k and n. Then a ⅃-*diagram* $(\lambda, D)_{k,n}$ is a partition λ contained in a $k \times (n-k)$ rectangle (which we will denote by $(n-k)^k$), together with a filling $D : Y_\lambda \to \{0, 1\}$ which has the ⅃-*property*: there is no 0 which has a 1 above it and a 1 to its left. (Here, "above" means above and in the same column, and "to its left" means to the left and in the same row.) In Figure 2-2 we give an example of a ⅃-diagram. [1]

[1] The symbol ⅃ is meant to remind the reader of the shape of the forbidden pattern, and should be pronounced as [le], because of its relationship to the letter L. See [34] for some interesting numerological remarks on this symbol.

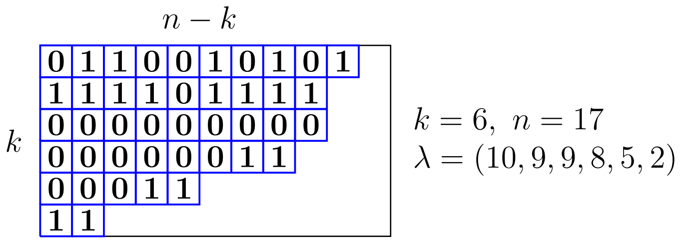


Figure 2-2: A ⅃-diagram $(\lambda, D)_{k,n}$

We define the rank of $(\lambda, D)_{k,n}$ to be the number of 1's in the filling D. Postnikov proved that there is a one-to-one correspondence between ⅃-diagrams (λ, D) contained in $(n-k)^k$, and totally positive cells in $Gr^+_{k,n}$, such that the dimension of a totally positive cell is equal to the rank of the corresponding ⅃-diagram. He proved this by providing a modified Gram-Schmidt algorithm A, which has the property that it maps a real $k \times n$ matrix of rank k with nonnegative maximal minors to another matrix whose entries are all positive or 0, which has the ⅃-property. In brief, the bijection between totally positive cells and ⅃-diagrams maps a matrix M (representing some totally positive cell) to a ⅃-diagram whose 1's represent the positive entries of $A(M)$. Figure A-1 shows the poset of cells of $Gr^+_{2,4}$ in terms of ⅃-diagrams.

Because of the correspondence between cells and ⅃-diagrams, in order to compute $A_{k,n}(q)$, we need to enumerate ⅃-diagrams contained in $(n-k)^k$ according to their number of 1's.

2.3 Decorated Permutations and the Cyclic Bruhat Order

The poset of decorated permutations (also called the cyclic Bruhat order) was introduced by Postnikov in [34]. A *decorated permutation* $\tilde{\pi} = (\pi, d)$ is a permutation π in the symmetric group S_n together with a coloring (decoration) d of its fixed points $\pi(i) = i$ by two colors. Usually we refer to these two colors as "clockwise" and "counterclockwise," for reasons which the next paragraph will make clear.

We represent a decorated permutation $\tilde{\pi} = (\pi, D)$, where $\pi \in S_n$, by its *chord diagram*, constructed as follows. Put n equally spaced points around a circle, and label these points from 1 to n in clockwise order. If $\pi(i) = j$ then this is represented as a directed arrow, or chord, from i to j. If $\pi(i) = i$ then we draw a chord from i to i (i.e. a loop), and orient it either clockwise or counterclockwise, according to d. We refer to the chord which begins at position i as $\text{Chord}(i)$, and we use ij to denote the directed chord from i to j. Also, if $i, j \in \{1, \ldots, n\}$, we use $\text{Arc}(i, j)$ to denote the set of points that we would encounter if we were to travel clockwise from i to j, including i and j.

For example, the decorated permutation $(3, 1, 5, 4, 8, 6, 7, 2)$ (written in list notation) with the fixed points 4, 6, and 7 colored in counterclockwise, clockwise, and counterclockwise, respectively, is represented by the chord diagram in Figure 2-3.

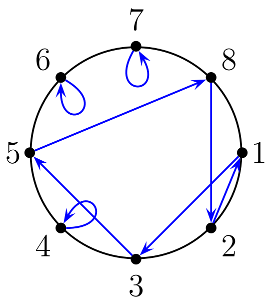


Figure 2-3: A chord diagram for a decorated permutation

The symmetric group S_n acts on the permutations in S_n by conjugation. This action naturally extends to an action of S_n on decorated permutations, if we specify that the action of S_n sends a clockwise (respectively, counterclockwise) fixed point to a clockwise (respectively, counterclockwise) fixed point.

We say that a pair of chords in a chord diagram forms a *crossing* if they intersect inside the circle or on its boundary.

Every crossing looks like Figure 2-4, where the point A may coincide with the point B, and the point C may coincide with the point D. A crossing is called a *simple crossing* if there are no other chords that go from $\text{Arc}(C, A)$ to $\text{Arc}(B, D)$. Say that two chords are *crossing* if they form a crossing.

Let us also say that a pair of chords in a chord diagram forms an *alignment* if they are not crossing and they are relatively located as in Figure 2-5. Here, again, the

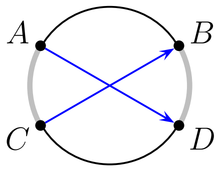


Figure 2-4: A crossing

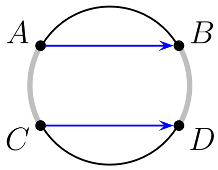


Figure 2-5: An alignment

point A may coincide with the point B, and the point C may coincide with the point D. If A coincides with B then the chord from A to B should be a counterclockwise loop in order to be considered an alignment with $\mathrm{Chord}(C)$. (Imagine what would happen if we had a piece of string pointing from A to B, and then we moved the point B to A). And if C coincides with D then the chord from C to D should be a clockwise loop in order to be considered an alignment with $\mathrm{Chord}(A)$. As before, an alignment is a *simple alignment* if there are no other chords that go from $\mathrm{Arc}(C, A)$ to $\mathrm{Arc}(B, D)$. We say that two chords are *aligned* if they form an alignment.

We now define a partial order on the set of decorated permutations. For two decorated permutations π_1 and π_2 of the same size n, we say that π_1 *covers* π_2, and write $\pi_1 \to \pi_2$, if the chord diagram of π_1 contains a pair of chords that forms a simple crossing and the chord diagram of π_2 is obtained by changing them to the pair of chords that forms a simple alignment (see Figure 2-6). If the points A and B happen to coincide then the chord from A to B in the chord diagram of π_2 degenerates to a counterclockwise loop. And if the points C and D coincide then the chord from C to D in the chord diagram of π_2 becomes a clockwise loop. These degenerate situations are illustrated in Figure 2-7.

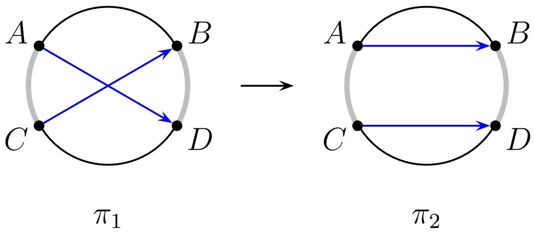


Figure 2-6: Covering relation

Let us define two statistics A and K on decorated permutations. For a decorated permutation π, the numbers $A(\pi)$ and $K(\pi)$ are given by

$$A(\pi) = \#\{\text{pairs of chords forming an alignment}\},$$

$$K(\pi) = \#\{i \mid \pi(i) > i\} + \#\{\text{counterclockwise loops}\}.$$

In our previous example $\pi = (3, 1, 5, 4, 8, 6, 7, 2)$ we have $A = 11$ and $K = 5$. The 11 alignments in π are $(13, 66)$, $(21, 35)$, $(21, 58)$, $(21, 44)$, $(21, 77)$, $(35, 44)$, $(35, 66)$, $(44, 66)$, $(58, 77)$, $(66, 77)$, $(66, 82)$.

Lemma 2.3.1. *[34] If π_1 covers π_2 then $A(\pi_1) = A(\pi_2) - 1$ and $K(\pi_1) = K(\pi_2)$.*

Note that if π_1 covers π_2 then the number of crossings in π_1 is greater then the number of crossings in π_2. But the difference of these numbers is not always 1.

Lemma 2.3.1 implies that the transitive closure of the covering relation "$\rightarrow$" has the structure of a partially ordered set and this partially ordered set decomposes into $n + 1$ incomparable components. For $0 \le k \le n$, we define the *cyclic Bruhat order* CB_{kn} as the set of all decorated permutations π of size n such that $K(\pi) = k$ with the partial order relation obtained by the transitive closure of the covering relation "$\rightarrow$". By Lemma 2.3.1 the function A is the corank function for the cyclic Bruhat order CB_{kn}.

The definitions of the covering relation and of the statistic A will not change if we rotate a chord diagram. The definition of K depends on the order of the boundary points $1, \ldots, n$, but it is not hard to see that the statistic K is invariant under the

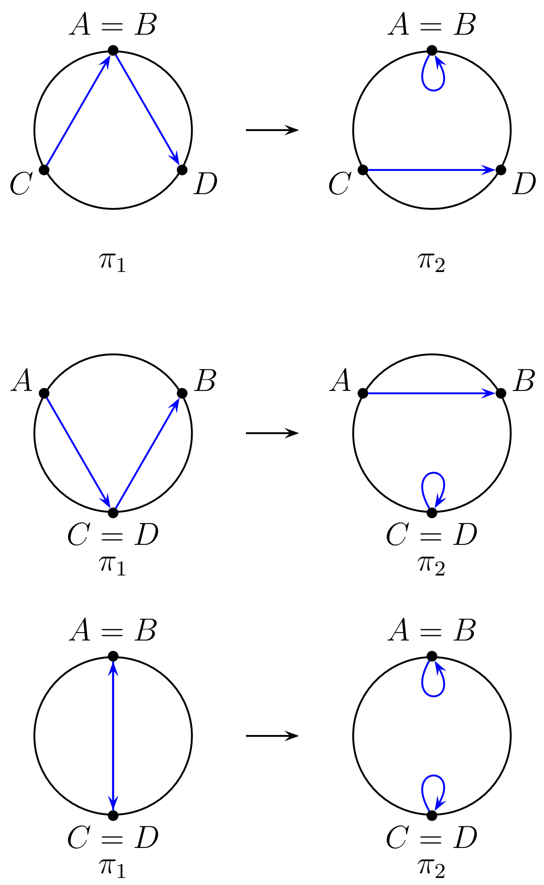


Figure 2-7: Degenerate covering relations

cyclic shift conj_σ for the long cycle $\sigma = (1, 2, \ldots, n)$. Thus the order CB_{kn} is invariant under the action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ on decorated permutations.

In [34], Postnikov proved that the number of totally positive cells in $Gr^+_{k,n}$ of dimension r is equal to the number of decorated permutations in CB_{kn} of rank r. Thus, $A_{k,n}(1)$ is the cardinality of CB_{kn}, and the coefficient of $q^{k(n-k)-\ell}$ in $A_{k,n}(q)$ is the number of decorated permutations in CB_{kn} with ℓ alignments.

Figure A-2 shows the poset of cells of $Gr^+_{2,4}$ in terms of decorated permutations.

2.4 The Rank Generating Function of $Gr^+_{k,n}$

Recall that the coefficient of q^r in $A_{k,n}(q)$ is the number of cells of dimension r in the cellular decomposition of $Gr^+_{k,n}$. In this section we use the Ⅎ-diagrams to find an explicit expression for $A_{k,n}(q)$. Additionally, we will find explicit expressions for the generating functions $A_k(q,x) := \sum_n A_{k,n}(q)x^n$ and $A(q,x,y) := \sum_{k\geq 1}\sum_n A_{k,n}(q)x^n y^k$. Our main theorem is the following:

Theorem 2.4.1.

$$\begin{aligned}
A(q,x,y) &= \frac{-y}{q(1-x)} + \sum_{i\geq 1} \frac{y^i(q^{2i+1}-y)}{q^{i^2+i+1}(q^i - q^i[i+1]x + [i]xy)} \\
A_k(q,x) &= \sum_{i=0}^{k-1} (-1)^{i+k} \frac{x^{k-i-1}[i]^{k-i-1}}{q^{ki+i+1}(1-[i+1]x)^{k-i}} + \sum_{i=0}^{k} (-1)^{i+k} \frac{x^{k-i}[i]^{k-i}}{q^{ki}(1-[i+1]x)^{k-i+1}} \\
A_{k,n}(q) &= q^{-k^2} \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} (q^{ki}[k-i]^i[k-i+1]^{n-i} - q^{(k+1)i}[k-i-1]^i[k-i]^{n-i}) \\
&= \sum_{i=0}^{k-1} \binom{n}{i} q^{-(k-i)^2} ([i-k]^i[k-i+1]^{n-i} - [i-k+1]^i[k-i]^{n-i}).
\end{aligned}$$

Note that it is not obvious from the above formulas that $A_{k,n}(q)$ is either polynomial or nonnegative.

Since the expressions for $A_k(q,x)$ and $A_{k,n}(q)$ follow easily from the formula for $A(q,x,y)$, we will concentrate on proving the formula for $A(q,x,y)$.

Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$. Let $F_\lambda(q)$ be the polynomial in q such that the coefficient of q^r is the number of Ⅎ-fillings of the Young diagram Y_λ which contain r 1's. As Figure 2-8 illustrates, there is a simple recurrence for $F_\lambda(q)$.

Explicitly, any Ⅎ-filling of λ is obtained in one of the following ways: adding a 1 to the last row of a Ⅎ-filling of $(\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k - 1)$; adding a row containing λ_k 0's to a Ⅎ-filling of $(\lambda_1, \ldots, \lambda_{k-1})$; or inserting an all-zero column after the $(\lambda_k - 1)$st column of a Ⅎ-filling of $(\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_k - 1)$. Note, however, that the second and third cases are not exclusive, so that our resulting recurrence must subtract off a term corresponding to their overlap.

Remark 2.4.2.

$$F_\lambda(q) = qF_{(\lambda_1,\ldots,\lambda_{k-1},\lambda_k-1)}(q) + F_{(\lambda_1,\ldots,\lambda_{k-1})}(q) + F_{(\lambda_1-1,\ldots,\lambda_k-1)}(q) - F_{(\lambda_1-1,\ldots,\lambda_{k-1}-1)}(q).$$

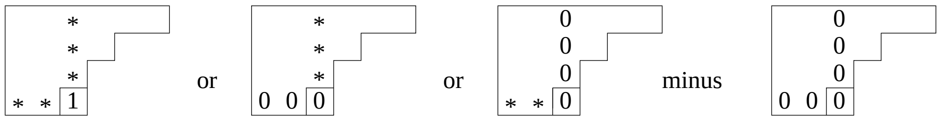


Figure 2-8: Recurrence for $F_\lambda(q)$

From the definition, or using the recurrence, it is easy to compute the first few formulas. Here are $F_{(\lambda_1)}(q)$ and $F_{(\lambda_1,\lambda_2)}(q)$.

Proposition 2.4.3.

$$\begin{aligned} F_{(\lambda_1)}(q) &= [2]^{\lambda_1} \\ F_{(\lambda_1,\lambda_2)}(q) &= -q^{-1}[2]^{\lambda_1} + q^{-1}[2]^{\lambda_1-\lambda_2+1}[3]^{\lambda_2}. \end{aligned}$$

In general, we have the following formula.

Theorem 2.4.4. *Fix* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. *Then*

$$F_\lambda(q) = \sum_{i=1}^{k} \sum_{1=t_1<\cdots<t_i\le k} M(t_1,\ldots,t_i : k)[i+1]^{\lambda_{t_i}} \prod_{j=2}^{i} [j]^{\lambda_{t_{j-1}}-\lambda_{t_j}+1},$$

where $M(t_1,\ldots,t_i : k) = (-1)^{k+i} q^{-ik+\sum_{j=1}^{i} t_j} [i]^{k-t_i} \prod_{j=1}^{i-1} [j]^{t_{j+1}-t_j-1}$.

Before beginning the proof of the theorem, we state two lemmas which follow immediately from the formula for $M(t_1,\ldots,t_i : k)$.

Lemma 2.4.5. $M(t_1,\ldots,t_i : k) = (-1)^{k-t_i} q^{-i(k-t_i)} [i]^{k-t_i} M(t_1,\ldots,t_i : t_i)$.

Lemma 2.4.6. $M(t_1,\ldots,t_i : t_i) = -[i-1]^{-1} M(t_1,\ldots,t_{i-1} : t_i)$.

Proof. To prove the theorem, we must show that the expression for $F_\lambda(q)$ holds for $\lambda = (\lambda_1)$, and that it satisfies the recurrence of Remark 2.4.2. Also, we must show that $F_{(\lambda_1,\lambda_2,\ldots,\lambda_k)}(q) = F_{(\lambda_1,\lambda_2,\ldots,\lambda_k,0)}(q)$.

The formula $F_{(\lambda_1)}(q) = [2]^{\lambda_1}$ clearly agrees with the expression in the theorem. To show that the recurrence is satisfied, we will fix $(t_1,\ldots,t_i)$ where $1 = t_1 < \cdots < t_i \le k$, and calculate the coefficient of $[2]^{\lambda_{t_1}-\lambda_{t_2}+1}[3]^{\lambda_{t_2}-\lambda_{t_3}+1}\ldots[i+1]^{\lambda_{t_i}}$ in each of the five terms of 2.4.2. We will then show that these coefficients satisfy the recurrence.

The coefficient in $F_{(\lambda_1,\ldots,\lambda_k)}(q)$ is $M(t_1,\ldots,t_i : k)$.

The coefficient in $F_{(\lambda_1,\lambda_2,\ldots,\lambda_k-1)}(q)$ is $M(t_1,\ldots,t_i : k)$ if $t_i < k$, because the term we are looking at together with its coefficient do not involve λ_k. The coefficient is $[i][i+1]^{-1}M(t_1,\ldots,t_i : k)$ if $t_i = k$.

The coefficient in $F_{(\lambda_1,\lambda_2,\ldots,\lambda_{k-1})}(q)$ is $M(t_1,\ldots,t_i : k-1)$ if $t_i < k$, which is equal to $-q^i[i]^{-1}M(t_1,\ldots,t_i : k)$. But if $t_i = k$, no such term appears, so the coefficient is 0.

The coefficient in $F_{(\lambda_1-1,\lambda_2-1,\ldots,\lambda_k-1)}(q)$ is always $M(t_1,\ldots,t_i : k)[i+1]^{-1}$.

The coefficient in $F_{(\lambda_1-1,\lambda_2-1,\ldots,\lambda_{k-1}-1)}(q)$ is $-q^i[i]^{-1}[i+1]^{-1}M(t_1,\ldots,t_i : k)$ if $t_i < k$, and 0 if $t_i = k$.

Let us abbreviate $M(t_1,\ldots,t_i : k)$ by M. We need to show that the coefficients we have just calculated satisfy the recurrence of Remark 2.4.2. For $t_i < k$, this amounts to showing that $M = qM - q^i[i]^{-1}M + M[i+1]^{-1} + q^i[i]^{-1}[i+1]^{-1}M$. And for $t_i = k$, we must show that $M = q[i][i+1]^{-1}M + M[i+1]^{-1}$. Both of these are easily seen to be true. Thus, we have shown that our expression for $F_\lambda(q)$ satisfies Remark 2.4.2.

Now we will show that $F_{(\lambda_1,\lambda_2,\ldots,\lambda_{k-1},0)}(q) = F_{(\lambda_1,\lambda_2,\ldots,\lambda_{k-1})}(q)$. It is sufficient to show that the coefficient of $[2]^{\lambda_{t_1}-\lambda_{t_2}+1}[3]^{\lambda_{t_2}-\lambda_{t_3}+1}\ldots[i+1]^{\lambda_{t_i}}$ in $F_{(\lambda_1,\ldots,\lambda_k)}(q)$, plus $[i+1]$ times the coefficient of $[2]^{\lambda_{t_1}-\lambda_{t_2}+1}[3]^{\lambda_{t_2}-\lambda_{t_3}+1}\ldots[i+1]^{\lambda_{t_i}-\lambda_k+1}[i+2]^{\lambda_k}$ in $F_{(\lambda_1,\ldots,\lambda_k)}(q)$, is equal to the coefficient of $[2]^{\lambda_{t_1}-\lambda_{t_2}+1}\ldots[i+1]^{\lambda_{t_i}}$ in $F_{(\lambda_1,\ldots,\lambda_{k-1})}(q)$.

In other words, we need

$$M(t_1,\ldots,t_i : k-1) = M(t_1,\ldots,t_i : k) + M(t_1,\ldots,t_i,k : k)[i+1].$$

From the formula for M, we have $M(t_1,\ldots,t_i : k-1) = -q^i[i]^{-1}M(t_1,\ldots,t_i : k)$. And from Lemma 2.4.6, $M(t_1,\ldots,t_i,k : k) = -[i]^{-1}M(t_1,\ldots,t_i : k)$. The proof follows. $\square$

Recall that $A_k(q,x)$ is the polynomial in q and x such that $[q^r x^n]A_k(q,x)$ is equal to the number of totally positive cells of dimension r in $Gr^+_{k,n}$. This is equal to the number of Ꞁ-diagrams $(\lambda, D)_{k,n}$ of rank r. We can compute these numbers by using Theorem 2.4.4.

Corollary 2.4.7.

$$A_k(q,x) = \sum_{i=1}^{k} \sum_{1=t_1<\cdots<t_{i+1}=k+1} \frac{(-1)^{k+i} q^{-ik+\sum_{j=1}^{i} t_j} x^k}{(1-x)} \prod_{j=1}^{i} \left(\frac{[j]}{1-[j+1]x} \right)^{t_{j+1}-t_j}.$$

To compute $A_k(q,x)$, we must sum $F_{(\lambda_1,\dots,\lambda_k)}(q)x^n$, as λ varies over all partitions which fit into a $k \times (n-k)$ rectangle. To do this, we use the following simple lemmas, the second of which follows immediately from the first.

Lemma 2.4.8.

$$\sum_{\lambda_1=0}^{\infty} \sum_{\lambda_2=0}^{\lambda_1} \cdots \sum_{\lambda_d=0}^{\lambda_{d-1}} x_1^{\lambda_1} x_2^{\lambda_2} \dots x_d^{\lambda_d} = \frac{1}{(1-x_1)(1-x_1x_2)\dots(1-x_1x_2\dots x_d)}.$$

Lemma 2.4.9. *Fix a set of positive integers $t_1 < t_2 < \cdots < t_d < n+1$. Then*

$$\sum_{n=0}^{\infty} \sum_{\lambda_1=0}^{n} \sum_{\lambda_2=0}^{\lambda_1} \cdots \sum_{\lambda_d=0}^{\lambda_{d-1}} [2]^{\lambda_{t_1}-\lambda_{t_2}} \dots [d]^{\lambda_{t_{d-1}}-\lambda_{t_d}} [d+1]^{\lambda_{t_d}} x^n$$

is equal to

$$\frac{1}{(1-x)(1-[2]x)^{t_2-t_1} \dots (1-[d]x)^{t_d-t_{d-1}} (1-[d+1]x)^{n+1-t_d}}.$$

Proof. For the proof of the corollary, apply Theorem 2.4.4 and Lemma 2.4.9 to the fact that

$$A_k(q,x) = \sum_{m=0}^{\infty} \sum_{\lambda_1=0}^{m} \sum_{\lambda_2=0}^{\lambda_1} \cdots \sum_{\lambda_k=0}^{\lambda_{k-1}} F_{(\lambda_1,\dots,\lambda_k)}(q) x^m.$$

□

Corollary 2.4.10. *The Euler characteristic of the totally non-negative part of the Grassmannian $Gr^+_{k,n}$ is* 1*.*

Proof. Recall that the Euler characteristic of a cell complex is defined to be $\sum_i (-1)^i f_i$, where f_i is the number of cells of dimension i. So if we set $q = -1$ in Corollary 2.4.7, we will obtain a polynomial in x such that the coefficient of x^n is the Euler characteristic of $Gr^+_{k,n}$. Notice that $[i]$ is equal to 0 if i is even, and 1 if i is odd.

So all terms of $A_k(-1,x)$ vanish except the term for $i=1$, which becomes $\frac{x^k}{1-x} = x^k + x^{k+1} + x^{k+2} + \dots$. $\square$

Note that this corollary also follows from Lusztig's result that the totally nonnegative part of a real flag variety is contractible.

Now our goal will be to simplify our expressions. To do so, it is helpful to work with the "master" generating function $A(q,x,y) := \sum_{k\geq 1} A_k(q,x)y^k$. As a first step, we compute the following expression for $A(q,x,y)$:

Proposition 2.4.11.

$$A(q,x,y) = \sum_{i=1}^{\infty} q^i[i]!x^iy^i \prod_{j=0}^{i} \frac{1}{q^j - q^j[j+1]x + [j]xy}.$$

Note that $\frac{1}{q^j - q^j[j+1]x+[j]xy}$ is not a well-defined formal power series because it is not clear how to expand it. In this chapter, for reasons which will become clear in the following proof, we shall always use $\frac{1}{q^j - q^j[j+1]x+[j]xy}$ as shorthand for the formal power series whose expansion is implied by the expression

$$\frac{1}{q^j(1-[j+1]x)(1-\frac{q^{-j}[j]y}{1-[j+1]x})}.$$

See [43, Example 6.3.4] for remarks on the subtleties of such power series.

Proof. From Corollary 2.4.7, we know that $A_k(q,x)$ is equal to

$$\frac{(-x)^k}{1-x} \sum_{i=1}^{k} \sum_{1=t_1<\cdots<t_{i+1}=k+1} (-1)^i q^{-ik+\sum_{j=1}^{i} t_j} \prod_{j=1}^{i} \left(\frac{[j]}{1-[j+1]x}\right)^{t_{j+1}-t_j}.$$

If we make the substitution $\alpha_j = t_{j+1}-t_j$, we then get

$$A_k(q,x) = \frac{(-x)^k}{1-x} \sum_{i=1}^{k} (-1)^i q^i \sum_{\substack{\alpha_j\geq 1\\ \alpha_1+\cdots+\alpha_i=k}} \prod_{j=1}^{i} \left(\frac{[j]}{q^j(1-[j+1]x)}\right)^{\alpha_j}.$$

Now let $f_j(p) = (\frac{[j]}{q^j(1-[j+1]x)})^p$. For future use, define $F_j(y) := \sum_{p\geq 1} f_j(p)y^p$, which is equal to $\frac{[j]y}{q^j - q^j[j+1]x - [j]y}$. We get

$$A_k(q,x) = \frac{(-x)^k}{1-x} \sum_{i=1}^{k} (-1)^i q^i \sum_{\substack{\alpha_j \geq 1 \\ \alpha_1 + \cdots + \alpha_i = k}} \prod_{j=1}^{i} f_j(\alpha_j),$$

and we can now easily compute $A(q,x,y) := \sum_{k\geq 1} A_k(q,x) y^k$.

$$\begin{aligned} A(q,x,y) &= \frac{1}{1-x} \sum_{k\geq 1} (-x)^k \sum_{i=1}^{k} (-1)^i q^i \sum_{\substack{\alpha_j \geq 1 \\ \alpha_1 + \cdots + \alpha_i = k}} \prod_{j=1}^{i} f_j(\alpha_j) y^{\alpha_j} \\ &= \frac{1}{1-x} \sum_{i=1}^{\infty} \sum_{k\geq i} \sum_{\substack{\alpha_j \geq 1 \\ \alpha_1 + \cdots + \alpha_i = k}} (-x)^k (-1)^i q^i \prod_{j=1}^{i} f_j(\alpha_j) y^{\alpha_j}. \end{aligned}$$

Actually, we can replace $k \geq i$ above with $k \geq 0$, since if $k < i$ there will be no set of α_j satisfying the conditions of the third sum. So we have

$$\begin{aligned} A(q,x,y) &= \frac{1}{1-x} \sum_{i=1}^{\infty} \sum_{k\geq 0} \sum_{\substack{\alpha_j \geq 1 \\ \alpha_1 + \cdots + \alpha_i = k}} (-x)^k (-1)^i q^i \prod_{j=1}^{i} f_j(\alpha_j) y^{\alpha_j} \\ &= \frac{1}{1-x} \sum_{i=1}^{\infty} (-1)^i q^i \sum_{k\geq 0} \sum_{\substack{\alpha_j \geq 1 \\ \alpha_1 + \cdots + \alpha_i = k}} \prod_{j=1}^{i} f_j(\alpha_j) (-xy)^{\alpha_j} \\ &= \frac{1}{1-x} \sum_{i=1}^{\infty} (-1)^i q^i \prod_{j=1}^{i} F_j(-xy) \\ &= \frac{1}{1-x} \sum_{i=1}^{\infty} (-1)^i q^i \prod_{j=1}^{i} \frac{-[j]xy}{q^j - q^j[j+1]x + [j]xy} \\ &= \frac{1}{1-x} \sum_{i=1}^{\infty} q^i [i]! x^i y^i \prod_{j=1}^{i} \frac{1}{q^j - q^j[j+1]x + [j]xy} \\ &= \frac{1}{1-x} \sum_{i=1}^{\infty} q^i [i]! x^i y^i \prod_{j=1}^{i} \frac{1}{q^j - q^j[j+1]x + [j]xy} \\ &= \sum_{i=1}^{\infty} q^i [i]! x^i y^i \prod_{j=0}^{i} \frac{1}{q^j - q^j[j+1]x + [j]xy}. \end{aligned}$$

□

Now we will prove the following identity. This identity combined with Proposition 2.4.11 will complete the proof of Theorem 2.4.1.

Theorem 2.4.12.

$$\sum_{i=1}^{\infty} q^i[i]!x^iy^i \prod_{j=0}^{i} \frac{1}{q^j - q^j[j+1]x + [j]xy} = \frac{-y}{q(1-x)} + \sum_{i\geq 1} \frac{q^{-i^2-i-1}y^i(q^{2i+1}-y)}{q^i - q^i[i+1]x + [i]xy}$$

Proof. Observe that the expression on the right-hand side can be thought of as a partial fraction expansion in terms of x, since all denominators are distinct, and the numerators are free of x. Also note that the i-summand of the left-hand side should be easy to express in partial fractions with respect to x, since all factors of the denominator are distinct and the x-degree of the numerator is smaller than the x-degree of the denominator.

Thus, our strategy will be to put the left-hand side into partial fractions with respect to x, and then show that this agrees with the right-hand side.

To this end, define $\beta_i(j)$ by the equation

$$\frac{x^i}{\prod_{j=0}^{i} q^j - q^j[j+1]x + [j]xy} = \sum_{j=0}^{i} \frac{\beta_i(j)}{q^j - q^j[j+1]x + [j]xy}.$$

Clearing denominators, we obtain

$$x^i = \sum_{j=0}^{i} \beta_i(j) \prod_{\substack{r=0 \\ r\neq j}}^{i} (q^r - q^r[r+1]x + [r]xy). \tag{2.1}$$

Fix j. Notice that $(q^j - q^j[j+1]x + [j]xy)$ vanishes when $x = \frac{q^j}{q^j[j+1]-[j]y}$, so substitute $x = \frac{q^j}{q^j[j+1]-[j]y}$ into (2.1). We get

$$\frac{q^{ij}}{(q^j[j+1]-[j]y)^i} = \beta_i(j) \prod_{\substack{r=0 \\ r\neq j}}^{i} \frac{q^r(q^j[j+1]-[j]y) + q^j([r]y - q^r[r+1])}{q^j[j+1]-[j]y}.$$

Solving for $\beta_i(j)$ and simplifying, we arrive at

$$\beta_i(j) = \frac{(-1)^{i+j} q^{\frac{j^2+3j-i^2-3i-2ij}{2}}}{[j]![i-j]! \prod_{\substack{r=0 \\ r\neq j}}^{i} (1 - q^{-r-j-1} y)}.$$

Thus the partial fraction expansion with respect to x of the left-hand side of Theorem 2.4.12 is

$$\sum_{i=1}^{\infty} \sum_{j=0}^{i} \frac{\beta_i(j) q^i [i]! y^i}{q^j - q^j [j+1] x + [j] x y},$$

which is equal to

$$\sum_{j=0}^{\infty} \frac{(-1)^j q^{\frac{j^2+3j}{2}} \sum_{\substack{i \geq j \\ i \neq 0}} \begin{bmatrix} i \\ j \end{bmatrix} q^{-\binom{i+1}{2} - ij} (-y)^i \prod_{\substack{r=0 \\ r \neq j}}^{i} (1 - q^{-r-j-1} y)^{-1}}{q^j - q^j [j+1] x + [j] x y}. \tag{2.2}$$

Now it remains to show that the numerator of $(q^j - q^j[j+1]x + [j]xy)$ in (2.2) is equal to the numerator of $(q^j - q^j[j+1]x + [j]xy)$ in the right-hand side of Theorem 2.4.12. For $j = 0$, we must show that

$$(1 - \frac{y}{q}) \sum_{i \geq 1} (-1)^i q^{-\binom{i+1}{2}} y^i \prod_{r=0}^{i} (1 - q^{-r-1} y)^{-1} = \frac{-y}{q}. \tag{2.3}$$

And for $j > 0$, we must show that

$$(-1)^j q^{\frac{3j^2+j}{2}} y^{-j} \sum_{i \geq j} \begin{bmatrix} i \\ j \end{bmatrix} q^{-\binom{i+1}{2} - ij} (-y)^i \prod_{r=0}^{i} (1 - q^{-r-j-1} y)^{-1} = 1. \tag{2.4}$$

If we make the substitution $q \to q^{-1}$ and $r \to r - 1$ into (2.3) and then add the $i = 0$ term to both sides, we obtain

$$\sum_{i \geq 0} (-1)^i y^i q^{\binom{i+1}{2}} \prod_{r=1}^{i+1} \frac{1}{1 - q^r y} = 1. \tag{2.5}$$

And if we make the same substitution into (2.4), we get

$$(-1)^j q^{-\binom{j+1}{2}} y^{-j} \sum_{i \geq j} (-1)^i q^{\binom{i+1}{2}} \begin{bmatrix} i \\ j \end{bmatrix} y^i \prod_{r=1}^{i+1} \frac{1}{1-q^{r+j}y} = 1. \tag{2.6}$$

Since (2.5) is a special case of (2.6), it suffices to prove (2.6). We will prove this as a separate lemma below; modulo this lemma, we are done. □

Lemma 2.4.13.

$$(-1)^j q^{-\binom{j+1}{2}} y^{-j} \sum_{i \geq j} (-1)^i q^{\binom{i+1}{2}} \begin{bmatrix} i \\ j \end{bmatrix} y^i \prod_{r=1}^{i+1} \frac{1}{1-q^{r+j}y} = 1.$$

Proof. Christian Krattenthaler has pointed out to us that this lemma is actually a special case of the ${}_1\phi_1$ summation described in Appendix II.5 of [24]. Here, we give two additional proofs of this lemma. The first method is to show that the infinite sum actually telescopes (we thank Ira Gessel for suggesting this to us). The second method is to interpret the lemma as a statement about partitions, and to prove it combinatorially.

Let us sketch the first method. We use induction to show that

$$(-1)^j q^{-\binom{j+1}{2}} y^{-j} \sum_{i=j}^{m-1} (-1)^i q^{\binom{i+1}{2}} \begin{bmatrix} i \\ j \end{bmatrix} y^i \prod_{r=1}^{i+1} \frac{1}{1-q^{r+j}y}$$

is equal to

$$1 + \frac{(-1)^{m-1} q^{jm+\binom{m+1}{2}} y^m \sum_{p=0}^{j} (-1)^p \begin{bmatrix} m \\ p \end{bmatrix} q^{\binom{p}{2}-pj-pm} y^{-p}}{\prod_{r=1}^{m}(1-q^{r+j}y)}.$$

Then we take the limit as m goes to ∞, obtaining the statement of the lemma.

Now let us give a combinatorial proof of the lemma. For clarity, we prove the $j=0$ case in detail and then explain how to generalize this proof.

First we claim that $(-1)^i y^i q^{\binom{i+1}{2}} \prod_{r=1}^{i+1} \frac{1}{1-q^r y}$ is a generating function for partitions λ with $i+1$ parts, all distinct, where the smallest part may be zero. In this formal power series, the coefficient of $y^m q^n$ is equal to the number of such partitions with

m columns and n total boxes. The generating function is multiplied by 1 or -1, according to the parity of the number of rows (including zero).

To prove the claim, note that each term of $\prod_{r=1}^{i+1} \frac{1}{1-q^r y}$ corresponds to a (normal) partition where rows have lengths between 1 and $i+1$, inclusive. The exponent of y enumerates the number of rows and the exponent of q enumerates the number of boxes. Now take the transpose of this partition, so that it is a partition with exactly $i+1$ rows (possibly zero). Now the exponent of y is the length of the longest row. Add $i, i-1, \ldots, 1$ and 0 boxes to the first, second, ... , and $(i+1)$st rows, respectively. Finally we have a partition with $i+1$ parts, all distinct, where the smallest part may be zero. Since we've added a total of $\binom{i+1}{2}$ boxes to the original partition, the generating function for this type of partition is $q^{\binom{i+1}{2}} y^i \prod_{r=1}^{i+1} \frac{1}{1-q^r y}$. Figure 2-9 illustrates the steps in this paragraph. In the figure, the rows and columns of the partitions are indicated by solid and dashed lines, respectively.

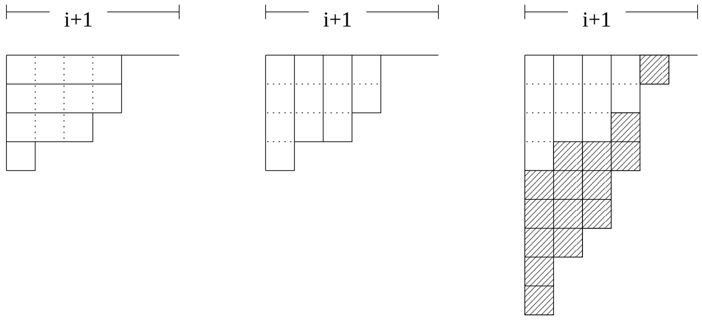


Figure 2-9: A combinatorial interpretation for $y^i q^{\binom{i+1}{2}} \prod_{r=1}^{i+1} \frac{1}{1-q^r y}$

Now we need to find an involution ϕ which explains why all of the terms on the left-hand side of (2.5) cancel out, except for the 1. This involution is very simple: if $(\lambda_1, \ldots, \lambda_k)$ is a partition such that $\lambda_k \neq 0$, then $\phi(\lambda_1, \ldots, \lambda_k) = (\lambda_1, \ldots, \lambda_k, 0)$. And if $\lambda_k = 0$, then $\phi(\lambda_1, \ldots, \lambda_k) = (\lambda_1, \ldots, \lambda_{k-1})$. Clearly both $(\lambda_1, \ldots, \lambda_k)$ and $\phi(\lambda_1, \ldots, \lambda_k)$ contribute the same powers of y and q to the generating function; the only difference is the sign. Only the 0 partition has no partner under the involution, so all terms cancel except for 1.

For the proof of the general case, we will show that

$$q^{\binom{i+1}{2}} \begin{bmatrix} i \\ j \end{bmatrix} y^{i-j} \prod_{r=1}^{i+1} \frac{1}{1-q^{r+j}y} \tag{2.7}$$

enumerates certain pairs of partitions, $(\lambda, \hat{\lambda})$. First, note that $\prod_{r=1}^{i+1} \frac{1}{1-q^{r+j}y}$ is a generating function for partitions with rows of lengths $j+1$ through $i+j+1$, inclusive. It is well-known that $\begin{bmatrix} i \\ j \end{bmatrix}$ is a polynomial in q whose q^r coefficient is the number of partitions of r which fit inside a $j \times (i-j)$ rectangle. To account for the $\begin{bmatrix} i \\ j \end{bmatrix}$ term in (2.7), let us take a partition which fits inside a $j \times (i-j)$ rectangle, and place it underneath a partition with rows of lengths $j+1$ through $i+j+1$, giving us a partition with row lengths between 0 and $i+j+1$, inclusive. We consider this partition to have exactly $i+j+1$ columns, possibly zero. Finally, to account for the $q^{\binom{i+1}{2}}$ term in (2.7) let us add $0, 1, \ldots, i$ boxes to the last $i+1$ columns of our partition, so that that the last $i+1$ columns have distinct lengths (possibly zero). We now view the boxes in the first j columns of our figure to comprise one partition λ, and the boxes in the last $i+1$ columns of our figure to comprise the transpose of a second partition $\hat{\lambda}$. Let $\hat{\lambda}_1$ denote the length of the first row of $\hat{\lambda}$, and let $r_j(\lambda)$ denote the number of rows of λ which have length j. Then the pair $(\lambda, \hat{\lambda})$ satisfies the following conditions: λ has rows with lengths between 0 and j, inclusive; $\hat{\lambda}$ has exactly $i+1$ rows, all distinct, where the smallest row can have length 0; and $r_j(\lambda) + i - j = \hat{\lambda}_1$. (See Figure 2-10 for an illustration of $(\lambda, \hat{\lambda})$.) The term in (2.7) that corresponds to this pair of partitions is $q^{|\lambda|+|\hat{\lambda}|} y^{\text{numparts}(\lambda)}$.

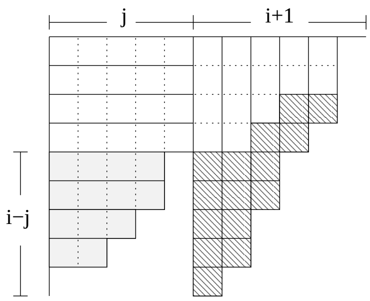


Figure 2-10: $(\lambda, \hat{\lambda})$, where $\lambda = (5, 5, 5, 5, 4, 4, 3, 2, 0)$ and $\hat{\lambda} = (9, 8, 6, 4, 3, 0)$

Our involution ϕ is a simple generalization of the involution we used before. This time, ϕ fixes λ, and either adds or subtracts a trailing zero to $\hat{\lambda}$.

$\square$

This completes the proof of Theorem 2.4.1.

Remark 2.4.14. *Discovering the formulas which appear in Theorem 2.4.1 was non-trivial. In our early work on this subject, we were able to compute by hand closed expressions for $A_1(q,x)$, $A_2(q,x)$, $A_3(q,x)$, and $A_4(q,x)$. By looking at the partial fraction expansion of these expressions we were able to see enough patterns to conjecture the formula for $A_k(q,x)$ in Theorem 2.4.1.*

In Table 2.1, we have listed some of the values of $A_{k,n}(q)$ for small k and n. It is easy to see from the definition of ⅃-diagrams that $A_{k,n}(q) = A_{n-k,n}(q)$: one can reflect a ⅃-diagram $(\lambda, D)_{k,n}$ of rank r over the main diagonal to get another ⅃-diagram $(\lambda', D')_{n-k,n}$ of rank r. Alternatively, one should be able to prove the claim directly from the expression in Theorem 2.4.1, using some q-analog of Abel's identity.

$A_{1,1}(q)$	1
$A_{1,2}(q)$	$q+2$
$A_{1,3}(q)$	q^2+3q+3
$A_{1,4}(q)$	q^3+4q^2+6q+4
$A_{2,4}(q)$	$q^4+4q^3+10q^2+12q+6$
$A_{2,5}(q)$	$q^6+5q^5+15q^4+30q^3+40q^2+30q+10$
$A_{2,6}(q)$	$q^8+6q^7+21q^6+50q^5+90q^4+120q^3+110q^2+60q+15$
$A_{3,6}(q)$	$q^9+6q^8+21q^7+56q^6+114q^5+180q^4+215q^3+180q^2+90q+20$
$A_{3,7}(q)$	$q^{12}+7q^{11}+28q^{10}+84q^9+203q^8+406q^7+679q^6+938q^5+1050q^4+910q^3+560q^2+210q+35$

Table 2.1: $A_{k,n}(q)$

Note that it is possible to see directly from the definition that $Gr^+_{1,n}$ is just some deformation of a simplex with n vertices. This explains the simple form of $A_{1,n}(q)$.

2.5 A New q-Analog of the Eulerian Numbers

If $\pi \in S_n$, we say that π has a *weak excedence* at position i if $\pi(i) \geq i$. The *Eulerian number* $E_{k,n}$ is the number of permutations in S_n which have k weak excedences. (One can define the Eulerian numbers in terms of other statistics, such as descent, but this will not concern us here.)

Now that we have computed the rank generating function for CB^+_{kn} (which is the rank generating function for the poset of decorated permutations), we can use this result to enumerate (regular) permutations according to two statistics: weak excedences and alignments. This gives us a new q-analog of the Eulerian numbers.

Recall that the statistic K on decorated permutations was defined as

$$K(\pi) = \#\{i \mid \pi(i) > i\} + \#\{\text{counterclockwise loops}\}.$$

Note that K is related to the notion of weak excedence in permutations. In fact, we can extend the definition of weak excedence to decorated permutations by saying that a decorated permutation has a weak excedence in position i, if $\pi(i) > i$, or if $\pi(i) = i$ and $d(i)$ is counterclockwise. This makes sense, since the limit of a chord from 1 to 2 as 1 approaches 2, is a counterclockwise loop. Then $K(\pi)$ is the number of weak excedences in π.

We will call a decorated permutation *regular* if all of its fixed points are oriented counterclockwise. Thus, a fixed point of a regular permutation will always be a weak excedence, as it should be. Recall that the Eulerian number $E_{k,n}$ is the number of permutations of $[n]$ with k weak excedences. Earlier, we saw that the coefficient of $q^{k(n-k)-\ell}$ in $A_{k,n}(q)$ is the number of decorated permutations in CB_{kn} with ℓ alignments. By analogy, let $E_{k,n}(q)$ be the polynomial in q whose coefficient of $q^{k(n-k)-\ell}$ is the number of (regular) permutations with k weak excedences and ℓ alignments. Thus, the family $E_{k,n}(q)$ will be a q-analog of the Eulerian numbers.

We can relate decorated permutations to regular permutations via the following lemma.

Lemma 2.5.1. $A_{k,n}(q) = \sum_{i=0}^{n} \binom{n}{i} E_{k,n-i}(q)$.

Proof. To prove this lemma we need to figure out how the number of alignments changes, if we start with a regular permutation on $[n-i]$ with k weak excedences, and then add i clockwise fixed points. Note that adding a clockwise fixed point adds exactly k alignments, since a clockwise fixed point is aligned with all of the weak excedences. Since clockwise fixed points are not in alignment with each other, it follows that adding i clockwise fixed points adds exactly ik alignments.

This shows that the new number of alignments is equal to ki plus the old number of alignments, or equivalently, that $k(n-i-k)$ minus the old number of alignments is equal to $k(n-k)$ minus the new number of alignments. In other words, the rank of the permutation on $[n-i]$ is equal to the rank of the new decorated permutation on $[n]$. Both permutations have k weak excedences. Since there are $\binom{n}{i}$ ways to pick i entries of a permutation on $[n]$ to be designated as clockwise fixed points, we have that $A_{k,n}(q) = \sum_{i=0}^{n} \binom{n}{i} E_{k,n}(q)$.

□

Observe that we can invert the formula given in the lemma, deriving the following corollary.

Corollary 2.5.2.

$$E_{k,n}(q) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} A_{k,n-i}(q).$$

Putting this together with Theorem 2.4.1, we get the following.

Corollary 2.5.3.

$$\begin{aligned} E_{k,n}(q) &= q^{n-k^2} \sum_{i=0}^{k-1} \binom{n}{i} (-1)^i (q^{ki-i}[k-i]^n - q^{ki}[k-i-1]^n) \\ &= q^{n-k^2} \sum_{i=0}^{k-1} (-1)^i [k-i]^n q^{ki-k} (\binom{n}{i} q^{k-i} + \binom{n}{i-1}). \end{aligned}$$

Notice that by substituting $q = 1$ into the second formula, we get

$$E_{k,n} = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k-i)^n,$$

the well-known exact formula for the Eulerian numbers.

Now we will investigate the properties of $E_{k,n}(q)$. Actually, since $E_{k,n}(q)$ is a multiple of q^{n-k}, we first define $\hat{E}_{k,n}(q)$ to be $q^{k-n}E_{k,n}(q)$, and then work with this renormalized polynomial. Table 2.2 lists $\hat{E}_{k,n}(q)$ for $n = 4, 5, 6, 7$.

$\hat{E}_{1,4}(q)$	1
$\hat{E}_{2,4}(q)$	$6 + 4q + q^2$
$\hat{E}_{3,4}(q)$	$6 + 4q + q^2$
$\hat{E}_{4,4}(q)$	1
$\hat{E}_{1,5}(q)$	1
$\hat{E}_{2,5}(q)$	$10 + 10q + 5q^2 + q^3$
$\hat{E}_{3,5}(q)$	$20 + 25q + 15q^2 + 5q^3 + q^4$
$\hat{E}_{4,5}(q)$	$10 + 10q + 5q^2 + q^3$
$\hat{E}_{5,5}(q)$	1
$\hat{E}_{1,6}(q)$	1
$\hat{E}_{2,6}(q)$	$15 + 20q + 15q^2 + 6q^3 + q^4$
$\hat{E}_{3,6}(q)$	$50 + 90q + 84q^2 + 50q^3 + 21q^4 + 6q^5 + q^6$
$\hat{E}_{4,6}(q)$	$50 + 90q + 84q^2 + 50q^3 + 21q^4 + 6q^5 + q^6$
$\hat{E}_{5,6}(q)$	$15 + 20q + 15q^2 + 6q^3 + q^4$
$\hat{E}_{6,6}(q)$	1
$\hat{E}_{1,7}(q)$	1
$\hat{E}_{2,7}(q)$	$21 + 35q + 35q^2 + 21q^3 + 7q^4 + q^5$
$\hat{E}_{3,7}(q)$	$105+245q+308q^2+259q^3+161q^4+77q^5+28q^6+7q^7+q^8$
$\hat{E}_{4,7}(q)$	$175 + 441q + 588q^2 + 532q^3 + 364q^4 + 196q^5 + 84q^6 + 28q^7 + 7q^8 + q^9$
$\hat{E}_{5,7}(q)$	$105+245q+308q^2+259q^3+161q^4+77q^5+28q^6+7q^7+q^8$
$\hat{E}_{6,7}(q)$	$21 + 35q + 35q^2 + 21q^3 + 7q^4 + q^5$
$\hat{E}_{7,7}(q)$	1

Table 2.2: $\hat{E}_{k,n}(q)$

We can make a number of observations about these polynomials. For example, we can generalize the well-known result that $E_{k,n} = E_{n+1-k,n}$, where $E_{k,n}$ is the Eulerian number corresponding to the number of permutations of S_n with k weak excedences.

Proposition 2.5.4. $\hat{E}_{k,n}(q) = \hat{E}_{n+1-k,n}(q)$.

Proof. To prove this, we define an alignment-preserving bijection on the set of permutations in S_n, which maps permutations with k weak excedences to permutations with $n+1-k$ weak excedences. If $\pi = (a_1, a_2, \ldots, a_n)$ is a permutation written in list notation, then the bijection maps π to $(b_1, b_2, \ldots, b_n)$, where $b_i = n - a_{n+1-i}$ modulo n. □

The reader will probably have noticed from the table that the coefficients of $\hat{E}_{2,n}(q)$ are binomial coefficients. Indeed, we have the following proposition, which follows from Corollary 2.5.3.

Proposition 2.5.5. $\hat{E}_{2,n}(q) = \sum_{i=0}^{n-2} \binom{n}{i+2} q^i$.

Proposition 2.5.6. *[34] The coefficient of the highest degree term of $\hat{E}_{k,n}(q)$ is* 1.

Proof. This is because there is a unique permutation in S_n with k weak excedences and no alignments, as proved in [34]. That unique permutation is $\pi_k : i \mapsto i + k$ modulo n. □

Proposition 2.5.7. $\hat{E}_{k,n}(-1) = \pm\binom{n-1}{k-1}$.

Proof. If we substitute $q = -1$ into the first expression for $E_{k,n}(q)$, we eventually get $(-1)^{n+1} \sum_{i=0}^{k-1} \binom{n}{i} (-1)^i$. It is known (see [1]) that this expression is equal to $\binom{n-1}{k-1}$. □

Proposition 2.5.8. *$\hat{E}_{k,n}(q)$ is a polynomial of degree $(k-1)(n-k)$, and $\hat{E}_{k,n}(0)$ is the Narayana number $N_{k,n} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$.*

We will prove Proposition 2.5.8 in Section 2.6.

Corollary 2.5.9. *$\hat{E}_{k,n}(q)$ interpolates between the Eulerian numbers, the Narayana numbers, and the binomial coefficients, at $q = 1, 0,$ and -1, respectively.*

Proof. This follows from the fact that $\hat{E}_{k,n}(q)$ is a q-analog of the Eulerian numbers, together with Propositions 2.5.7 and 2.5.8. □

Based on experimental evidence, we formulated the following conjecture about the coefficient of q in $\hat{E}_{k,n}(q)$. However, nice expressions for coefficients of other terms have eluded us so far.

Conjecture 2.5.10. *The coefficient of q in $\hat{E}_{k,n}(q)$ is $\binom{n}{k+1}\binom{n}{k-2}$.*

Remark 2.5.11. *The coefficients of $\hat{E}_{k,n}(q)$ appear to be unimodal. However, these polynomials do not in general have real zeroes.*

Since it may be helpful to have formulas which enumerate permutations by alignments (rather than $k(n-k)$ minus the number of alignments), we let $\widetilde{E}_{k,n}(q)$ be the polynomial in q such that the coefficient of q^l is the number of permutations on $\{1,\dots n\}$ with k weak excedences and l alignments. Note that by using Corollary 2.5.3 and performing a transformation which sends q to q^{-1}, we get the following expressions.

$$\begin{aligned}\widetilde{E}_{k,n}(q) &= \sum_{i=0}^{k-1}\binom{n}{i}(-1)^i q^{i(n-k)}(q^i[k-i]^n - q^n[k-i-1]^n)\\ &= \sum_{i=0}^{k-1}(-1)^i[k-i]^n q^{i(n-k)}(\binom{n}{i}q^i + \binom{n}{i-1}q^k).\end{aligned}$$

Remark 2.5.12. *An occurrence of the generalized pattern $13-2$ in a permutation π is a triple of indices $(i,i+1,j)$ where $i+1<j$ such that $\pi_i < \pi_j < \pi_{i+1}$. Together with E. Steingrimsson [45], we conjectured that the polynomials $\hat{E}_{k,n}(q)$ enumerated permutations according to descents and occurrences of the generalized pattern p, where p is any one of the patterns $13-2, 31-2, 2-13, 2-31$. This conjecture was subsequently proved by Sylvie Corteel [13]. Additionally, she showed that the polynomials $\hat{E}_{k,n}(q)$ arise in the study of the ASEP model in statistical physics [13].*

Theorem 2.5.13. *[13] The coefficient of q^r in $\hat{E}_{k,n}(q)$ is the number of permutations on n letters with $k-1$ descents and r occurrences of the generalized pattern $13-2$.*

2.6 Connection with Narayana Numbers

A *noncrossing partition* of the set $[n]$ is a partition π of the set $[n]$ with the property that if $a < b < c < d$ and some block B of π contains both a and c, while some block B' of π contains both b and d, then $B = B'$. Graphically, we can represent a noncrossing partition on a circle which has n labeled points equally spaced around it. We represent each block B as the polygon whose vertices are the elements of B. Then the condition that π is noncrossing just means that no two blocks (polygons) intersect each other.

It is known that the number of noncrossing partitions of $[n]$ which have k blocks is equal to the Narayana number $N_{k,n} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ (see Exercise 68e in [43]).

To prove the following proposition we will find a bijection between permutations of S_n with k excedences and the maximal number of alignments, and noncrossing partitions on $[n]$.

Proposition 2.6.1. *Fix k and n. Then $(k-1)(n-k)$ is the maximal number of alignments that a permutation in S_n with k weak excedences can have. The number of permutations in S_n with k weak excedences that achieve the maximal number of alignments is the Narayana number $N_{k,n} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$.*

Proof. Recall the bijection between Ⅎ-diagrams and decorated permutations. The Ⅎ-diagrams which correspond to regular permutations with k weak excedences are the Ⅎ-diagrams (λ, D) contained in a k by $n-k$ rectangle, such that each column of the rectangle contains at least one 1. The squares of the rectangle which do not contain a 1 correspond to alignments, so the maximal number of alignments is $(k-1)(n-k)$. (It is also straightforward to prove this using decorated permutations.)

In order to prove that the number of permutations which achieve the maximum number of alignments is $N_{k,n}$, we put these permutations in bijection with noncrossing partitions of $[n]$ which have k blocks.

To figure out what the maximal-alignment permutations look like, imagine starting from any given permutation and applying the covering relations in the cyclic Bruhat order as many times as possible, such that the result is a regular permutation. Note

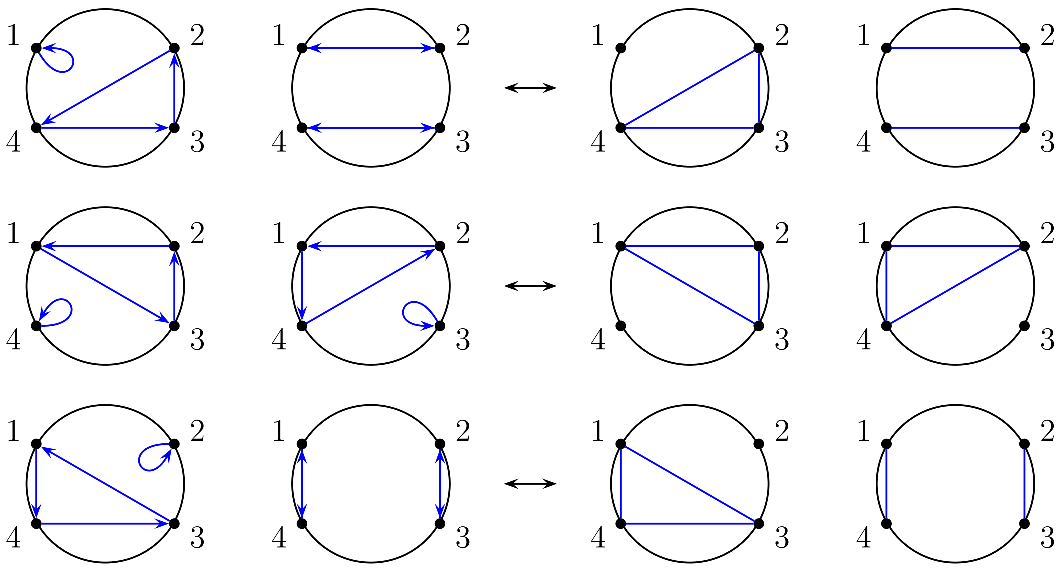


Figure 2-11: The bijection between maximal-alignment permutations and noncrossing partitions

that of the four cases of the covering relation (illustrated in section 2.3), we can use only the first and second cases. We cannot use the third and fourth operations because these add clockwise fixed points, which are not allowed in regular permutations. It is easy to see that after applying the first two operations as many times as possible, the resulting permutation will have no crossings among its chords and all cycles will be directed counterclockwise.

The map from maximal-alignment permutations to noncrossing partitions is now obvious. We simply take our permutation and then erase the directions on the edges. Since the covering relations in the cyclic Bruhat order preserve the number of weak excedences, and since each counterclockwise cycle in a permutation contributes one weak excedence, the resulting noncrossing partitions will all have k blocks. In Figure 2-11 we show the permutations in S_4 which have 2 weak excedences and 2 alignments, along with the corresponding noncrossing partitions.

Conversely, if we start with a noncrossing partition on $[n]$ which has k blocks, and then orient each cycle counterclockwise, then this gives us a maximal-alignment permutation with k weak excedences.

The map from maximal-alignment permutations to noncrossing permutations is obvious. Note that a maximal-alignment permutation must correspond to a noncrossing partition because, if there were a crossing of chords, we could uncross them to increase the number of alignments (while preserving the number of excedences).

□

Corollary 2.6.2. *The number of permutations in S_n which have the maximal number of alignments, given their weak excedences, is $C_n = \frac{1}{n}\binom{2n}{n+1}$, the nth Catalan number.*

Proof. It is known that $\sum_k N_{k,n} = C_n$. □

Remark 2.6.3. *The bijection between maximal-alignment permutations and noncrossing partitions is especially interesting because the connection gives a way of incorporating noncrossing partitions into a larger family of "crossing" partitions; this family of crossing partitions is a ranked poset, graded by alignments.*

2.7 Connections with the Permanent

Let $M_n(x)$ denote the permanent of the $n \times n$ matrix

$$\begin{pmatrix} 1+x & x & x & \dots & x \\ 1 & 1+x & x & \dots & x \\ 1 & 1 & 1+x & \dots & x \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \dots & 1+x \end{pmatrix}.$$

Clearly $[x^k]M_n(x)$ is equal to the number of decorated permutations on $[n]$ which have k weak excedences, i.e. $[x^k]M_n(x) = A_{k,n}(1)$. It would be interesting to find some q-analog of the above matrix whose permanent encodes $A_{k,n}(q)$.

Chapter 3

The Tropical Totally Positive Grassmannian

This part of my thesis is based on joint work with David Speyer [41].

3.1 Introduction

Tropical algebraic geometry is the geometry of the tropical semiring $(\mathbb{R}, \min, +)$. Its objects are polyhedral cell complexes which behave like complex algebraic varieties. Although this is a very new field in which many basic questions have not yet been addressed (see [35] for a nice introduction), tropical geometry has already been shown to have remarkable applications to enumerative geometry (see [32]), as well as connections to representation theory (see [21], [22], [29]).

In this chapter we introduce the totally positive part (or positive part, for short) of the tropicalization of an arbitrary affine variety over the ring of Puiseux series, and then investigate what we get in the case of the Grassmannian $Gr_{k,n}$. First we give a parameterization of the totally positive part of the Grassmannian, largely based on work of Postnikov [34], and then we compute its tropicalization, which we denote by $\mathrm{Trop}^+ Gr_{k,n}$. We identify $\mathrm{Trop}^+ Gr_{k,n}$ with a polyhedral subcomplex of the $\binom{n}{k}$-dimensional Gröbner fan of the ideal of Plücker relations, and then show that this fan, modulo its n-dimensional lineality space, is combinatorially equivalent to an $(n-$

$k-1)(k-1)$-dimensional fan which we explicitly describe. As a special case, we show that $\mathrm{Trop}^+ Gr_{2,n}$ is a fan which appeared in the work of Stanley and Pitman (see [44]), which parameterizes certain binary trees, and which is combinatorially equivalent to the (type A_{n-3}) associahedron. We also show that $\mathrm{Trop}^+ Gr_{3,6}$ and $\mathrm{Trop}^+ Gr_{3,7}$ are fans which are closely related to the fans of the types D_4 and E_6 associahedra, which were first introduced in [23]. These results are strikingly reminiscent of the results of Fomin and Zelevinsky [22], and Scott [38], who showed that the Grassmannian has a natural cluster algebra structure which is of type A_{n-3} for $Gr_{2,n}$, type D_4 for $Gr_{3,6}$, and type E_6 for $Gr_{3,7}$. (Fomin and Zelevinsky proved the $Gr_{2,n}$ case and stated the other results; Scott worked out the cluster algebra structure of all Grassmannians in detail.) Finally, we suggest a general conjecture about the positive part of the tropicalization of a cluster algebra.

3.2 Definitions

In this section we will define the tropicalization and positive part of the tropicalization of an arbitrary affine variety over the ring of Puiseux series. We will then describe the tropical varieties that will be of interest to us.

Let $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$ and $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathbb{R}((t^{1/n}))$ be the fields of Puiseux series over $\mathbb{C}$ and $\mathbb{R}$. Every Puiseux series $x(t)$ has a unique lowest term at^u where $a \in \mathbb{C}^*$ and $u \in \mathbb{Q}$. Setting $\mathrm{val}(f) = u$, this defines the *valuation map* $\mathrm{val} : (\mathcal{C}^*)^n \to \mathbb{Q}^n, (x_1, \dots, x_n) \mapsto (\mathrm{val}(x_1), \dots, \mathrm{val}(x_n))$. We define $\mathcal{R}^+$ to be $\{x(t) \in \mathcal{C} \mid$ the coefficient of the lowest term of $x(t)$ is real and positive$\}$. We will discuss the wisdom of this definition later; for practically all purposes, the reader may think of $\mathcal{C}$ as if it were $\mathbb{C}$ and of $\mathcal{R}^+$ as if it were $\mathbb{R}^+$.

Let $I \subset \mathcal{C}[x_1, \dots, x_n]$ be an ideal. We define the tropicalization of $V(I)$, denoted $\mathrm{Trop}\, V(I)$, to be the closure of the image under val of $V(I) \cap (\mathcal{C}^*)^n$, where $V(I)$ is the variety of I. Similarly, we define the positive part of $\mathrm{Trop}\, V(I)$, which we will denote as $\mathrm{Trop}^+ V(I)$, to be the closure of the image under val of $V(I) \cap (\mathcal{R}^+)^n$. Note that $\mathrm{Trop}\, V$ and $\mathrm{Trop}^+ V$ are slight abuses of notation; they depend on the affine space in which V is embedded and not solely on the variety V.

If $f \in \mathcal{C}[x_1, \ldots, x_n] \setminus \{0\}$, let the *initial form* $\mathrm{in}(f) \in \mathbb{C}[x_1, \ldots, x_n]$ be defined as follows: write $f = t^a g$ for $a \in \mathbb{Q}$ chosen as large as possible such that all powers of t in g are nonnegative. Then $\mathrm{in}(f)$ is the polynomial obtained from g by plugging in $t = 0$. If $f = 0$, we set $\mathrm{in}(f) = 0$. If $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ then $\mathrm{in}_w(f)$ is defined to be $\mathrm{in}(f(x_i t^{w_i}))$. If $I \subset \mathcal{C}[x_1, \ldots, x_n]$ then $\mathrm{in}_w(I)$ is the ideal generated by $\mathrm{in}_w(f)$ for all $f \in I$. It was shown in [40] that $\mathrm{Trop}\, V(I)$ consists of the collection of w for which $\mathrm{in}_w(I)$ contains no monomials. The essence of this proof was the following:

Proposition 3.2.1. *[40] If $w \in \mathbb{Q}^n$ and $\mathrm{in}_w(I)$ contains no monomial then $V(\mathrm{in}_w(I)) \cap (\mathbb{C}^*)^n$ is nonempty and any point $(a_1, \ldots, a_n)$ of this variety can be lifted to a point $(\tilde{a}_1, \ldots, \tilde{a}_n) \in V(I)$ with the leading term of $\tilde{a}_i$ equal to $a_i t^{w_i}$.*

We now prove a similar criterion to characterize the points in $\mathrm{Trop}^+ V(I)$.

Proposition 3.2.2. *A point $w = (w_1, \ldots, w_n)$ lies in $\mathrm{Trop}^+ V(I)$ if and only if $\mathrm{in}_w(I)$ does not contain any nonzero polynomials in $\mathbb{R}^+[x_1, \ldots, x_n]$.*

In order to prove this proposition, we will need the following result of [15], which relies heavily on a result of [26].

Proposition 3.2.3. *[15] An ideal I of $\mathbb{R}[x_1, \ldots, x_n]$ contains a nonzero element of $\mathbb{R}^+[x_1, \ldots, x_n]$ if and only if $(\mathbb{R}^+)^n \cap V(\mathrm{in}_\eta(I)) = \emptyset$ for all $\eta \in \mathbb{R}^n$.*

We are now ready to prove Proposition 3.2.2.

Proof. Define $T \subset \mathbb{Q}^n$ to be the image of $V(I) \cap (\mathcal{R}^+)^n$ under val. Let U denote the subset of $\mathbb{R}^n$ consisting of those w for which $\mathrm{in}_w(I)$ contains no polynomials with all positive terms. By definition, $\mathrm{Trop}^+ V(I)$ is the closure of T in $\mathbb{R}^n$. We want to show that the closure of T is U.

It is obvious that T lies in U. U is closed, as the property that $\mathrm{in}_w(f)$ has only positive terms is open as w varies. Thus, the closure of T lies in U.

Conversely, suppose that $w \in U$. Then, by Proposition 3.2.3, for some $\eta \in \mathbb{R}^n$, $(\mathbb{R}^+)^n \cap V(\mathrm{in}_\eta(\mathrm{in}_w(I))) \neq \emptyset$. For $\epsilon > 0$ sufficently small, we have $\mathrm{in}_\eta(\mathrm{in}_w(I)) =$

$\mathrm{in}_{\epsilon\eta+w}(I)$. Therefore we can find a sequence w_1, w_2, … approaching w with $(\mathbb{R}^+)^n \cap V(\mathrm{in}_{w_i}(I)) \neq \emptyset$.

As w varies, $\mathrm{in}_w(I)$ takes on only finitely many values, and the subsets of $\mathbb{R}^n$ on which $\mathrm{in}_w(I)$ takes a specific value form the relative interiors of the faces of a complete rational complex known as the Gröbner complex (see [47]). These complexes are actually fans when I is defined over $\mathbb{R}$ ([46]). Therefore, we may perturb each w_i, while preserving $\mathrm{in}_{w_i}(I)$, in order to assume that the $w_i \in \mathbb{Q}^n$ and we still have $w_i \to w$. Then, by Proposition 3.2.1, each $w_i \in T$, so w is in the closure of T as desired. □

Corollary 3.2.4. $\mathrm{Trop}\, V(I)$ *and* $\mathrm{Trop}^+ V(I)$ *are closed subcomplexes of the Gröbner complex. In particular, they are polyhedral complexes. If I is defined over* $\mathbb{R}$*, then* $\mathrm{Trop}\, V(I)$ *and* $\mathrm{Trop}^+ V(I)$ *are closed subfans of the Gröbner fan.*

One might wonder whether it would be better to modify the definition of $\mathcal{R}^+$ to require that our power series lie in $\mathcal{R}$. This definition, for example, is more similar to the appearance of the ring of formal powers series in [29]. One can show that in the case of the Grassmanian, this difference is unimportant. Moreover, the definition used here has the advantage that it makes it easy to prove that the positive part of the tropicalization is a fan.

Suppose $V(I) \subset \mathcal{C}^m$ and $V(J) \subset \mathcal{C}^n$ are varieties and we have a rational map $f : \mathcal{C}^m \to \mathcal{C}^n$ taking $V(I) \to V(J)$. Unfortunately, knowing $\mathrm{val}(x_i)$ for $1 \leq i \leq m$ does not in general determine $\mathrm{val}(f(x_1, \ldots, x_m))$, so we don't get a nice map $\mathrm{Trop}\, V(I) \to \mathrm{Trop}\, V(J)$. However, suppose that f takes the positive points of $V(I)$ surjectively onto the positive points of $V(J)$ and suppose that $f = (f_1, \ldots, f_n)$ is *subtraction-free*, that is, the formulas for the f_i's are rational functions in the x_i's whose numerators and denominators have positive coefficients. Define $\mathrm{Trop}\, f : \mathbb{R}^m \to \mathbb{R}^n$ by replacing every $\times$ in f with a $+$, every $/$ with a $-$, every $+$ with a min and every constant a with $\mathrm{val}(a)$.

Proposition 3.2.5. *Suppose $V(I) \subset \mathcal{C}^m$ and $V(J) \subset \mathcal{C}^n$ are varieties. Let $f : \mathcal{C}^m \to \mathcal{C}^n$ be a subtraction-free rational map taking $V(I)$ to $V(J)$ such that $V(I) \cap$*

$(\mathcal{R}^+)^m$ *surjects onto* $V(J) \cap (\mathcal{R}^+)^n$. *Then* $\operatorname{Trop} f$ *takes* $\operatorname{Trop}^+ V(I)$ *surjectively onto* $\operatorname{Trop}^+ V(J)$.

Proof. This follows immediately from the formulas $\operatorname{val}(x+y) = \min(\operatorname{val}(x), \operatorname{val}(y))$ and $\operatorname{val}(xy) = \operatorname{val}(x) + \operatorname{val}(y)$ for x and $y \in \mathcal{R}^+$. $\square$

We now define the objects that we will study in this chapter. Fix k and n, and let $N = \binom{n}{k}$. Fix a polynomial ring S in N variables with coefficients in a commutative ring. The *Plücker ideal* $I_{k,n}$ is the homogeneous prime ideal in S consisting of the algebraic relations (called *Plücker relations*) among the $k \times k$ minors of any $k \times n$-matrix with entries in a commutative ring.

Classically, the *Grassmannian* $Gr_{k,n}$ is the projective variety in $\mathbb{P}_{\mathbb{C}}^{N-1}$ defined by the ideal $I_{k,n}$ of Plücker relations. We write $Gr_{kn}(\mathcal{C})$ for the variety in $\mathbb{P}_{\mathcal{C}}^{N-1}$ defined by the same equations. Similarly, we write $Gr_{k,n}(\mathbb{R})$ for the real points of the Grassmannian, $Gr_{k,n}(\mathbb{R}^+)$ for the real positive points, $Gr_{k,n}(\mathcal{R}^+)$ for those points of $Gr_{k,n}(\mathcal{C})$ all of whose coordinates lie in $\mathcal{R}^+$ and so on. We write $Gr_{k,n}(\mathbb{C})$ when we want to emphasize that we are using the field $\mathbb{C}$, and use $Gr_{k,n}$ when discussing results that hold with no essential modification for any field. The *totally positive Grassmannian* is the set $Gr_{k,n}(\mathbb{R}^+)$.

An element of $Gr_{k,n}$ can be represented by a full rank $k \times n$ matrix A. If $K \in \binom{[n]}{k}$ we define the *Plücker coordinate* $\Delta_K(A)$ to be the minor of A corresponding to the columns of A indexed by K. We identify the element of the Grassmannian with the matrix A and with its set of Plücker coordinates (which satisfy the Plücker relations).

Our primary object of study is the *tropical positive Grassmannian* $\operatorname{Trop}^+ Gr_{k,n}$, which is a fan, by Corollary 3.2.4. As in [40], this fan has an n-dimensional lineality space. Let ϕ denote the map from $(\mathbb{C}^*)^n$ into $(\mathbb{C}^*)^{\binom{n}{k}}$ which sends $(a_1, \ldots, a_n)$ to the $\binom{n}{k}$-vector whose $(i_1, \ldots, i_k)$-coordinate is $a_{i_1} a_{i_2} \cdots a_{i_k}$. We abuse notation by also using ϕ for the same map $(\mathcal{C}^*)^n \to (\mathcal{C}^*)^{\binom{n}{k}}$. Let $\operatorname{Trop} \phi$ denote the corresponding linear map which sends $(a_1, \ldots, a_n)$ to the $\binom{n}{k}$-vector whose $(i_1, \ldots, i_k)$-coordinate is $a_{i_1} + a_{i_2} + \cdots + a_{i_k}$. The map $\operatorname{Trop} \phi$ is injective, and its image is the common lineality space of all cones in $\operatorname{Trop} Gr_{k,n}$.

3.3 Parameterizing the totally positive Grassmannian $Gr_{k,n}(\mathbb{R}^+)$

In this section we explain two equivalent ways to parameterize $Gr_{k,n}(\mathbb{R}^+)$, as well as a way to parameterize $Gr_{k,n}(\mathbb{R}^+)/\phi((\mathbb{R}^+)^n)$. The first method, due to Postnikov [34], uses a certain directed graph $\mathrm{Web}_{k,n}$ with variables associated to each of its $2k(n-k)$ edges. The second method is closely related to the first and uses the same graph, but this time variables are associated to each of its $k(n-k)$ regions. This has the advantage of giving a bijection between $(\mathbb{R}^+)^{k(n-k)}$ and $Gr_{k,n}(\mathbb{R}^+)$. Finally, we use $\mathrm{Web}_{k,n}$ with variables labelling each of its $(k-1)(n-k-1)$ inner regions in order to give a bijective parameterization of $Gr_{k,n}(\mathbb{R}^+)/\phi((\mathbb{R}^+)^n)$.

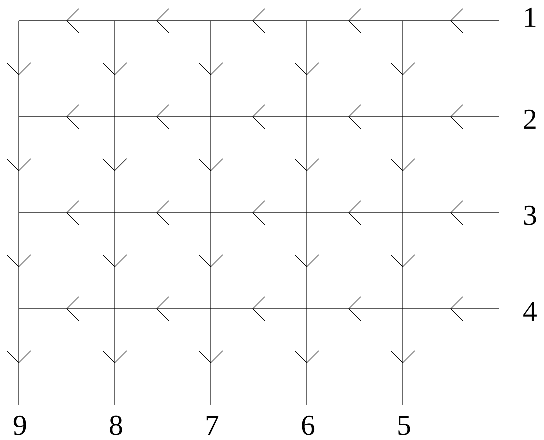


Figure 3-1: $\mathrm{Web}_{k,n}$ for $k = 4$ and $n = 9$

Let $\mathrm{Web}_{k,n}$ be the directed graph which is obtained from a k by $n-k$ grid, as shown in Figure 3-1. It has k incoming edges on the right and $n-k$ outgoing edges on the bottom, and the vertices attached to these edges are labelled clockwise from 1 to n. We denote the set of $2k(n-k)$ edges by E. Let us associate a formal variable x_e with each edge $e \in E$, and if there is no ambiguity, we abbreviate the collection $\{x_e\}$ by x. If p is a path on $\mathrm{Web}_{k,n}$ (compatible with the directions of the edges), then we let $\mathrm{Prod}_p(x)$ denote $\prod_{e \in p} x_e$. And if S is a set of paths on $\mathrm{Web}_{k,n}$, then we let $\mathrm{Prod}_S(x)$ denote $\prod_{p \in S} \mathrm{Prod}_p(x)$.

As in [34], we define a $k \times n$ matrix $A_{k,n}(x)$, whose entries $a_{ij}(x)$ are polynomials in the variables x_e, by the following equation:

$$a_{ij}(x) = (-1)^{i+1} \sum_p \mathrm{Prod}_p(x),$$

where the sum is over all directed paths p from vertex i to vertex j. Note that the $k \times k$ submatrix of $A_{k,n}(x)$ obtained by restricting to the first k columns is the identity matrix. In particular, $A_{k,n}(x)$ is a full rank matrix and hence we can identify it with an element of $Gr_{k,n}$. Also note that every element of the totally positive Grassmannian $Gr_{k,n}(\mathbb{R}^+)$ has a unique matrix representative whose leftmost $k \times k$ submatrix is the identity. We shall see that as the $\{x_e\}$ vary over $(\mathbb{R}^+)^{2k(n-k)}$, the $A_{k,n}(x)$ range over all of $Gr_{k,n}(\mathbb{R}^+)$.

We now show it is possible to express the maximal minors (Plücker coordinates) of $A_{k,n}(x)$ as subtraction-free rational expressions in the x_e, as shown in [34]. If $K \in \binom{[n]}{k}$, then let $\mathrm{Path}(K)$ denote the set

$$\{S : S \text{ is a set of pairwise vertex-disjoint paths from } [k]\backslash(K \cap [k]) \text{ to } K\backslash([k] \cap K)\}.$$

Note that for $K = [k]$, we consider the empty set to be a legitimate set of pairwise vertex-disjoint paths.

Applied to $\mathrm{Web}_{k,n}$, Theorem 15.4 of [34] implies the following.

Proposition 3.3.1. *The Plücker coordinates of $A_{k,n}(x)$ are given by*

$$\Delta_K(A_{k,n}(x)) = \sum_{S \in \mathrm{Path}(K)} \mathrm{Prod}_S(x).$$

Proof. We give a brief proof of this result: the main idea is to use the well-known Gessel-Viennot trick [25]. First note that $a_{ij}(x)$ has a combinatorial interpretation: it is a generating function keeping track of paths from i to j. Thus, the determinant of a $k \times k$ submatrix of $A_{k,n}(x)$ corresponding to the column set K also has a combinatorial interpretation: it is a generating function for all sets of paths from $[k]\backslash(K \cap [k])$ to $K\backslash([k] \cap K)$, with the sign of each term keeping track of the number of crossings in the corresponding path set. What we need to show is that this is equal to the sum of the contributions from path sets which are pairwise vertex-disjoint. To see this, consider a path set which does have an intersection. Look at its lexicographically last intersection, and compare this path set to the one obtained from it by switching the two path tails starting at that point of intersection. These two path sets get different signs, but have equal weights, and hence they cancel each other out. □

Proposition 3.3.1 allows us to define a map $\Phi_0 : (\mathbb{R}^+)^{2k(n-k)} \to Gr_{k,n}(\mathbb{R}^+)$ as follows. Let $K \in \binom{[n]}{k}$, and define $P_K : (\mathbb{R}^+)^{2k(n-k)} \to \mathbb{R}^+$ by

$$P_K(x) := \sum_{S \in \mathrm{Path}(K)} \mathrm{Prod}_S(x).$$

Clearly if we substitute positive values for each x_e, then $P_K(x)$ will be positive. We now define Φ_0 by

$$\Phi_0(x) = \{P_K(x)\}_{K \in \binom{[n]}{k}}.$$

In other words, Φ_0 is the map which sends a collection of positive real numbers $\{x_e\}$ to the element of $Gr_{k,n}$ with Plücker coordinates $P_K(x)$ (which is identified with the matrix $A_{k,n}(x)$).

By Theorem 19.1 of [34], the map Φ_0 is actually surjective: any point in $Gr_{k,n}(\mathbb{R}^+)$ can be represented as $A_{k,n}(x)$ for some positive choices of $\{x_e\}$. In summary, we have the following result (which will also be a consequence of our Theorem 3.3.3).

Proposition 3.3.2. *The map $\Phi_0 : (\mathbb{R}^+)^{2k(n-k)} \to Gr_{k,n}(\mathbb{R})$ is a surjection onto $Gr_{k,n}(\mathbb{R}^+)$.*

Unfortunately, the method we have just described uses $2k(n-k)$ variables to parameterize a space of dimension $k(n-k)$. We will now explain how to do a substitution of variables which will reduce the number of variables to $k(n-k)$.

We define an *inner region* of $\mathrm{Web}_{k,n}$ to be a bounded component of the complement of $\mathrm{Web}_{k,n}$ (viewed as a subset of $\mathbb{R}^2$). And we define an *outer region* of $\mathrm{Web}_{k,n}$ to be one of the extra inner regions we would obtain if we were to connect vertices i and $i+1$ by a straight line, for i from 1 to $n-1$. A *region* is an inner or outer region. Note that there are $k(n-k)$ regions, which we denote by R, and there are $(k-1)(n-k-1)$ inner regions.

Let us label each region $r \in R$ with a new variable x_r, which we define to be the product of its counterclockwise edge variables divided by the product of its clockwise edge variables.

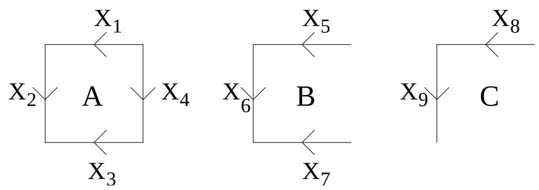


Figure 3-2: Labels for regions

For example, the new variables A, B, C shown in Figure 3-2 would be defined by

$$A := \frac{x_1 x_2}{x_3 x_4},\ B := \frac{x_5 x_6}{x_7},\ C := x_8 x_9.$$

It is easy to check that for a path p on $\mathrm{Web}_{k,n}$, $\mathrm{Prod}_p(x)$ is equal to the product of the variables attached to all regions below p. Since $\mathrm{Prod}_S(x)$ and $A_{k,n}(x)$ were defined in terms of the $\mathrm{Prod}_p(x)$'s, we can redefine these expressions in terms of the $k(n-k)$ region variables. Proposition 3.3.2 still holds, but our map is now a map Φ_1 from $(\mathbb{R}^+)^{k(n-k)}$ onto $Gr_{k,n}(\mathbb{R}^+)$, taking the region variables $\{x_r\}$ to the element of $Gr_{k,n}(\mathbb{R}^+)$ represented by $A_{k,n}(x)$.

Since we are now parameterizing a space of dimension $k(n-k)$ with $k(n-k)$ variables, we should have a bijection. We shall prove that this is so by constructing the inverse map.

Theorem 3.3.3. *The map $\Phi_1 : (\mathbb{R}^+)^{k(n-k)} \to Gr_{k,n}(\mathbb{R}^+)$, which maps $\{x_r\}_{r\in R}$ to the Grassmannian element represented by $A_{k,n}(x)$, is a bijection.*

Before we prove this theorem, we need a lemma about matrices and their minors. We use a very slight generalization of a lemma which appeared in [20]. For completeness, we include the proof of this lemma. First we must define some terminology. Let M be a $k \times n$ matrix. Let $\Delta_{I,J}$ denote the minor of M which uses row set I

and column set J. We say that $\Delta_{I,J}$ is *solid* if I and J consist of several consecutive indices; if furthermore $I \cup J$ contains 1, we say that $\Delta_{I,J}$ is *initial*. Thus, an initial minor is a solid minor which includes either the first column or the first row.

Lemma 3.3.4. *[20] A matrix M is uniquely determined by its initial minors provided that all these minors are nonzero.*

Proof. Let us show that each matrix entry x_{ij} of M is uniquely determined by the initial minors. If $i = 1$ or $j = 1$, there is nothing to prove, since x_{ij} is an initial minor. Assume that $\min(i, j) > 1$. Let Δ be the initial minor whose last row is i and last column is j, and let Δ' be the initial minor obtained from Δ by deleting this row and column. Then $\Delta = \Delta' x_{ij} + P$, where P is a polynomial in the matrix entries $x_{i'j'}$ with $(i', j') \neq (i, j)$ and $i' \leq i, j' \leq j$. Using induction on $i + j$, we can assume that each $x_{i'j'}$ that occurs in P is uniquely determined by the initial minors, so the same is true of $x_{ij} = (\Delta - P)/\Delta'$. ∎

We now define a *reflected initial* minor to be a solid minor $\Delta_{I,J}$ such that I contains k or J contains 1. Thus, a reflected initial minor is a solid minor which includes either the first column or the last row. A trivial corollary of Lemma 3.3.4 is the following.

Corollary 3.3.5. *A matrix M is uniquely determined by its reflected initial minors provided that all these minors are nonzero.*

Now we are ready to prove Theorem 3.3.3.

Proof. To prove the theorem, we will construct an explicit inverse map $\Psi : Gr_{k,n}(\mathbb{R}^+) \to (\mathbb{R}^+)^{k(n-k)}$. The first step is to prove that $\Psi\Phi_1 = \mathrm{id}$.

Let us index the regions in $\mathrm{Web}_{k,n}$ by ordered pairs (i, j) as follows. Given a region, we choose i to be the label of the horizontal wire which forms the upper boundary of the region, and choose j to be the label of the vertical wire which forms the left boundary of the region. Now we define a map K from the set of regions to $\binom{[n]}{k}$ by

$$K(i, j) := \{1, 2, \ldots, i-1\} \cup \{i+j-k, i+j-k+1, \ldots, j-1, j\}.$$

If (i, j) is not a region of $\mathrm{Web}_{k,n}$, then we define $K(i, j) := \emptyset$.

Let A be a $k \times n$ matrix whose initial $k \times k$ minor is the identity. We define $\Psi(A)$ by

$$(\Psi(A))_{(i,j)} := \frac{\Delta_{K(i,j)}(A)\Delta_{K(i+1,j-2)}(A)\Delta_{K(i+2,j-1)}(A)}{\Delta_{K(i,j-1)}(A)\Delta_{K(i+1,j)}(A)\Delta_{K(i+2,j-2)}(A)}. \tag{3.1}$$

Note that by convention, we define $\Delta_\emptyset$ to be 1.

See Figure 3-3 for the definition of Ψ in the case of $Gr_{3,6}(\mathbb{R}^+)$. Note that for brevity, we have omitted the A's from each term.

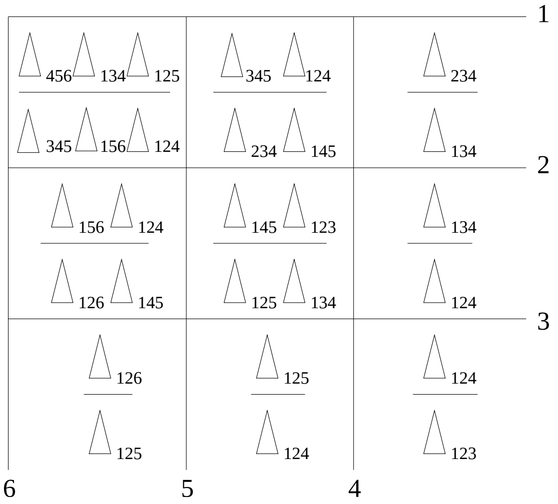


Figure 3-3: $\mathrm{Web}_{3,6}$

We claim that if $A = \Phi_1 x$, then $\Psi\Phi_1 x = x$. To prove this, we note that the variable in region (i, j) can be expressed in terms of vertex-disjoint paths as follows.

First observe that if $K(i, j) \neq \emptyset$ then there is a unique set of pairwise vertex-disjoint paths from $[k] \setminus ([k] \cap K(i, j))$ to $K(i, j) \setminus ([k] \cap K(i, j))$. If one examines the terms in (3.1) and draws in the six sets of pairwise vertex-disjoint paths on $\mathrm{Web}_{k,n}$ (say the three from the numerator in red and the three from the denominator in blue) then it is clear that every region in $\mathrm{Web}_{k,n}$ lies underneath an equal number of red and blue paths – except the region (i, j), which lies underneath only one red path. Thus, by definition of the maps P_K, it follows that

$$(\Psi\Phi_1(x))_{(i,j)} = \frac{P_{K(i,j)}(x)P_{K(i+1,j-2)}(x)P_{K(i+2,j-1)}(x)}{P_{K(i,j-1)}(x)P_{K(i+1,j)}(x)P_{K(i+2,j-2)}(x)} = x_{(i,j)}.$$

To complete the proof, it remains to show that Ψ is injective. This will complete the proof because we know that $\Psi\Phi_1\Psi = \Psi$, and Ψ injective then implies that $\Phi_1\Psi =$ id.

Choose an element of $Gr_{k,n}(\mathbb{R}^+)$, which we identify with its unique matrix representative A whose leftmost $k \times k$ minor is the identity. Let $\mathcal{V}$ denote the set of rational expressions which appear in the right-hand side of (3.1) for all regions (i, j) in $\mathrm{Web}_{k,n}$. Let $\mathcal{P}$ denote the set of all individual Plucker coordinates which appear in $\mathcal{V}$. We prove that Ψ is injective in two steps. First we show that the values of the expressions in $\mathcal{V}$ uniquely determine the values of the Plücker coordinates in $\mathcal{P}$. Next we show that the values of the Plücker coordinates in $\mathcal{P}$ uniquely determine the matrix A.

The first step is clear by inspection. We illustrate the proof in the case of $Gr_{3,6}(\mathbb{R}^+)$. By the choice of A, $\Delta_{123} = 1$. Looking at the rational expressions in Figure 3-3, we see that knowing the value $\frac{\Delta_{124}}{\Delta_{123}}$ determines Δ_{124}; the value Δ_{124} together with the value $\frac{\Delta_{134}}{\Delta_{124}}$ determines Δ_{134}; and similarly for $\Delta_{234}, \Delta_{125}, \Delta_{126}$. Next, these values together with the value $\frac{\Delta_{145}\Delta_{123}}{\Delta_{125}\Delta_{134}}$ determines Δ_{145}, and so on.

For the second step of the proof, let A' denote the $k \times (n - k)$ matrix obtained from A by removing the leftmost $k \times k$ identity matrix. Note that the values of the Plücker coordinates $\Delta_{K(i,j)}(A)$ (which are all elements of $\mathcal{P}$) determine the values of all of the reflected initial minors of A'. (Each such Plücker coordinate is equal to one of the reflected initial minors, up to sign.) Thus, by Corollary 3.3.5, they uniquely determine the matrix A' and hence A. This completes the proof of Theorem 3.3.3. $\square$

Now let us parameterize $Gr_{k,n}(\mathbb{R}^+)/\phi((\mathbb{R}^+)^n)$. We shall show that we can do this by using variables corresponding to only the $(k-1)(n-k-1)$ inner regions of $\mathrm{Web}_{k,n}$.

First recall that the n-dimensional torus acts on $Gr_{k,n}(\mathbb{R}^+)$ by scaling columns of a matrix representative for $A \in Gr_{k,n}(\mathbb{R}^+)$. (Although the torus has dimension n, this is actually just an $(n-1)$-dimensional action as the scalars act trivially.) Namely,

$$(\lambda_1, \ldots, \lambda_n) \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \ldots & a_{kn} \end{pmatrix} := \begin{pmatrix} \lambda_1 a_{11} & \ldots & \lambda_n a_{1n} \\ \vdots & & \vdots \\ \lambda_1 a_{k1} & \ldots & \lambda_n a_{kn} \end{pmatrix}$$

If $A \in Gr_{k,n}(\mathbb{R}^+)$ then we let $\overline{A}$ denote the torus orbit of A under this action. Note that if $K = \{i_1, \ldots, i_k\}$, then $\Delta_K(\lambda A) = \lambda_{i_1}\lambda_{i_2} \ldots \lambda_{i_k}\Delta_K(A)$.

We will now determine the corresponding torus action on $(\mathbb{R}^+)^{k(n-k)}$ such that the above bijection commutes with the actions. If r is an internal region then x_r is a ratio of Plücker coordinates with the same indices appearing on the top and bottom, so x_r is not modified by the torus action. A simple computation shows that the torus acts transitively on the values of the outer region variables. Thus, taking the quotient by $\phi((\mathbb{R}^+)^n)$ on the right hand side of the equation corresponds to forgetting the outer variables on the left.

Define a map $\Phi_2 : (\mathbb{R}^+)^{(k-1)(n-k-1)} \to Gr_{k,n}(\mathbb{R}^+)/\phi((\mathbb{R}^+)^n)$ by lifting a point $c \in (\mathbb{R}^+)^{(k-1)(n-k-1)}$ to any arbitrarily chosen point $\tilde{c} \in (\mathbb{R}^+)^{k(n-k)}$ and then mapping c to $\overline{\Phi_1(\tilde{c})}$. We have just proven:

Theorem 3.3.6. *The map $\Phi_2 : (\mathbb{R}^+)^{(k-1)(n-k-1)} \to Gr_{k,n}(\mathbb{R}^+)/\phi((\mathbb{R}^+)^n)$ is a bijection.*

3.4 A fan associated to the tropical positive Grassmannian

In this section we will construct a lower-dimensional fan associated to the tropical positive Grassmannian $\text{Trop}^+ Gr_{k,n}$. By methods precisely analogous to those above, we can prove an analogue of Theorem 3.3.6 for the field of Puiseux series.

Theorem 3.4.1. *The map $\Phi_2 : (\mathcal{R}^+)^{(k-1)(n-k-1)} \to Gr_{k,n}(\mathcal{R}^+)/\phi((\mathcal{R}^+)^n)$ is a bijection.*

This theorem allows us to compute $\operatorname{Trop}^+ Gr_{k,n}/(\operatorname{Trop}\phi)(\mathbb{R}^n)$ by applying the valuation map to the image of Φ_2. By Proposition 3.2.5 we can tropicalize the map Φ_2, obtaining the following surjective map.

$$\operatorname{Trop}\Phi_2 : \mathbb{R}^{(k-1)(n-k-1)} \to \operatorname{Trop}^+ Gr_{k,n}/(\operatorname{Trop}\phi)(\mathbb{R}^n)$$

The map $\operatorname{Trop}\Phi_2$ is the map we get by replacing multiplication with addition and addition with minimum in the definition of Φ_2. Explicitly, it is defined as follows. Let $K \in \binom{[n]}{k}$, and let inner region variables take on values $\{x_r\}$ in $\mathbb{R}$. Outer region variables are chosen arbitrarily. If p is a path on $\mathrm{Web}_{k,n}$ then let $\mathrm{Sum}_p(x)$ denote the sum of all variables which label regions below p. Similarly, if S is a set of paths, then let $\mathrm{Sum}_S(x)$ denote $\sum_{p\in S}\mathrm{Sum}_p(x)$. We define $\operatorname{Trop} P_K(x) : \mathbb{R}^{(k-1)(n-k-1)} \to \mathbb{R}$ by

$$\operatorname{Trop} P_K(x) := \min\{\mathrm{Sum}_S(x) : S \in \mathrm{Path}(K)\}.$$

The map $\operatorname{Trop}\Phi_2$ is the map

$$\operatorname{Trop}\Phi_2 : \mathbb{R}^{(k-1)(n-k-1)} \to \operatorname{Trop}^+ Gr_{k,n}/(\operatorname{Trop}\phi)(\mathbb{R}^n) \subset \mathbb{R}^N/(\operatorname{Trop}\phi)(\mathbb{R}^n)$$

given by

$$(\operatorname{Trop}\Phi_2(x))_K = P_K(x).$$

Definition 3.4.2. *The fan $F_{k,n}$ is the complete fan in $\mathbb{R}^{(k-1)(n-k-1)}$ whose maximal cones are the domains of linearity of the piecewise linear map* $\operatorname{Trop}\Phi_2$*.*

Because $\operatorname{Trop}\Phi_2$ surjects onto $\operatorname{Trop}^+ Gr_{k,n}/(\operatorname{Trop}\phi)(\mathbb{R}^n)$, the fan $F_{k,n}$ reflects the combinatorial structure of the fan $\operatorname{Trop}^+ Gr_{k,n}/(\operatorname{Trop}\phi)(\mathbb{R}^n)$, which differs from $\operatorname{Trop}^+ Gr_{k,n}$ only through modding out by the linearity space. However, $F_{k,n}$ is much easier to work with, as it lives in $(k-1)(n-k-1)$-dimensional space as opposed to $\binom{n}{k}$-dimensional space.

Since the maps $\operatorname{Trop} P_K$ are piecewise linear functions, to each one we can associate a fan $F(P_K)$ whose maximal cones are the domains of linearity for $\operatorname{Trop} P_K$. It is clear that the fan $F_{k,n}$ is the simultaneous refinement of all of the fans $F(P_K)$.

From now on, we will refer to $\operatorname{Trop}^+ Gr_{k,n}/(\operatorname{Trop}\phi)(\mathbb{R}^n)$ as $\operatorname{Trop}^+ Gr_{k,n}$.

3.5 $\operatorname{Trop}^+ Gr_{2,n}$ and the associahedron

In this section we will describe the fan $F_{2,n}$ associated to $\operatorname{Trop}^+ Gr_{2,n}$. We show that this fan is exactly the Stanley-Pitman fan F_{n-3}, which appeared in the work of Stanley and Pitman in [44]. In particular, the face poset of $F_{2,n}$, with a top element $\hat{1}$ adjoined, is isomorphic to the face lattice of the normal fan of the associahedron, a polytope whose vertices correspond to triangulations of the convex n-gon. (In the language of [11], this is the associahedron of type A_{n-3}.)

Let us first do the example of $\operatorname{Trop}^+ Gr_{2,5}$.

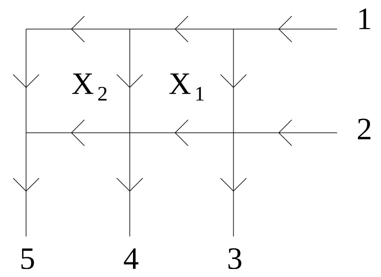


Figure 3-4: $\mathrm{Web}_{2,5}$

We use the web diagram $\mathrm{Web}_{2,5}$, as shown in Figure 3-4. The maps $\operatorname{Trop} P_K$ are given by:

$$\operatorname{Trop} P_{1j} = 0 \text{ for all } j$$
$$\operatorname{Trop} P_{23} = 0$$
$$\operatorname{Trop} P_{24} = \min(x_1, 0)$$
$$\operatorname{Trop} P_{25} = \min(x_1 + x_2, x_1, 0)$$
$$\operatorname{Trop} P_{34} = x_1$$
$$\operatorname{Trop} P_{35} = \min(x_1, x_1 + x_2)$$
$$\operatorname{Trop} P_{45} = x_1 + x_2$$

Each map $\operatorname{Trop} P_K : \mathbb{R}^2 \to \mathbb{R}$ is piecewise linear and so gives rise to the complete fan $F(P_K)$. For example, the map Trop P_{24} is linear on the region $\{(x_1, x_2) : x_1 \geq 0\}$, where it is the function $(x_1, x_2) \mapsto 0$, and on the region $\{(x_1, x_2) : x_1 \leq 0\}$, where it is the function $(x_1, x_2) \mapsto x_1$. Thus, $F(P_{24})$ is simply the subdivision of the real plane into the regions $x_1 \geq 0$ and $x_1 \leq 0$. The three nontrivial fans that we get from the maps Trop P_J are shown in Figure 3-5. In each picture, the maximal cones of each fan are separated by solid lines. $F_{2,5}$, which is the simultaneous refinement of the three nontrivial fans, is shown in Figure 3-6.

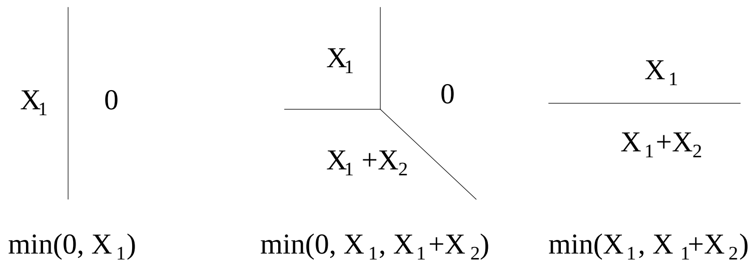


Figure 3-5: Fans for Trop P_J

In [40], it was shown that maximal cones of the fan Trop $Gr_{2,n}$ correspond to trivalent trees on n labelled leaves. It turns out that maximal cones of the fan Trop $Gr^+_{2,n}$ correspond to trivalent planar trees on n labelled leaves, as is illustrated in Figure 3-6.

We will now describe the fan that appeared in [44], but first, we must review some notions about trees. A *plane binary tree* is a rooted tree such that each vertex has either two children designated as left and right, or none at all; and an *internal vertex* of a binary tree is a vertex which is not a leaf. A *trivalent planar tree* is an (unrooted) tree such that every vertex has degree three, and such that the leaves are labelled in a clockwise fashion. It is known that both plane binary trees with $n-1$ leaves, and trivalent planar trees with n labelled leaves, are counted by the Catalan number $c_{n-2} = \frac{1}{n-1}\binom{2(n-2)}{n-2}$.

There is a simple bijection between such trivalent planar trees and plane binary trees: if T a trivalent planar tree, then simply contract the edge whose leaf is labelled 1, and make this the root. This bijection is illustrated in Figure 3-6.

Let us now define the Stanley-Pitman fan F_{n-3} in $\mathbb{R}^{n-3}$. (Note that we use different

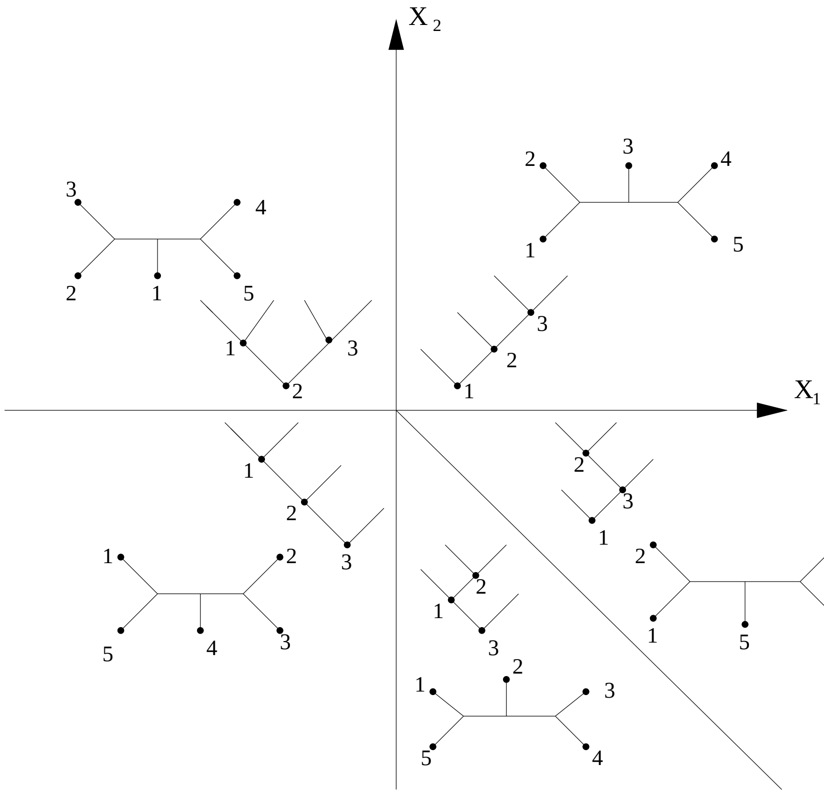


Figure 3-6: The fan of $\text{Trop}^+ Gr_{2,5}$

indices than are used in [44]). The maximal cones of F_{n-3} are indexed by plane binary trees with $n-1$ leaves, in the following manner. Let T be a plane binary tree with $n-1$ leaves. Label the internal vertices of T with the numbers $1, 2, \ldots n-2$ in the order of the first time we drop down to them from a child when doing a depth-first search from left to right starting at the root. (See Figure 3-6 for examples.) Let $x_1, \ldots, x_{n-3}$ denote the coordinates in $\mathbb{R}^{n-3}$. If the internal vertex i of T is the parent of vertex j, and $i < j$, then associate with the pair (i, j) the inequality

$$x_i + \cdots + x_{j-1} \geq 0,$$

while if $i > j$ then associate with (i, j) the inequality

$$x_i + \cdots + x_{j-1} \leq 0.$$

These $n-3$ inequalities define a simplicial cone C_T in $\mathbb{R}^{n-3}$.

The result proved in [44] is the following.

Theorem 3.5.1 ([44])**.** *The c_{n-2} cones C_T, as T ranges over all plane binary trees with $n-1$ leaves, form the chambers of a complete fan in $\mathbb{R}^{n-3}$. Moreover, the face poset of F_{n-3}, with a top element $\hat{1}$ adjoined, is dual to the face lattice of the associahedron which parameterizes triangulations of the convex n-gon.*

The key step in proving that our fan $F_{2,n}$ is equal to the Stanley-Pitman fan F_{n-3} is the following lemma, also proved in [44].

Lemma 3.5.2 ([44])**.** *Let $D_i = \{(x_1, \dots, x_{n-3}) \in \mathbb{R}^{n-3} : x_1 + \cdots + x_{i-1} = \min(0, x_1, x_1 + x_2, \dots, x_1 + \cdots + x_{n-3})\}$. Let $\mathcal{T}_i$ consist of all plane binary trees with $n-1$ leaves and root i. Then $D_i = \cup_{T \in \mathcal{T}_i} C_T$.*

Proposition 3.5.3. *The fan $F_{2,n}$ is equal to the fan F_{n-3}.*

Proof. First let us describe the fan $F_{2,n}$ as explicitly as possible. Note that if we label the regions of $\mathrm{Web}_{2,n}$ with the variables $x_1, \dots, x_{n-3}$ from right to left, then all of the maps $\operatorname{Trop} P_K$ are of the form

$$\min(x_1 + x_2 + \cdots + x_i, x_1 + x_2 + \cdots + x_{i+1}, \dots, x_1 + x_2 + \cdots + x_j),$$

where $0 \le i \le j \le n$. Since this map has the same domains of linearity as the map

$$\theta_{ij} := \min(x_i, x_i + x_{i+1}, \dots, x_i + \cdots + x_j),$$

we can work with the maps θ_{ij} instead. Let $F(i,j)$ be the fan whose cones are the domains of linearity of θ_{ij}. Then $F_{2,n}$ is the simultaneous refinement of all fans $F(i,j)$ where $0 \le i \le j \le n$.

Now note that the previous lemma actually gives us an algorithm for determining which cone C_T a generic point $(x_1, \dots, x_{n-3}) \in \mathbb{R}^{n-3}$ lies in. Namely, if we are given such a point, compute the partial sums of $x_1 + \cdots + x_{i-1}$, for $1 \le i \le n-2$. Choose i such that $x_1 + \cdots + x_{i-1}$ is the minimum of these sums. (If $i = 1$, the sum is 0.) Then the root of the tree T is i. The left subtree of T consists of vertices $\{1, \dots, i-1\}$, and the right subtree of T consists of vertices $\{i+1, \ldots, n-2\}$. We now compute $\min\{0, x_1, x_1+x_2, \ldots, x_1+\cdots+x_{i-2}\}$ and $\min\{x_i, x_i+x_{i+1}, \ldots, x_i+\cdots+x_{n-3}\}$ in order to compute the roots of these two subtrees and so forth.

Now take a point $(x_1, \ldots, x_{n-3})$ in a cone C of $F_{2,n}$. This means that the point is in a domain of linearity for all of the piecewise linear functions $\theta_{ij} = \min\{x_i, x_i + x_{i+1}, \ldots, x_i + \cdots + x_j\}$ where $0 \leq i \leq j \leq n$, and we take x_0 to be 0. In other words, for each i and j, there is a unique k such that $x_i + \cdots + x_k = \min\{x_i, x_i + x_{i+1}, \ldots, x_i + \cdots + x_j\}$. In particular, we can reconstruct the tree T such that $(x_1, \ldots, x_{n-3}) \in C_T$, and *every* point $x \in C$ belongs to this same cone C_T.

Finally, we can show by induction that $C_T \subset C$. (We need to show that all of the functions θ_{ij} are actually linear on C_T.) This shows that each cone C in $F_{2,n}$ is actually equal to a cone C_T in F_{n-3}, and conversely.

□

3.6 $\operatorname{Trop}^+ Gr_{3,6}$ and the type D_4 associahedron

In connection with their work on cluster algebras, Fomin and Zelevinsky [23] recently introduced certain polytopes called *generalized associahedra* corresponding to each Dynkin type, of which the usual associahedron is the type A example. When we computed $F_{3,6}$, the fan associated with $\operatorname{Trop}^+ Gr_{3,6}$, we found that it was closely related to the normal fan of the type D_4 associahedron, in a way which we will now make precise. (We defer the explanation of our computations to the end of this section.)

Proposition 3.6.1. *The f-vector of $F_{3,6}$ is $(16, 66, 98, 48)$. The rays of $F_{3,6}$ are listed in Table 3.1, along with the inequalities defining the polytope that $F_{3,6}$ is normal to.*

Using the formulas of [23], we calculated the f-vector of the normal fan to the type D_4 associahedron: it is $(16, 66, 100, 50)$. More specifically, our fan has two cones which are of the form of a cone over a bipyramid. (Type FFFGG in the language of [40].) If we subdivide these two bipyramids into two tetrahedra each, then we get precisely the D_4 associahedron.

e_1	$x_1 \leq 5$
e_2	$x_2 \leq 7$
e_3	$x_3 \leq 7$
e_4	$x_4 \leq 10$
$-e_1$	$-x_1 \leq 0$
$-e_2$	$-x_2 \leq -2$
$-e_3$	$-x_3 \leq -2$
$-e_4$	$-x_4 \leq -5$
$e_1 - e_2$	$x_1 - x_2 \leq 0$
$e_1 - e_3$	$x_1 - x_3 \leq 0$
$e_1 - e_4$	$x_1 - x_4 \leq -1$
$-e_1 + e_4$	$-x_1 + x_4 \leq 9$
$e_2 - e_4$	$x_2 - x_4 \leq 0$
$e_3 - e_4$	$x_3 - x_4 \leq 0$
$e_1 - e_2 - e_3$	$x_1 - x_2 - x_3 \leq -3$
$e_2 + e_3 - e_4$	$x_2 + x_3 - x_4 \leq 6$

Table 3.1: Rays and inequalities for $F_{3,6}$

In Section 3.8, we will give some background on cluster algebras and formulate a conjecture which explains the relation of $F_{3,6}$ to the normal fan to the type D_4 associahedron.

We depict the intersection of $F_{3,6}$ with a sphere in Figures 3-7 and 3-8. Each of the figures is homeomorphic to a solid torus, and the two figures glue together to form the sphere S^3. The bipyramids in question have vertices $\{e_2+e_3-e_4, -e_1, e_2, e_3, -e_1+e_4\}$ and $\{e_1 - e_2 - e_3, -e_4, e_1 - e_2, e_1 - e_3, e_1 - e_4\}$.

Now we will explain how we computed $F_{3,6}$. We used two methods: the first method was to use computer software (we used both cdd+ and Polymake) to compute the fan which we described in Section 3.4. The second method was to figure out which subfan of Trop $Gr_{3,6}$ (which was explicitly described in [40]) was positive.

To implement our first method, we used the well-known result that if F_1 and F_2 are fans which are normal to polytopes Q_1 and Q_2, then the fan which is the refinement of F_1 and F_2 is normal to the Minkowski sum of Q_1 and Q_2. Since the fan $F_{k,n}$ is the simultaneous refinement of all the fans $F(P_K)$, we found explicit coordinates for polytopes $Q(P_K)$ whose normal fans were the fans $F(P_K)$, and had the programs cdd+ and Polymake compute the Minkowski sum $Q_{k,n}$ of all of these polytopes. We then got explicit coordinates for the fan which was normal to the resulting polytope.

For the second method, we used the results in [40]: we checked which of the rays of $\operatorname{Trop} Gr_{3,6}$ did *not* lie $\operatorname{Trop}^+ Gr_{3,6}$, and checked which facets of $\operatorname{Trop} Gr_{3,6}$ *did* lie in $\operatorname{Trop}^+ Gr_{3,6}$. As $\operatorname{Trop}^+ Gr_{3,6}$ is a closed subfan of $\operatorname{Trop} Gr_{3,6}$, this implied that every face of $\operatorname{Trop} Gr_{3,6}$ which lay in a totally positive facet was in $\operatorname{Trop}^+ Gr_{3,6}$ and every face of $\operatorname{Trop} Gr_{3,6}$ which contained a non-totally positive ray was not in $\operatorname{Trop}^+ Gr_{3,6}$; for every face of $\operatorname{Trop} Gr_{3,6}$, this proved sufficient to determine whether it was in $\operatorname{Trop}^+ Gr_{3,6}$ or not.

3.7 $\operatorname{Trop}^+ Gr_{3,7}$ and the type E_6 associahedron

As in the case of $F_{3,6}$, we used computer software to compute $F_{3,7}$, the fan associated to $\operatorname{Trop}^+ Gr_{3,7}$.

Proposition 3.7.1. *The f-vector of $F_{3,7}$ is* $(42, 392, 1463, 2583, 2163, 693)$. *Its rays are listed in Table 3.2, along with the inequalities defining the polytope that $F_{3,7}$ is normal to. Of the facets of this fan,* 595 *are simplicial,* 63 *have 7 vertices,* 28 *have 8 vertices and* 7 *have 9 vertices. All faces not of maximal dimension are simplicial.*

Using the formulas of [23], we calculated the f-vector of the fan normal to the type E_6 associahedron: it is $(42, 399, 1547, 2856, 2499, 833)$.

In Section 3.8, we will explain why $F_{3,7}$ differs from the E_6 fan, and how one can refine $F_{3,7}$ to get a fan combinatorially equivalent to the fan dual to the type E_6 associahedron. In this refinement, the simplicial facets remain facets. The 7, 8 and 9 vertex facets split into 2, 3 and 4 simplices respectively. The following table shows how the vertices of the 7, 8, and 9 vertex facets are grouped into simplices.

$$
\begin{aligned}
< ABCDEFG > &\implies < ABCDEF > \cup < ABCDEG > \\
< ABCDEFGH > &\implies < ABCDEF > \cup < ABCDFG > \\
&\qquad\qquad\qquad\qquad \cup < ABCDGH > \\
< ABCDEFGHI > &\implies < ABCDEF > \cup < ABCEFG > \cup \\
&\quad\;\; < ABCFGH > \cup < ABCGHI >
\end{aligned}
$$

3.8 Cluster Algebras

Cluster algebras are commutative algebras endowed with a certain combinatorial structure, introduced in [21] and expected to be relevant in studying total positivity and homogeneous spaces, such as Grassmannians.

We will not attempt to give a precise definition of a cluster algebra here, but will rather describe their key properties. Slightly varying definitions can be found in [21], [22] and [38]; we follow [38] but do not believe these small variations are important.

A cluster algebra is an algebra $\mathcal{A}$ over a field k, which in our examples can be thought of as $\mathbb{R}$. Additionally, a cluster algebra carries two subsets C and $X \subset \mathcal{A}$, known as the coefficient variables and the cluster variables. C is finite, but X may be finite or infinite. If X is finite, $\mathcal{A}$ is known as a cluster algebra of finite type. There is also a nonnegative integer r associated to a cluster algebra and known as the rank of the algebra.

There is a pure $(r-1)$-dimensional simplicial complex called the *cluster complex* whose vertices are the elements of X and whose maximal simplices are called *clusters*. We will denote the cluster complex by $S(\mathcal{A})$. If $x \in X$ and $\Delta \in S(\mathcal{A})$ is a cluster containing x, there is always a unique cluster Δ' with $\Delta \cap \Delta' = \Delta \setminus \{x\}$. Let $\Delta' = (\Delta \setminus \{x\}) \cup \{x'\}$. Then there is a relation $xx' = B$ where B is a binomial in the variables of $(\Delta \cap \Delta') \cup C$.

For any $x \in X$ and any cluster Δ, x is a subtraction-free rational expression in the members of $\Delta \cup C$ and is also a Laurent polynomial in the members of $\Delta \cup C$. Conjecturally, this Laurent polynomial has non-negative coefficients. Note that this conjecture does not follow from the preceding sentence: $\frac{x^3+y^3}{x+y} = x^2 - xy + y^2$ is a subtraction-free expression in x and y, and a polynomial in x and y, but it is not a polynomial with positive coefficients.

It was demonstrated in [38] that the coordinate rings of Grassmannians have natural cluster algebra structures. Usually these cluster algebras are of infinite type, making them hard to work with in practice, but in the cases of $Gr_{2,n}$, $Gr_{3,k}$ for $k \leq 8$ and their duals, we get cluster algebras of finite type.

In the case of $Gr_{2,n}$, the coefficient set C is $\{\Delta_{12}, \Delta_{23}, \ldots, \Delta_{(n-1)n}, \Delta_{1n}\}$ and the set of cluster variables X is $\{\Delta_{ij} : i < j \text{ and } i - j \not\equiv \pm 1 \mod n\}$. (Note that these Δ's are Plücker coordinates and not simplices.) Label the vertices of an n-gon in clockwise order with the indices $\{1, 2, \cdots, n\}$ and associate to each member of $X \cup C$ the corresponding chord of the n-gon. The clusters of $Gr_{2,n}$ correspond to the collections of chords which triangulate the n-gon. Thus, $S(\mathcal{A})$ in this example is (as an abstract simplicial complex) isomorphic to the dual of the associahedron. Since we have shown that the fan of $\mathrm{Trop}^+ Gr_{2,n}$ is combinatorially equivalent to the normal fan of the associahedron, it follows that $\mathrm{Trop}^+ Gr_{2,n}$ is (combinatorially) the cone on $S(\mathcal{A})$.

In the case of $Gr_{3,6}$, the coefficient set C is equal to $\{\Delta_{123}, \Delta_{234}, \Delta_{345}, \Delta_{456}, \Delta_{561}, \Delta_{612}\}$. X contains the other 14 Plücker coordinates, but it also contains two unexpected elements: $\Delta_{134}\Delta_{256} - \Delta_{156}\Delta_{234}$ and $\Delta_{236}\Delta_{145} - \Delta_{234}\Delta_{156}$. By definition, all Plücker coordinates are positive on the totally positive Grassmannian, so by the results above on subtraction-free rational expressions, these new coordinates are positive on the totally positive Grassmannian as well.

The new coordinates turn out to be Laurent polynomials with positive coefficients in the region variables of Section 3.3. Thus, we can tropicalize these Laurent polynomials and associate a fan to each of them. When we refine $F_{3,6}$ by these fans, the refinement subdivides the two bipyramids and yields precisely the normal fan to the D_4 associahedron, which is again the cone over $S(\mathcal{A})$.

In the case of $Gr_{3,7}$, C again consists of $\{\Delta_{i(i+1)(i+2)}\}$ where indices are modulo 7. X contains all of the other Plücker variables and the pullbacks to $Gr_{3,7}$ of the two new cluster variables of $Gr_{3,6}$, along with the 7 rational coordinate projections $Gr_{3,7} \to Gr_{3,6}$. Thus, X contains 28 Plücker variables and 14 other variables.

As in the case of $Gr_{3,6}$, the 14 new variables are Laurent polynomials with positive coefficients in the region variables of Section 3.3, so to each one we can associate a corresponding fan. When we refine $F_{3,7}$ by these 14 new fans, we get a fan combinatorially equivalent to the fan normal to the E_6 associahedron.

We can describe what we have seen in each of these Grassmannian examples in terms of the general language of cluster algebras as follows:

Observation when $\mathcal{A}$ is the Coordinate Ring of a Grassmannian. *Embed* $\operatorname{Spec} \mathcal{A}$ *in affine space by the variables $X \sqcup C$. Then* $\operatorname{Trop}^+ \operatorname{Spec} \mathcal{A}$ *is a fan with lineality space of dimension $|C|$. After taking the quotient by this lineality space, we get a simplical fan abstractly isomorphic to the cone over $S(\mathcal{A})$.*

This observation does not quite hold for an arbitrary cluster algebra of finite type. For example, if we take the cluster algebra of $Gr_{2,6}$ and set all coefficient variables equal to 1, we get a different cluster algebra which is still of type A_3. However, when we compute the positive part of the corresponding tropical variety, we get a fan whose lineality space has dimension 1, not 0 as the above would predict. Our fan is a cone over a hexagon cross a 1-dimensional lineality space, which is a coarsening of the fan normal to the type A_3 associahedron. Based on this and other small examples, it seems that in order to see the entire cone over $S(\mathcal{A})$, one needs to use "enough" coefficients.

Conjecture 3.8.1. *Let $\mathcal{A}$ be a cluster algebra of finite type over $\mathbb{R}$ and $S(\mathcal{A})$ its associated cluster complex. If the lineality space of* $\operatorname{Trop}^+ \operatorname{Spec} \mathcal{A}$ *has dimension $|C|$ then* $\operatorname{Trop}^+ \operatorname{Spec} \mathcal{A}$ *modulo its lineality space is a simplicial fan abstractly isomorphic to the cone over $S(\mathcal{A})$. If the condition on the lineality space does not hold, the resulting fan is a coarsening of the cone over $S(\mathcal{A})$.*

Remark: The condition on the lineality space can be restated without mentioning tropicalizations. Consider the torus $(\mathbb{R}^*)^{X \cup C}$ acting on the affine space $\mathbb{R}^{X \cup C}$ and let G be the subgroup taking $\operatorname{Spec} \mathcal{A}$ to itself. We want to require that $\dim G = |C|$.

Remark: In the notation of [21] and [22], the condition on the dimension of the lineality space is equivalent to requiring that the matrix $\tilde{B}$ be of full rank. We thank Andrei Zelevinsky for pointing this out to us.

Note how surprising this conjecture is in light of how the two complexes are computed. The fan described in the conjecture is computed as the refinement of a number of fans, indexed by the vertices of $S(\mathcal{A})$. That the rays of this fan, which arise as the intersections of many hypersurfaces, should again be in bijection with the vertices of $S(\mathcal{A})$ is quite unexpected.

We expect an analogous statement to hold for infinite type cluster algebras.

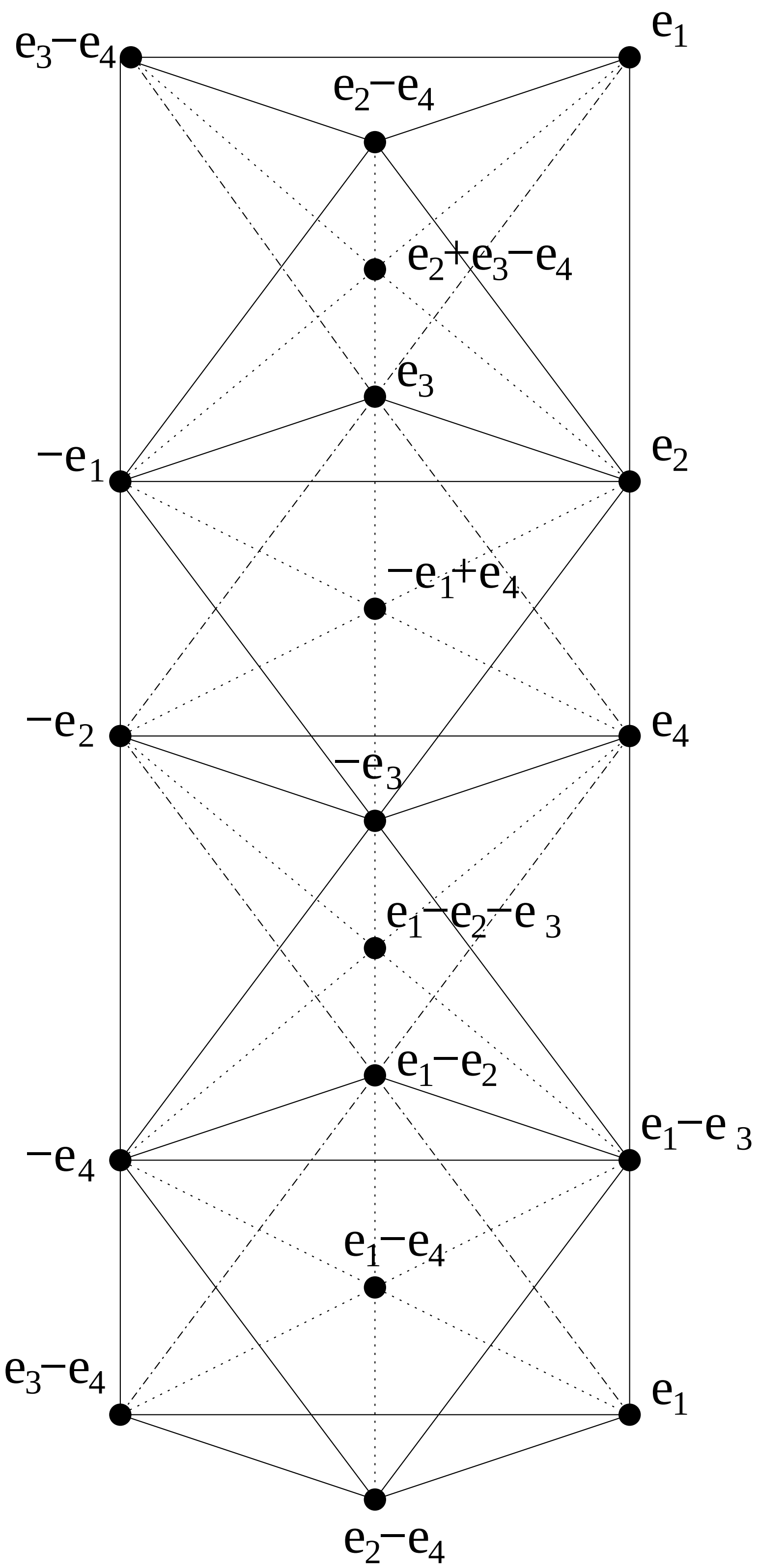


Figure 3-7: The intersection of $F_{3,6}$ with a sphere (solid torus)

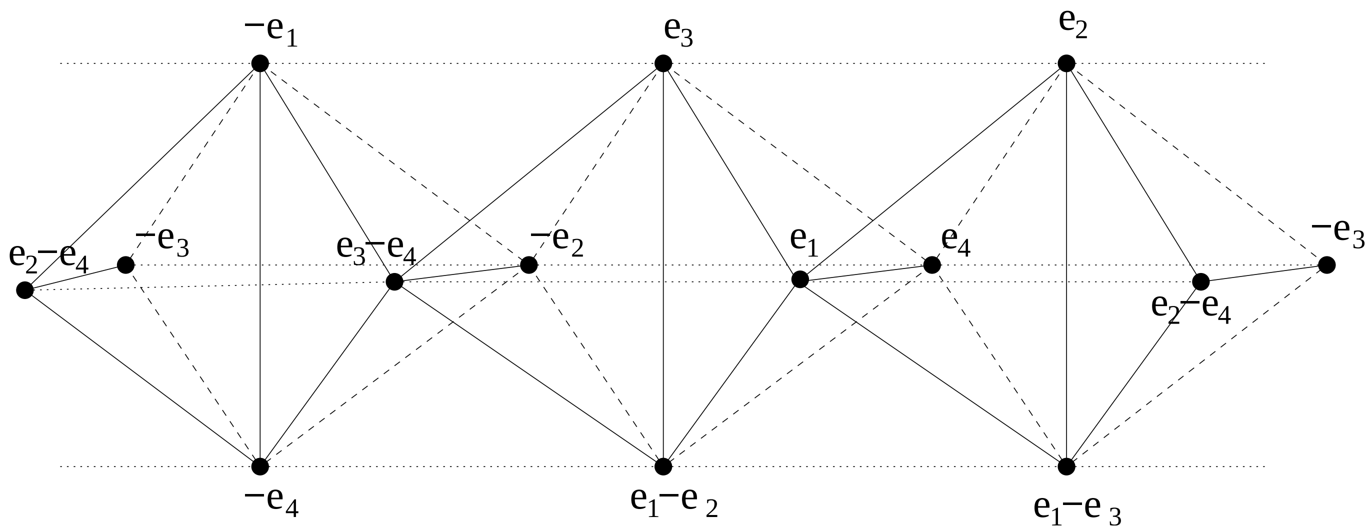


Figure 3-8: The intersection of $F_{3,6}$ with a sphere

e_1	$x_1 \leq 10$
e_2	$x_2 \leq 16$
e_3	$x_3 \leq 19$
e_4	$x_4 \leq 14$
e_5	$x_5 \leq 26$
e_6	$x_6 \leq 35$
$-e_1$	$-x_1 \leq -1$
$-e_2$	$-x_2 \leq -4$
$-e_3$	$-x_3 \leq -10$
$-e_4$	$-x_4 \leq -5$
$-e_5$	$-x_5 \leq -17$
$-e_6$	$-x_6 \leq -26$
$e_1 - e_2$	$x_1 - x_2 \leq -1$
$e_1 - e_3$	$x_1 - x_3 \leq -4$
$e_1 - e_4$	$x_1 - x_4 \leq -1$
$e_1 - e_5$	$x_1 - x_5 \leq -7$
$e_1 - e_6$	$x_1 - x_6 \leq -17$
$e_2 - e_3$	$x_2 - x_3 \leq -1$
$e_2 - e_5$	$x_2 - x_5 \leq -4$
$e_2 - e_6$	$x_2 - x_6 \leq -12$
$e_3 - e_6$	$x_3 - x_6 \leq -10$
$e_4 - e_5$	$x_4 - x_5 \leq -5$
$e_4 - e_6$	$x_4 - x_6 \leq -14$
$e_5 - e_6$	$x_5 - x_6 \leq -5$
$-e_1 + e_5$	$-x_1 + x_5 \leq 23$
$-e_2 + e_5$	$-x_2 + x_5 \leq 21$
$-e_2 + e_6$	$-x_2 + x_6 \leq 28$
$e_2 + e_4 - e_5$	$x_2 + x_4 - x_5 \leq 8$
$e_2 + e_4 - e_6$	$x_2 + x_4 - x_6 \leq 1$
$e_3 + e_4 - e_6$	$x_3 + x_4 - x_6 \leq 3$
$e_3 + e_5 - e_6$	$x_3 + x_5 - x_6 \leq 13$
$e_1 - e_2 + e_6$	$x_1 - x_2 + x_6 \leq 33$
$e_1 - e_2 - e_4$	$x_1 - x_2 - x_4 \leq -7$
$e_1 - e_3 - e_4$	$x_1 - x_3 - x_4 \leq -12$
$e_1 - e_3 - e_5$	$x_1 - x_3 - x_5 \leq -19$
$e_2 - e_3 - e_5$	$x_2 - x_3 - x_5 \leq -17$
$-e_1 + e_5 - e_6$	$-x_1 + x_5 - x_6 \leq -7$
$e_1 + e_2 - e_3 - e_5$	$x_1 + x_2 - x_3 - x_5 \leq -9$
$e_2 + e_4 - e_5 - e_6$	$x_2 + x_4 - x_5 - x_6 \leq -19$
$e_1 - e_2 - e_4 + e_6$	$x_1 - x_2 - x_4 + x_6 \leq 26$
$e_2 - e_3 + e_4 - e_5$	$x_2 - x_3 + x_4 - x_5 \leq -7$
$-e_1 + e_3 + e_5 - e_6$	$-x_1 + x_3 + x_5 - x_6 \leq 11$

Table 3.2: Rays and inequalitiesfor $F_{3,7}$

Chapter 4

The Positive Bergman Complex of an Oriented Matroid

This chapter is based on joint work with Federico Ardila and Carly Klivans [3].

4.1 Introduction

Bergman [6] defined the *logarithmic limit-set* of an algebraic variety in order to study its exponential behavior at infinity. We follow [46] in calling this set the *Bergman complex* of the variety. Bergman complexes have recently received considerable attention in several areas, such as tropical algebraic geometry and dynamical systems. They are the *non-Archimedean amoebas* of [15] and the *tropical varieties* of [40].

When the variety is a linear space, so that the defining ideal I is generated by linear forms, Sturmfels [46] showed that the Bergman complex can be described solely in terms of the matroid associated to the linear ideal. He used this description to define the Bergman complex $\mathcal{B}(M)$ of an arbitrary matroid M. Ardila and Klivans [2] showed that, appropriately subdivided, the Bergman complex of a matroid M is the order complex of the proper part of the lattice of flats L_M of the matroid. This result implies that the Bergman complex of an arbitrary matroid M is a finite, pure polyhedral complex, which is homotopy equivalent to a wedge of spheres.

Sturmfels [47] suggested the notion of a positive Bergman complex $\mathcal{B}^+(M)$ of an oriented matroid M and conjectured its relation to the Las Vergnas face lattice of M. We define the positive Bergman complex and positive Bergman fan so that given a linear ideal I with associated oriented matroid M_I, the positive tropical variety associated to I is equal to the positive Bergman fan of M_I.

We prove that appropriately subdivided, $\mathcal{B}^+(M)$ is a geometric realization of the order complex of the proper part of the Las Vergnas face lattice of M. $\mathcal{B}^+(M)$ is homeomorphic to a sphere and naturally sits inside $\mathcal{B}(\underline{M})$, the Bergman complex of the underlying unoriented matroid of M. We conclude by showing that, for the oriented matroid of the complete graph K_n, the face poset of a certain "coarse" subdivision of $\mathcal{B}^+(K_n)$ is dual to the face poset of the associahedron A_{n-2}.

The chapter is organized as follows. In Section 4.2 we introduce a certain oriented matroid M_ω which will play an important role in our work. In Section 4.3 we define the positive Bergman complex and prove our main theorem. In Section 4.4 we explain the relation between the positive Bergman complex of an oriented matroid and the positive tropical variety of a linear ideal. In Section 4.5 we describe the topology of the positive Bergman complex of an oriented matroid. Finally, in Sections 4.6 and 4.7 we describe in detail the positive Bergman complex of the oriented matroid of K_n: we relate it to the associahedron, and we give a formula for the number of full-dimensional fine cells within a full-dimensional coarse cell.

Throughout this chapter we will abuse notation and use M to denote either a matroid or oriented matroid, depending on the context. Similarly, we will use the term "circuits" to describe either unsigned or signed circuits. When the distinction between matroids and oriented matroids is important, we will use $\underline{M}$ to denote the underlying matroid of an oriented matroid M.

4.2 The Oriented Matroid M_w

Let M be an oriented matroid on the ground set $[n] = \{1, 2, \ldots, n\}$ whose collection of signed circuits is $\mathcal{C}$. Let $\omega \in \mathbb{R}^n$ and regard ω as a weight function on $[n]$. For any circuit $C \in \mathcal{C}$ define $\mathrm{in}_\omega(C)$ to be the ω-maximal subset of the circuit C – in other words, the collection of elements of C which have the largest weight. We will say that the circuit C *achieves its largest weight with respect to* ω at $\mathrm{in}_\omega(C)$. Define $\mathrm{in}_\omega(\mathcal{C})$ to be the collection of inclusion-minimal sets of the collection $\{\mathrm{in}_\omega(C) \mid C \in \mathcal{C}\}$. We then define M_ω to be the oriented matroid on $[n]$ whose collection of circuits is $\mathrm{in}_\omega(\mathcal{C})$.

It is not clear that M_ω is a well-defined oriented matroid; we will prove this shortly.

Given $\omega \in \mathbb{R}^n$, let $\mathcal{F}(\omega)$ denote the unique flag of subsets $\emptyset = F_0 \subset F_1 \subset \cdots \subset F_k \subset F_{k+1} = [n]$ such that ω is constant on each set $F_i \setminus F_{i-1}$ and satisfies $\omega|_{F_i \setminus F_{i-1}} < \omega|_{F_{i+1} \setminus F_i}$ for all $1 \le i \le k$. We call $\mathcal{F}(\omega)$ the *flag* of ω, and we say that the *weight class* of ω or of the flag $\mathcal{F}$ is the set of vectors ν such that $\mathcal{F}(\nu) = \mathcal{F}$.

It is clear that M_ω depends only on the flag $\mathcal{F} := \mathcal{F}(\omega)$ and so we also refer to this oriented matroid as $M_{\mathcal{F}}$.

Example 4.2.1. *Let M be the oriented matroid of the digraph D shown in Figure 4-1. Equivalently, let M be the oriented matroid of the point configuration shown in Figure 4-2.*

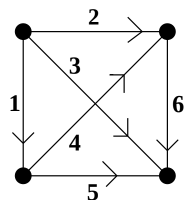


Figure 4-1: The digraph D

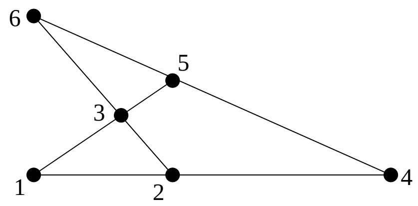


Figure 4-2: A point configuration

Note that D is an acyclic orientation of K_4, the complete graph on 4 vertices. The signed circuits $\mathcal{C}$ of M are $\{1\bar{2}4, 1\bar{3}5, 2\bar{3}6, 4\bar{5}6, 1\bar{2}5\bar{6}, 1\bar{3}46, 2\bar{3}4\bar{5}\}$, together with the negatives of every set of this collection. Choose ω such that $\omega_6 < \omega_1 = \omega_2 = \omega_3 = \omega_4 = \omega_5$, which corresponds to the flag $\emptyset \subset \{6\} \subset \{1, 2, 3, 4, 5, 6\}$.

If we calculate $\mathrm{in}_\omega(C)$ for each $C \in \mathcal{C}$ we get $\{1\bar{2}4, 1\bar{3}5, 2\bar{3}, 4\bar{5}, 1\bar{2}5, 1\bar{3}4, 2\bar{3}\bar{4}5\}$, together with the negatives of every set of this collection. However, $2\bar{3}\bar{4}5$ is not inclusion-minimal in this collection, as it contains $2\bar{3}$ and $\bar{4}5$. Thus $\mathrm{in}_\omega(\mathcal{C})$ is equal to $\{1\bar{2}4, 1\bar{3}5, 2\bar{3}, 4\bar{5}, 1\bar{2}5, 1\bar{3}4\}$, together with the negatives of every set, and M_ω is the oriented matroid whose collection of signed circuits is $\mathrm{in}_\omega(\mathcal{C})$. Notice that in this case M_ω is the oriented matroid of the digraph D' in Figure 4-3.

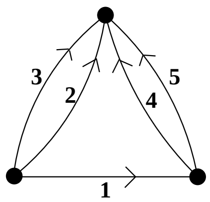


Figure 4-3: The digraph D'.

We must show that M_ω is well-defined. For convenience, we review here the *circuit axioms for oriented matroids* [9]:

C1. $\emptyset$ is not a signed circuit.

C2. If X is a signed circuit, then so is $-X$.

C3. No proper subset of a circuit is a circuit.

C4. If X_0 and X_1 are circuits with $X_1 \neq -X_0$ and $e \in X_0^+ \cap X_1^-$, then there is a third circuit $X \in \mathcal{C}$ with $X^+ \subseteq (X_0^+ \cup X_1^+) \setminus \{e\}$ and $X^- \subseteq (X_0^- \cup X_1^-) \setminus \{e\}$.

We will also need the following stronger characterization of oriented matroids:

Theorem 4.2.2. *[9, Theorem 3.2.5] Let $\mathcal{C}$ be a collection of signed subsets of a set E satisfying* C1, C2, C3. *Then* C4 *is equivalent to* C4′*:*

C4′. for all $X_0, X_1 \in \mathcal{C}$, $e \in X_0^+ \cap X_1^-$ and $f \in (X_0^+ \setminus X_1^-) \cup (X_0^- \setminus X_1^+)$, there is a $Z \in \mathcal{C}$ such that $Z^+ \subseteq (X_0^+ \cup X_1^+) \setminus \{e\}$, $Z^- \subseteq (X_0^- \cup X_1^-) \setminus \{e\}$, and $f \in Z$.

Proposition 4.2.3. *Let M be an oriented matroid on $[n]$ and $\omega \in \mathbb{R}^n$. Then M_ω is an oriented matroid.*

Proof. The strategy of our proof is to show that if C1, C2, C3 and C4′ hold for M, then C1, C2, C3 and C4 hold for M_ω. First note that it is obvious that C1 and C2 hold for M_ω. C3 holds for M_ω because we defined $\mathrm{in}_\omega(\mathcal{C})$ to consist of *inclusion-minimal* elements. It remains to show that C4 holds for M_ω. To do this, we start with two circuits in M_ω, lift them to circuits in M, and then use C4′ for M to show that C4 holds for M_ω.

Take Y_0 and Y_1 in $\mathrm{in}_\omega(\mathcal{C})$ such that $Y_1 \neq -Y_0$ and $e \in Y_0^+ \cap Y_1^-$. By definition, there exist circuits X_0 and X_1 of M such that $Y_0 = \mathrm{in}_\omega(X_0)$ and $Y_1 = \mathrm{in}_\omega(X_1)$. Notice that the presence of e in Y_0 and Y_1 guarantees that the maximum weights occurring in X_0 and in X_1 are both equal to ω_e.

Choose any $f \in (Y_0^+ \setminus Y_1^-) \cup (Y_0^- \setminus Y_1^+)$. Clearly such an f exists. Then $f \in (X_0^+ \setminus X_1^-) \cup (X_0^- \setminus X_1^+)$. By C4′ for M, there exists a circuit X in $\mathcal{C}$ such that

(a_1) $X^+ \subseteq (X_0^+ \cup X_1^+) \setminus \{e\}$,

(a_2) $X^- \subseteq (X_0^- \cup X_1^-) \setminus \{e\}$, and

(a_3) $f \in X$.

Look at $\mathrm{in}_\omega(X)$. We will prove that $\mathrm{in}_\omega(X)$ contains the third circuit of M_ω which we are looking for. We want to show that

(b_1) $\mathrm{in}_\omega(X)^+ \subseteq (Y_0^+ \cup Y_1^+) \setminus \{e\}$

(b_2) $\mathrm{in}_\omega(X)^- \subseteq (Y_0^- \cup Y_1^-) \setminus \{e\}$

First, it is obvious that e is not in $\mathrm{in}_\omega(X)^+$, since e was not in X^+. Clearly $\mathrm{in}_\omega(X)^+$ is a subset of $X_0^+ \cup X_1^+$. To show (b_1) and (b_2), we just need to show that the maximum weight which occurs in X is also equal to ω_e. By (a_1) and (a_2), this maximum weight is at most ω_e. By (a_3), equality is attained for $f \in X$, since $\omega_f = \omega_e$.

Note that if $\mathrm{in}_\omega(X)$ is not inclusion-minimal in the set $\{\mathrm{in}_\omega(C) \mid C \text{ a circuit of } M\}$, then it contains some inclusion-minimal $\mathrm{in}_\omega(W)$ for another circuit W of M. And since $\mathrm{in}_\omega(X)^+ \subseteq (Y_0^+ \cup Y_1^+) \setminus \{e\}$ and $\mathrm{in}_\omega(X)^- \subseteq (Y_0^- \cup Y_1^-) \setminus \{e\}$, it is clear that we also have $\mathrm{in}_\omega(W)^+ \subseteq (Y_0^+ \cup Y_1^+) \setminus \{e\}$ and $\mathrm{in}_\omega(W)^- \subseteq (Y_0^- \cup Y_1^-) \setminus \{e\}$.

□

In the following proposition, we describe the bases of M_ω and their orientations. Let us say that if $S \subset [n]$ and $\omega \in \mathbb{R}^n$, the *ω-weight* of S is the sum $\sum_{i \in S} \omega_i$.

Proposition 4.2.4. *The bases of M_ω are the bases of M which have minimal ω-weight. The basis orientations of M_ω are equal to their orientations in M.*

Proof. We know that if N is an oriented matroid on $[n]$ with signed circuits $\mathcal{C}$, then the bases of N are the maximal subsets of $[n]$ which contain no circuit. Thus, the bases $\mathcal{B}$ of M_ω are the maximal subsets of $[n]$ which do not contain a set of the collection $\{\mathrm{in}_\omega(C) \mid C \text{ a circuit of } M\}$. We want to show that $\mathcal{B}$ is exactly the set of bases of M which have minimal ω-weight.

First let us choose a basis B of M which has minimal ω-weight. We claim that B is independent in M_ω. Suppose that B contains a subset of the form $\mathrm{in}_\omega(C)$, where C is a circuit of M. Write $B = \mathrm{in}_\omega(C) \cup \{b_1, \dots, b_m\}$, and choose any $c \in \mathrm{in}_\omega(C)$. We now construct a new basis B'' of M of smaller weight than B, as follows. Start with the set C. We know that $C \cup \{b_1, \dots, b_m\}$ is a spanning set of M. Let B_0 be a minimal subset (possibly empty) of $\{b_1, \dots, b_m\}$ such that $C \cup B_0$ is still a spanning set of M. Since C is minimally dependent in M, the set $B' := (C \setminus \{c\}) \cup B_0$ will be a basis of M. Now by the basis exchange axiom, for some $b \in B' \setminus B$, $B'' := (B \setminus \{c\}) \cup \{b\}$ is a basis of M. Since $b \in B' \setminus B$, it follows that $b \in C \setminus \mathrm{in}_\omega(C)$. Thus, the weight of b is strictly less than the weight of c, which implies that the weight of B'' is less than the weight of B. This is a contradiction.

The previous argument shows that $r(M) \leq r(M_\omega)$. But every circuit of M_ω is contained in a circuit of M, so $r(M) \geq r(M_\omega)$. It follows that M and M_ω have the same rank.

Now let B be a basis of M_ω, *i.e.* B is a maximal subset of $[n]$ which does not contain an element of $\{\mathrm{in}_\omega(C) \mid C \text{ a circuit of } M\}$. We claim that B is a basis of M with minimal ω-weight. First note that B is clearly independent in M: if it were dependent in M, it would contain some circuit C of M and hence would contain $\mathrm{in}_\omega(C)$. Also, B has $r(M_\omega) = r(M)$ elements. Therefore it is a basis of M.

Finally, let us show that B has minimal ω-weight. Suppose not. Let $c_1, \dots, c_r$ be the elements of B with highest weight. We claim that $\{c_1, \dots, c_r\} \supseteq \mathrm{in}_\omega(C)$ for some circuit C of M, which will be a contradiction. Since B is a basis of M, adding to B any element of $[n] \setminus B$ creates a circuit. Since B is not a basis of minimal ω-weight, there must be an element $b \in [n] \setminus B$ such that the weight of b is strictly less than the weight of each of the elements $c_1, \dots, c_r$. Thus $B \cup b$ contains a circuit C, and $\mathrm{in}_\omega(C) \subseteq \{c_1, \dots, c_r\}$, as claimed.

To prove the claim about orientations, start with a basis B of minimal weight of M. Recall that an oriented matroid has exactly two basis orientations, which are opposite to each other. Therefore we can assume without loss of generality that B has the same orientation in M and M_ω.

For any two ordered bases $B_1 = (e, x_2, \dots, x_r)$ and $B_2 = (f, x_2 \dots, x_r)$ of M_ω with $e \neq f$, we have

$$\chi_\omega(e, x_2, \dots, x_r) = -C_\omega(e)C_\omega(f)\chi_\omega(f, x_2 \dots, x_r),$$

where χ_ω is the chirotope of M_ω, and C_ω is one of the two opposite signed circuits of M_ω in $\{e, f, x_2, \dots, x_r\}$. Now B_1 and B_2 are also bases of M; let C be one of the two opposite signed circuits of M in $\{e, f, x_2, \dots, x_r\}$. Then $\mathrm{in}_\omega(C)$ contains a circuit of M_ω; it must be either C_ω or $-C_\omega$. In any case, we have $C(e)C(f) = C_\omega(e)C_\omega(f)$, so $\chi_\omega(B_1)\chi_\omega(B_2) = \chi(B_1)\chi(B_2)$. It follows that if B_1 has the same orientation in M and M_ω, then so does B_2.

Recall that one can obtain any basis of a matroid from any other by a sequence of simple basis exchanges of the type above. Since B has the same orientation in M and M_ω, so does any other basis of M_ω. $\square$

4.3 The Positive Bergman Complex

Our goal in this section is to define the positive Bergman complex of an oriented matroid M and to relate it to the Las Vergnas face lattice of M, thus answering Sturmfels' question [47]. We begin by giving some background on the Bergman complex and fan of a (unoriented) matroid.

The *Bergman fan* of a matroid M on the ground set $[n]$ is the set

$$\widetilde{\mathcal{B}}(M) := \{\omega \in \mathbb{R}^n \ : \ M_\omega \text{ has no loops}\}.$$

The *Bergman complex* of M is

$$\mathcal{B}(M) := \{\omega \in S^{n-2} \ : \ M_\omega \text{ has no loops}\},$$

where S^{n-2} is the sphere $\{\, \omega \in \mathbb{R}^n \ : \ \omega_1 + \cdots + \omega_n = 0\,,\, \omega_1^2 + \cdots + \omega_n^2 = 1\}$.

For simplicity, in this section we will concentrate on the Bergman complex of M, but similar arguments hold for the Bergman fan of M.

Since the matroid M_ω depends only on the weight class that ω is in, the Bergman complex of M is a disjoint union of the weight classes of flags $\mathcal{F}$ such that $M_{\mathcal{F}}$ has no loops. We say that the weight class of a flag $\mathcal{F}$ is *valid* for M if $M_{\mathcal{F}}$ has no loops.

There are two polyhedral subdivisions of $\mathcal{B}(M)$, one of which is clearly finer than the other. The *fine subdivision* of $\mathcal{B}(M)$ is the subdivision of $\mathcal{B}(M)$ into valid weight classes: two vectors u and v of $\mathcal{B}(M)$ are in the same class if and only if $\mathcal{F}(u) = \mathcal{F}(v)$. The *coarse subdivision* of $\mathcal{B}(M)$ is the subdivision of $\mathcal{B}(M)$ into M_ω-equivalence classes: two vectors u and v of $\mathcal{B}(M)$ are in the same class if and only if $M_u = M_v$.

The following results give alternative descriptions of $\mathcal{B}(M)$:

Theorem 4.3.1. *[2] Given an (unoriented) matroid M on the ground set $[n]$ and $\omega \in \mathbb{R}^n$ which corresponds to a flag $\mathcal{F} := \mathcal{F}(\omega)$, the following are equivalent:*

1. *$M_{\mathcal{F}}$ has no loops.*

2. *For each circuit C of M, $\mathrm{in}_\omega(C)$ contains at least two elements of C.*

3. *$\mathcal{F}$ is a flag of flats of M.*

Corollary 4.3.2. *[2] Let M be a (unoriented) matroid. Then the fine subdivision of the Bergman complex $\mathcal{B}(M)$ is a geometric realization of $\Delta(L_M - \{\,\hat{0}\,,\hat{1}\,\}\,)$, the order complex of the proper part of the lattice of flats of M.*

We are now ready for the positive analogues of these concepts. The *positive Bergman fan* of an oriented matroid M on the ground set $[n]$ is

$$\widetilde{\mathcal{B}}^{+}(M) := \{\omega \in \mathbb{R}^n \ : \ M_\omega \text{ is acyclic}\}.$$

The *positive Bergman complex* of M is

$$\mathcal{B}^{+}(M) := \{\omega \in S^{n-2} \ : \ M_\omega \text{ is acyclic}\}.$$

Within each equivalence class of the coarse subdivision of $\mathcal{B}(M)$, the vectors ω give rise to the same unoriented M_ω. Since the orientation of M_ω is inherited from that of M, they also give rise to the same oriented matroid M_ω. Therefore each coarse cell of $B(M)$ is either completely contained in or disjoint from $\mathcal{B}^+(M)$. Thus $\mathcal{B}^+(M)$ inherits the coarse and the fine subdivisions from $\mathcal{B}(M)$, and each subdivision of $\mathcal{B}^+(M)$ is a subcomplex of the corresponding subdivision of $\mathcal{B}(M)$.

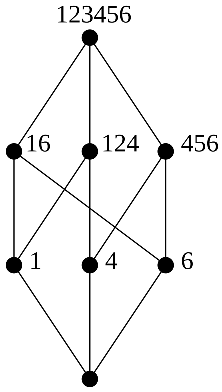
Let M be an acyclic oriented matroid on the ground set $[n]$. We say that a covector $v \in \{+, -, 0\}^n$ of M is *positive* if each of its entries is $+$ or 0. We say that a flat of M is *positive* if it is the 0-set of a positive covector. Additionally, we consider the set $[n]$ to be a positive flat. For example, if M is the matroid of Example 4.2.1, then 16 is a positive flat which is the 0-set of the positive covector $(0++++0)$.

The *Las Vergnas face lattice* $\mathcal{F}_{\ell v}(M)$ is the lattice of positive flats of M, ordered by containment. Note that the lattice of positive flats of the oriented matroid M sits inside L_M, the lattice of flats of M.

Example 4.3.3. *Let M be the oriented matroid from Example 4.2.1. The positive covectors of M are* $\{0++++0, 00+0++, +++000, 0+++++, +++0++, +++++0, ++++++\}$ *and the positive flats are* $\{16, 124, 456, 1, 4, 6, \emptyset, 123456\}$*. The lattice of positive flats and the lattice of positive flats of M are shown in Figure 4-4.*

We now give an analogue of Theorem 4.3.1.

Theorem 4.3.4. *Given an oriented matroid M and $\omega \in \mathbb{R}^n$ which corresponds to a flag $\mathcal{F} := \mathcal{F}(\omega)$, the following are equivalent:*



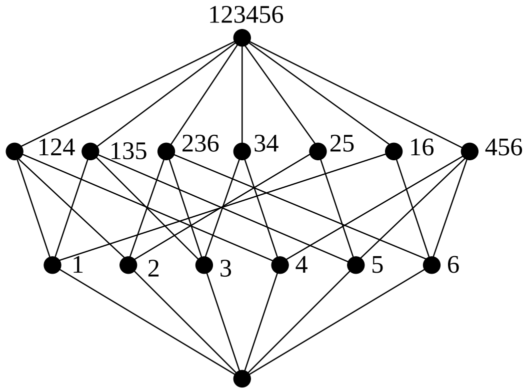


Figure 4-4: The lattice of positive flats and the lattice of flats.

1. *$M_{\mathcal{F}}$ is acyclic.*

2. *For each signed circuit C of M, $\mathrm{in}_\omega(C)$ contains a positive element and a negative element of C.*

3. *$\mathcal{F}$ is a flag of positive flats of M.*

Proof. First we will show that 1 and 2 are equivalent. The statement that M_ω is acyclic means that M_ω has no all-positive circuit: in other words, each circuit of M_ω contains a positive and a negative term. Since M_ω is the matroid whose circuits are the inclusion-minimal elements of the set $\{\mathrm{in}_\omega(C) \mid C \text{ a circuit of } M\}$, this means that for each circuit C of M, $\mathrm{in}_\omega C$ contains a positive and a negative term. Finally, this is equivalent to the statement that for each circuit C of M, C achieves its maximum value with respect to ω on both C^+ and C^-.

Next we show that 3 implies 2. Assume we have an ω such that $\mathcal{F}$ is a flag of positive flats. Let the flats of this flag be $F_1 \subset F_2 \subset \ldots \subset F_k$. For each F_i, $([n]-F_i)$ is a positive covector. By orthogonality of circuits and covectors, we know that for any circuit C and any covector Y, $(C^+ \cap Y)$ and $(C^- \cap Y)$ are either both empty or both non-empty. For any circuit C of M, consider the largest i such that $C \cap ([n] - F_i)$ is non-empty. Then clearly C will attain its maximum on $F_{i+1} - F_i$ and $\mathrm{in}_\omega(C)$ contains a positive element and a negative element of C.

Finally, assume that 1 and 2 hold, but 3 does not. From 1 we know that $M_{\mathcal{F}}$ is acyclic; therefore the unoriented $M_{\mathcal{F}}$ has no loops, and $\mathcal{F}$ is a flag of flats by Theorem

4.3.1. Let F_i be a flat which is not positive; by [9, Proposition 9.1.2] this is equivalent to saying that M/F_i is not acyclic. Let C be a positive circuit of M/F_i; then we can find a circuit X of M such that $C = X - F_i$. Then X has positive elements of weight greater than ω_i, and no negative elements of weight greater than ω_i. It follows that $\mathrm{in}_\omega(X)$ is positive, contradicting 2.

□

Corollary 4.3.5. *Let M be an oriented matroid. Then the fine subdivision of $\mathcal{B}^+(M)$ is a geometric realization of $\Delta(\mathcal{F}_{\ell v}(M) - \{\hat{0}, \hat{1}\})$, the order complex of the proper part of the Las Vergnas face lattice of M.*

4.4 Connection with Positive Tropical Varieties

In [41], the notion of the positive part of the tropicalization of an affine variety (or positive tropical variety, for short) was introduced, an object which has the structure of a polyhedral fan in $\mathbb{R}^n$. In order to describe this object, we must define an *initial ideal.*

Let $\mathcal{R} = \mathbb{R}[x_1, \ldots, x_n]$ and $\omega \in \mathbb{R}^n$. If $f = \sum c_i \mathbf{x}^{\mathbf{a_i}} \in \mathcal{R}$, define the *initial form* $\mathrm{in}_\omega(f) \in \mathcal{R}$ to be the sum of all terms $c_i \mathbf{x}^{\mathbf{a_i}}$ such that the inner product $\omega \cdot \mathbf{a_i}$ is maximal. For an ideal I of $\mathbb{R}[x_1, \ldots, x_n]$, define the *initial ideal* $\mathrm{in}_\omega(I)$ to be the ideal generated by $\mathrm{in}_\omega(f)$ for all $f \in I$.

If I is an ideal in a polynomial ring with n variables, the positive tropical variety associated to I is denoted by $\mathrm{Trop}^+ V(I)$ and can be characterized as follows:

$$\mathrm{Trop}^+ V(I) = \{\omega \in \mathbb{R}^n \mid \mathrm{in}_\omega(I) \text{ contains no nonzero polynomials in } \mathbb{R}^+[x_1, \ldots, x_n]\}.$$

Now recall that if I is a linear ideal (an ideal generated by linear forms), we can associate to it an oriented matroid M_I as follows. Write each linear form $f \in I$ in the form $a_1 x_{i_1} + a_2 x_{i_2} + \cdots + a_m x_{i_m} = b_1 x_{j_1} + b_2 x_{j_2} + \cdots + b_n x_{j_n}$, where $a_i, b_i > 0$ for all i. We then define M_I to be the oriented matroid whose set of signed circuits consists of all minimal collections of the form $\{i_1 i_2 \ldots i_m \bar{j}_1 \bar{j}_2 \ldots \bar{j}_n\}$. We now prove the following easy statement.

Proposition 4.4.1. *If I is a linear ideal and M_I is the associated oriented matroid, then* $\operatorname{Trop}^+ V(I) = \widetilde{B}^+(M_I)$.

Proof.

$$\begin{aligned}
\widetilde{B}^+(M_I) &= \{\omega \in \mathbb{R}^n \mid (M_I)_\omega \text{ is acyclic}\} \\
&= \{\omega \in \mathbb{R}^n \mid (M_I)_\omega \text{ has no all-positive circuit}\} \\
&= \{\omega \in \mathbb{R}^n \mid M_{\mathrm{in}_\omega(I)} \text{ has no all-positive circuit}\} \\
&= \{\omega \in \mathbb{R}^n \mid \mathrm{in}_\omega(I) \text{ contains no nonzero polynomial in } \mathbb{R}^+[x_1, \ldots, x_n]\} \\
&= \operatorname{Trop}^+ V(I).
\end{aligned}$$

□

4.5 Topology of the Positive Bergman Complex

The topology of the positive Bergman complex of an oriented matroid is very simple: it is homeomorphic to a sphere. This follows from Corollary 4.3.5 together with results about the Las Vergnas face lattice, which we will review here.

Theorem 4.5.1. *[9, Theorem 4.3.5] Let M be an acyclic oriented matroid of rank r. Then the Las Vergnas lattice $\mathcal{F}_{\ell v}(M)$ is isomorphic to the face lattice of a PL regular cell decomposition of the $(r-2)$-sphere.*

Proposition 4.5.2. *[9, Proposition 4.7.8] Let Δ be a regular cell complex. Then its geometric realization is homeomorphic to the geometric realization of the order complex of its face poset.*

The previous two results imply that the geometric realization of the order complex of the Las Vergnas lattice is homeomorphic to a sphere.

Putting this together with Corollary 4.3.5, we get the following result.

Corollary 4.5.3. *The positive Bergman complex of an oriented matroid is homeomorphic to a sphere.*

4.6 The positive Bergman complex of the complete graph

In this section, we wish to describe the positive Bergman complex $\mathcal{B}^+(K_n)$ of the graphical oriented matroid $M(K_n)$ of an acyclic orientation of the complete graph K_n. We start by reviewing the description of the Bergman complex $\mathcal{B}(K_n)$ of the unoriented matroid $M(K_n)$, obtained in [2]. For the moment we need to consider K_n as an unoriented graph.

An *equidistant n-tree* T is a rooted tree with n leaves labeled $1, \ldots, n$, and lengths assigned to each edge in such a way that the total distances from the root to each leaf are all equal. The internal edges are required to have positive lengths. Figure 4-5 shows an example of an equidistant 4-tree.

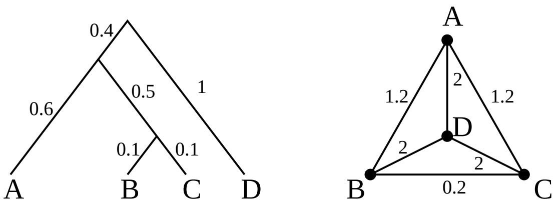


Figure 4-5: An equidistant tree and its corresponding distance vector.

To each equidistant n-tree T we assign a distance vector $d_T \in \mathbb{R}^{\binom{n}{2}}$: the distance d_{ij} is equal to the length of the path joining leaves i and j in T. Figure 4-5 also shows the distance vector of the tree, regarded as a weight function on the edges of K_4.

The Bergman fan $\widetilde{\mathcal{B}}(K_n)$ can be regarded as a space of equidistant n-trees, as the following theorem shows.

Theorem 4.6.1. *[2, 39] The distance vector of an equidistant n-tree, when regarded as a weight function on the edges of K_n, is in the Bergman fan $\widetilde{\mathcal{B}}(K_n)$. Conversely, any point in $\widetilde{\mathcal{B}}(K_n)$ is the distance vector of a unique equidistant n-tree.*

As mentioned earlier, the fine subdivision of $\widetilde{\mathcal{B}}(M)$ is well understood for any matroid M. The following theorem shows that the coarse subdivision of $\widetilde{\mathcal{B}}(K_n)$ also has a nice description: it is a geometric realization of the well-studied simplicial complex of trees T_n, sometimes called the Whitehouse complex [7, 37].

Theorem 4.6.2. *[2] Let $\omega, \omega' \in \widetilde{\mathcal{B}}(K_n)$. Let T and T' be the corresponding equidistant n-trees. The following are equivalent:*

1. *ω and ω' are in the same cell of the coarse subdivision.*

2. *T and T' have the same combinatorial type.*

Now we return to the setting of oriented matroids. The positive Bergman complex $\mathcal{B}^+(K_n)$ is defined in terms of an acyclic orientation of K_n. This graph has $n!$ acyclic orientations, corresponding to the $n!$ permutations of $[n]$. The orientation corresponding to the permutation π is given by $\pi_i \to \pi_j$ for $i < j$. Clearly the $n!$ orientations of K_n will give rise to positive Bergman complexes which are equal up to relabeling. Therefore, throughout this section, the edges of K_n will be oriented $i \to j$ for $i < j$.

As we go around a cycle C of K_n, C^+ is the set of edges which are crossed in the forward direction, and C^- is the set of edges which are crossed in the backward direction.

Proposition 4.6.3. *Let ω be a weight vector on the edges of the oriented complete graph K_n. Let T be the corresponding equidistant tree. The following are equivalent:*

1. *ω is in $\mathcal{B}^+(K_n)$.*

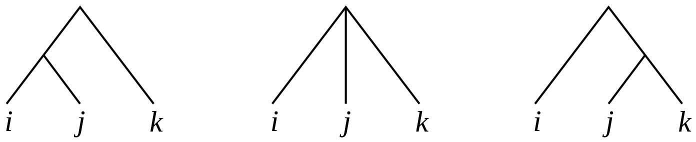
2. *T can be drawn in the plane without crossings in such a way that its leaves are numbered $1, 2, \ldots, n$ from left to right.*

Proof. We add three intermediate steps to the equivalence:

(a) *In any cycle C, the ω-maximum is achieved in C^+ and C^-.*

(b) *In any triangle C, the ω-maximum is achieved in C^+ and C^-.*

(c) *For any three leaves $i < j < k$ in T, the leaf j does not branch off before i and k;* i.e.,*their branching order is one of the following:*



The equivalence $1 \Leftrightarrow (a)$ follows from Theorem 4.3.4, and the implication $(a) \Rightarrow (b)$ is trivial. Now we show that $(b) \Rightarrow (a)$. Proceed by contradiction. Consider a cycle $C = v_1 \dots v_k$, with k minimal, such that (a) is not satisfied. Consider the cycles $T = v_1 v_{k-1} v_k$ and $C' = v_1 v_2 \dots v_{k-1}$, which do satisfy (a). Since C does not satisfy (a), the edge $v_1 v_{k-1}$ must be ω-maximum in T, along with another edge e of the opposite orientation. Similarly, the edge $v_{k-1} v_1$ must be ω-maximum in C', along with another edge f of the opposite orientation. Therefore, in C, the edges e and f are ω-maximum and have opposite orientations. This is a contradiction.

Let us now show $(b) \Leftrightarrow (c)$. In triangle ijk (where we can assume $i < j < k$), (b) holds if and only if we have one of the following:

$$\omega_{ij} < \omega_{jk} = \omega_{ik}, \quad \text{or} \quad \omega_{ij} = \omega_{jk} = \omega_{ik}, \quad \text{or} \quad \omega_{jk} < \omega_{ij} = \omega_{ik}.$$

These three conditions correspond, in that order, to the three possible branching orders of i, j and k in T prescribed by condition (c).

Finally we show $(c) \Leftrightarrow 2$. The backward implication is immediate. We prove the forward implication by induction on n. The case $n = 3$ is clear. Now let $n \geq 4$, and assume that condition (c) holds. Consider a lowest internal node v; it is incident to several leaves, which must have consecutive labels $i, i+1, \dots, j$ by (c). Let T' be the tree obtained from T by removing leaf i. This smaller tree satisfies (c), so it can be drawn in the plane with the leaves in order from left to right. Now we simply find node v in this drawing, and attach leaf i to it, putting it to the left of all the other leaves incident to v. This is a drawing of T satisfying 2. $\square$

The *associahedron* A_{n-2} is a well-known $(n-2)$-dimensional polytope whose vertices correspond to planar rooted trees [49]. There is a close relationship between $\mathcal{B}^+(K_n)$ and A_{n-2}.

Corollary 4.6.4. *The face poset of the coarse subdivision of $\mathcal{B}^+(K_n)$, with a $\widehat{1}$ attached, is dual to the face poset of the associahedron A_{n-2}.*

Proof. In the trees corresponding to the cells of $\mathcal{B}^+(K_n)$, the labeling of the leaves always increases from left to right. We can forget these labels and obtain the usual presentation of the dual to the associahedron, whose facets correspond to planar rooted trees. □

Figure 4-6 shows the positive Bergman complex of K_4 (in bold) within the Bergman complex of K_4. Vertices of the coarse subdivision are shown as black circles; vertices of the fine subdivision but not the coarse subdivision are shown as transparent circles. Observe that the coarse subdivision of $\mathcal{B}^+(K_4)$ is a pentagon, whose face poset is the face poset of the associahedron A_2 (which is self-dual).

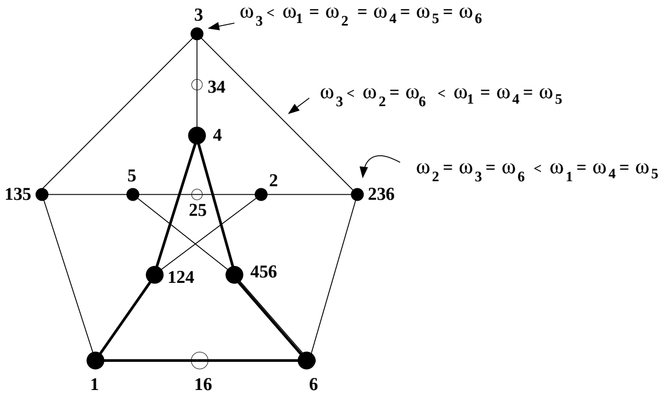


Figure 4-6: $\mathcal{B}^+(K_4) \subset \mathcal{B}(K_4)$

Now, recall that different orientations of K_n give rise to different positive Bergman complexes. Let us make two comments about the way in which these positive Bergman complexes fit together.

Consider the $n!$ different acyclic orientations $o(\pi)$ of K_n, each corresponding to a permutation π of $[n]$. Each orientation $o(\pi)$ gives rise to a positive Bergman complex: it consists of those weight vectors such that the corresponding tree can be drawn with the leaves labeled $\pi_1, \ldots, \pi_n$ from left to right. Clearly, each permutation and its reverse give the same positive Bergman complex. The $\frac{n!}{2}$ possible positive Bergman complexes $\mathcal{B}^+(K_n)$ give a covering of $\mathcal{B}(K_n)$, and each one of them is dual to the associahedron A_{n-2}. This corresponds precisely to the known covering of the space of trees with $\frac{n!}{2}$ polytopes dual to the associahedron, as described in [7].

Also, recall from [2] that the Bergman complex $\mathcal{B}(K_n)$ is homotopic to a wedge of $(n-1)!$ spheres. In fact, $\mathcal{B}(K_n)$ is covered by the $(n-1)!$ dual associahedra corresponding to the permutations π with $\pi_1 = 1$, because every tree can be drawn in the plane so that the leftmost leaf is labeled 1. This covering is optimal, since $\mathcal{B}(K_n)$ is homotopic to a wedge of $(n-1)!$ spheres.

4.7 The number of fine cells in $\mathcal{B}^+(K_n)$ and $\mathcal{B}(K_n)$.

Since $\mathcal{B}^+(K_n)$ and $\mathcal{B}(K_n)$ are $(n-2)$-dimensional, we will call the $(n-2)$-dimensional cells inside them *full-dimensional*. In this section we will give a formula reminiscent of the "hook-length" formula for the number of full-dimensional fine cells within a full-dimensional coarse cell of $\mathcal{B}(K_n)$.

Proposition 4.7.1. *Let τ be a rooted binary tree with n labeled leaves. For each internal vertex v of τ, let $d(v)$ be the number of internal vertices of τ which are descendants of v, including v. Let $C(\tau)$ be the coarse cell of $\mathcal{B}(K_n)$ corresponding to tree τ. There are exactly*

$$\frac{(n-1)!}{\prod_v d(v)}$$

full-dimensional fine cells in $C(\tau)$.

Proof. The cell $C(\tau)$ consists of the distance vectors $d \in \mathbb{R}^{\binom{n}{2}}$ of all equidistant n-trees T of combinatorial type τ. Notice that $d_{ij} = 2h - 2h(v)$, where v is the lowest common ancestor of leaves i and j in T, $h(v)$ is the distance from v to the root of T, and h is the distance from the root of T to any of its leaves.

To specify a full-dimensional fine cell in $C(\tau)$, one needs to specify the relative order of the d_{ij}'s. Equivalently, in the tree T that d comes from, one needs to specify the relative order of the heights of the internal vertices, consistently with the combinatorial type of tree τ. Therefore, the fine cells in $C(\tau)$ correspond to the labellings of the $n-1$ internal vertices of τ with the numbers $1, 2, \ldots, n-1$, such that the label of each vertex is smaller than the labels of its offspring. In the language of [42, Sec. 1.3], these are precisely the *increasing binary trees* of type τ', where τ' is the result of removing the leaves of tree τ, and the edges incident to them. Figure 4-7 shows a tree type τ and one of the increasing binary trees of type τ'.

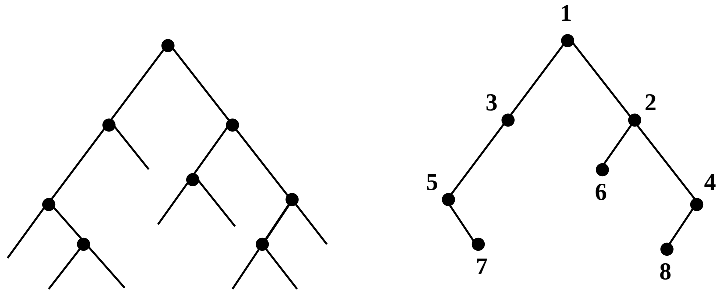


Figure 4-7: A type τ tree and an increasing binary tree of type τ'.

Suppose we choose one of the $(n-1)!$ labellings of τ' uniformly at random. Let $A_{\tau'}$ be the event that the chosen labeling L is increasing; it remains to show that $P(A_{\tau'}) = 1/\prod_v d(v)$.

Let τ_1' and τ_2' be the left and right subtrees of τ'. Let B_1 and B_2 be the events that τ_1' and τ_2' are labeled increasingly in L, and let B be the event that the root of τ is labeled 1. Then $A_{\tau'} = B \cap B_1 \cap B_2$. It is clear that B, B_1 and B_2 are independent events. It is also clear that $P(B_1) = P(A_{\tau_1'})$ and $P(B_2) = P(A_{\tau_2'})$. Therefore,

$$
\begin{aligned}
P(A_\tau) &= P(B)P(B_1)P(B_2) \\
&= \frac{1}{n-1} P(A_{\tau_1'})P(A_{\tau_2'}).
\end{aligned}
$$

The result follows by induction. □

It is also possible to obtain analogous formulas for the number of fine cells inside a lower-dimensional coarse cell, corresponding to a rooted tree which is not binary. We omit the details.

Notice that Proposition 4.7.1 is essentially equivalent to the formula for the number of linear extensions of a poset whose Hasse diagram is a tree [42, Supp. Ex. 3.1].

Corollary 4.7.2. *The positive Bergman complex $\mathcal{B}^+(K_n)$ contains exactly $(n-1)!$ full-dimensional fine cells. The Bergman complex $\mathcal{B}(K_n)$ contains exactly $n!\,(n-1)!\,/\,2^{n-1}$ full-dimensional fine cells.*

Proof. We recall the known bijection between increasing binary trees with vertices labeled $a_1 < \ldots < a_k$, and permutations of $\{a_1, \ldots, a_k\}$ [42, Sec. 1.3]. It is defined recursively: the permutation $\pi(T)$ corresponding to the increasing binary tree T is $\pi(T) = \pi(T_1)\, a_1\, \pi(T_2)$, where T_1 and T_2 are the left and right subtrees of T. For example, the tree of Figure 4-7 corresponds to the permutation 57316284. It is not difficult to see how T can be recovered uniquely from $\pi(T)$.

Since the full-dimensional fine cells of $\mathcal{B}^+(K_n)$ are in correspondence with the increasing binary trees with labels $1, \ldots, n-1$, the first result follows.

To show the second result, recall that the Bergman complex $\mathcal{B}(K_n)$ is covered by $n!$ positive Bergman complexes. Each permutation π of $[n]$ gives rise to a positive Bergman complex $\mathcal{B}^+(K_n)$; this complex parameterizes those trees which can be drawn in the plane so that its leaves are in the order prescribed by π. With $(n-1)!$ fine cells in each positive Bergman complex, we get a covering of $\mathcal{B}(K_n)$ with $n!(n-1)!$ fine cells. Each fine cell appears several times in this covering, since it sits inside several positive Bergman complexes.

More precisely, each binary tree with n labeled leaves can be drawn in the plane in exactly 2^{n-1} ways: at each internal vertex, we may or may not switch the left and right subtrees. Therefore, each fine cell of the Bergman complex $\mathcal{B}(K_n)$ is inside 2^{n-1} different positive Bergman complexes. The desired result follows. □

Recall that the maximum-dimensional fine cells of $\mathcal{B}(K_n)$ correspond to the maximal chains in the lattice of flats of K_n; *i.e.*, the partition lattice Π_n. Thus we have given an alternative proof of the fact that there are $n!\,(n-1)!\,/\,2^{n-1}$ maximal chains in Π_n [42, Supp. Ex. 3.3].

As an illustration of Corollary 4.7.2, notice that, in Figure 4-6, the positive Bergman complex $\mathcal{B}^+(K_4)$ consists of $3! = 6$ fine cells, while the Bergman complex $\mathcal{B}(K_4)$ consists of $4!\,3!\,/\,2^3 = 18$ fine cells.

Chapter 5

Bergman complexes, Coxeter arrangements, graph associahedra

This chapter is joint work with Federico Ardila and Victor Reiner [4].

5.1 Introduction

In this chapter we relate the Bergman complex and the positive Bergman complex of a Coxeter arrangement to the nested set complexes that arise in De Concini and Procesi's wonderful arrangement models [12, 16], and to the graph associahedra introduced by Carr and Devadoss [10], by Davis, Januszkiewicz, and Scott [14], and by Postnikov [33]. We will follow the notation of the previous chapter.

Graph associahedra are polytopes which generalize the associahedron, which were discovered independently by Carr and Devadoss [10], by Davis, Januszkiewicz, and Scott [14], and by Postnikov [33]. There is an intrinsic tiling by associahedra of the Deligne-Knudsen-Mumford compactification of the real moduli space of curves $\overline{M_0^n}(\mathbb{R})$, a space which is related to the Coxeter complex of type A. The motivation for Carr and Devadoss' work was the desire to generalize this phenomenon to all simplicial Coxeter systems.

Let $\mathcal{A}_\Phi$ be the Coxeter arrangement corresponding to the (possibly infinite, possibly non-crystallographic) root system Φ associated to a Coxeter system (W, S) with diagram Γ; see Section 5.4 below. Choose a region R of the arrangement, and let M_Φ be the oriented matroid associated to $\mathcal{A}_\Phi$ and R. In this chapter we prove:

Theorem 5.1.1. *The positive Bergman complex $\mathcal{B}^+(M_\Phi)$ of the arrangement $\mathcal{A}_\Phi$ is dual to the graph associahedron $P(\Gamma)$.*

In particular, the cellular sphere $\mathcal{B}^+(M_\Phi)$ is actually a simplicial sphere, and a *flag* (or *clique*) complex.

This result is also related to the *wonderful model of a hyperplane arrangement* and to *nested set complexes.* The wonderful model of a hyperplane arrangement is obtained by blowing up the non-normal crossings of the arrangement, leaving its complement unchanged. De Concini and Procesi [12] introduced this model in order to study the topology of this complement. They showed that the *nested sets* of the arrangement encode the underlying combinatorics. Feichtner and Kozlov [16] gave an abstract notion of the *nested set complex* for any meet-semilattice, and Feichtner and Müller [17] studied its topology. Recently, Feichtner and Sturmfels [18] studied the relation between the Bergman fan and nested set complexes (see Section 5.5 below).

In this chapter we also prove:

Theorem 5.1.2. *The Bergman complex $\mathcal{B}(M_\Phi)$ of $\mathcal{A}_\Phi$ equals its nested set complex.*

In particular, the cell complex $\mathcal{B}(M_\Phi)$ is actually a simplicial complex.

5.2 The matroid $M_\mathcal{F}$

In this section we give a detailed study of the matroid $M_\mathcal{F}$, which was introduced in the previous chapter. For convenience, we recall a few relevant definitions here.

Definition 5.2.1. *Let M be a matroid or oriented matroid of rank r on the ground set $[n]$, and let $\omega \in \mathbb{R}^n$. Regard ω as a weight function on M, so that the weight of a basis $B = \{b_1, \ldots, b_r\}$ of M is given by $\omega_B = \omega_{b_1} + \omega_{b_2} + \cdots + \omega_{b_r}$. Let B_ω be the collection of bases of M having minimum ω-weight. (If M is oriented, then bases in B_ω inherit orientations from bases of M.) This collection is itself the set of bases of a matroid (or oriented matroid) which we call M_ω.*

The matroid M_ω depends only on a certain flag associated to ω.

Definition 5.2.2. *Given $\omega \in \mathbb{R}^n$, let $\mathcal{F}(\omega)$ denote the unique flag of subsets*

$$\emptyset = F_0 \subset F_1 \subset \cdots \subset F_k \subset F_{k+1} = [n] \tag{5.1}$$

such that ω is constant on each set $F_i \setminus F_{i-1}$ and satisfies $\omega|_{F_i \setminus F_{i-1}} < \omega|_{F_{i+1} \setminus F_i}$. We call $\mathcal{F}(\omega)$ the flag of ω, and we say that the weight class of ω or of the flag $\mathcal{F}$ is the set of vectors ν such that $\mathcal{F}(\nu) = \mathcal{F}$.

It is shown in [2] that

$$M_\omega = \bigoplus_{i=1}^{k+1} F_i/F_{i-1} \tag{5.2}$$

where F_i/F_{i-1} is obtained from the matroid restriction of M to F_i by quotienting out the flat F_{i-1}. Hence we we also refer to this oriented matroid M_ω as $M_{\mathcal{F}}$.

We now make some observations about when two flags of flats in M correspond to the same cell of the coarse subdivision of $\mathcal{B}(M)$. Recall that the *connected components* of matroid M are the equivalence classes for the following equivalence relation on the ground set E of M: say $e \sim e'$ for two elements e, e' in E whenever they lie in a common circuit of M, and then take the transitive closure of $\sim$. Recall also that every connected component is a flat of M, and M decomposes (uniquely) as the direct sum of its connected components.

Definition 5.2.3. *To each flag $\mathcal{F}$ of flats of a matroid M indexed as in* (5.1)*, associate a forest $T_{\mathcal{F}}$ of rooted trees, in which each vertex v is labelled by a flat $F(v)$, as follows:*

- *For each connected component F of the matroid M, create a rooted tree (as specified below) and label its root vertex with F.*

- *For each vertex v already created, and already labelled by some flat $F(v)$ which is a connected component of some flat F_j in the flag $\mathcal{F}$, create children of v labelled by each of the connected components of F_{j-1} contained properly in $F(v)$.*

Alternatively, one can construct the forest $T_{\mathcal{F}}$ by listing all the connected components of all the flats in $\mathcal{F}$, and partially ordering them by inclusion.

Proposition 5.2.4. *For any flag $\mathcal{F}$ of flats in a matroid M, the labelled forest $T_{\mathcal{F}}$ determines the matroid $M_{\mathcal{F}}$.*

Proof. Recall the expression (5.2) for $M_{\mathcal{F}}$. By construction of $T_{\mathcal{F}}$, every component of F_i is $F(v)$ for some unique vertex v, and every component of F_{i-1} lying in $F(v)$ is $F(v')$ for some child v' of v. Since quotients commute with direct sums, this gives

$$M_{\mathcal{F}} = \bigoplus_{\text{vertices } v \text{ of } T_{\mathcal{F}}} \left(F(v) / \bigoplus_{\text{children } v' \text{ of } v} F(v') \right). \tag{5.3}$$

□

In general, the converse of this proposition does *not* hold; one can have $M_{\mathcal{F}} = M_{\mathcal{F}'}$ without $T_{\mathcal{F}} = T_{\mathcal{F}'}$. For example (cf. [18, Example 1.2]), in the matroid M on ground set $E = \{1,2,3,4,5\}$ having rank 3 and circuits $\{123, 145, 2345\}$, the two flags

$$\mathcal{F} := (\emptyset \subset 1 \subset 123 \subset 12345)$$
$$\mathcal{F}' := (\emptyset \subset 1 \subset 145 \subset 12345).$$

exhibit this possibility.

However, we can give at least one nice hypothesis that allows one to reconstruct $T_{\mathcal{F}}$ from $M_{\mathcal{F}}$. Given a base B of a matroid M on ground set E, and any element $e \in E \backslash B$, there is a unique circuit of M contained in $B \cup \{e\}$, called the *basic circuit* $\mathrm{circ}(B, e)$. Note that the flat spanned by $\mathrm{circ}(B, e)$ will always be a connected flat.

Definition 5.2.5. *Say that a base B of a matroid M is circuitous if every connected flat spanned by a subset of B is spanned by the basic circuit* $\mathrm{circ}(B, e)$ *for some $e \in E \backslash B$.*

Note that the basic circuit $\mathrm{circ}(B, e)$ spanning the connected flat F must be $(F \cap B) \cup e$. Before we state our proposition, we prove two useful lemmas.

Lemma 5.2.6. *Let F be a flat in a matroid, spanned by some independent set I. Then every connected component of F is spanned by some subset of I, namely, by the intersection of that component with I.*

Proof. Let F have components $F_1, \ldots, F_t$. Then

$$\sum_i r(F_i) = r(F) = |I| = \sum_i |F_i \cap I| = \sum_i r(F_i \cap I) \leq \sum_i r(F_i),$$

which means we must have an equality for each i: $r(F_i \cap I) = r(F_i)$. In other words, $F_i \cap I$ spans F_i. □

Lemma 5.2.7. *Let $F \subset G$ be flats of a matroid that are spanned by subsets of a circuitous base B. If G is connected, then G/F is also connected.*

Proof. Let $I_F = F \cap B$ and $I_G = G \cap B$; these are bases for F and G, respectively. Also, $I_F \subset I_G$, and $I_G - I_F$ is a base for the quotient G/F. Since G is a connected flat spanned by a subset of the circuitous base B, there exists e in $G - B$ such that $\mathrm{cl}(\mathrm{circ}(B, e)) = G$, and $\mathrm{circ}(B, e) = I_G \cup e$.

We now claim that $\mathrm{circ}_{G/F}(I_G - I_F, e) = I_G - I_F \cup e$. We need to check that $I_G - I_F \cup e - g$ is independent in G/F for any $g \in I_G - I_F \cup e$. Since I_F is a basis of F, this follows from the fact that $I_G \cup e - g$ is independent in G. We conclude by observing that G/F is the flat spanned by $\mathrm{circ}(I_G - I_F, e)$, so it is connected. □

Proposition 5.2.8. *Let B be a circuitous base of a matroid M. Then for any two flags $\mathcal{F}, \mathcal{F}'$ of flats spanned by subsets of B, one has $M_{\mathcal{F}} = M_{\mathcal{F}'}$ if and only if $T_{\mathcal{F}} = T_{\mathcal{F}'}$.*

Proof. We start by making two observations about the matroid $M_{\mathcal{F}}$ and the tree $T_{\mathcal{F}}$.

First we observe that, under these hypothesis, the expression (5.3) is actually the decomposition of $M_{\mathcal{F}}$ into its irreducible components. By Lemma 5.2.6, the $F(v)$'s are connected flats spanned by subsets of B. The direct sums $\oplus_{v'} F(v')$ are also spanned by subsets of B. Lemma 5.2.7 then guarantees that $F(v)/\oplus_{v'} F(v')$ is connected for each vertex v of the tree.

Secondly we show that, among the sets $\mathrm{cl}(\mathrm{circ}(B,e))$ with e in $F(v)\backslash \cup F(v')$ and not in B, there is a maximum one under containment, which is precisely $F(v)$.

Take any e in $F(v)\backslash \cup F(v')$ and not in B. The flat $F(v)$ is spanned by a subset I of B, and $I \cup e$ is dependent. Therefore $\mathrm{circ}(B,e) \subseteq I \cup e \subseteq F(v)$, which implies $\mathrm{cl}(\mathrm{circ}(B,e)) \subseteq F(v)$.

Now, since $F(v)$ is a connected flat spanned by a subset of B, $F(v) = \mathrm{cl}(\mathrm{circ}(B,e))$ for some $e \in E\backslash B$. Clearly $e \in F(v)$. If e was in $F(v')$ for some child v' of v, the argument of the previous paragraph would imply that $\mathrm{cl}(\mathrm{circ}(B,e)) \subseteq F(v')$. Therefore $e \in F(v)\backslash \cup F(v')$.

The two previous observations give us a procedure to recover the tree $T_{\mathcal{F}}$ from the matroid $M_{\mathcal{F}}$. The first step is to decompose $M_{\mathcal{F}}$ into its connected components $M_1, \ldots, M_t$, having accompanying ground set decomposition $E = E_1 \sqcup \cdots \sqcup E_t$. The second step is to recover the flat corresponding to each M_i, as the maximum $\mathrm{cl}(\mathrm{circ}(B,e))$ with $e \in E_i\backslash B$. The labelled forest $T_{\mathcal{F}}$ is simply the poset of inclusions among these flats. □

It will turn out that the simple roots Δ of a root system Φ always form a circuitous base for the associated matroid M_Φ; see Proposition 5.4.7(iii) below.

Remark 5.2.9. When the matroid M is connected, the forest $T_{\mathcal{F}}$ constructed above is a rooted tree. It coincides with the tree constructed by Feichtner and Sturmfels in [18, Proposition 3.1] when they choose the minimal building set for their lattice. In this way, Proposition 5.2.4 follows from [18, Theorem 4.4].

5.3 Graph associahedra

Graph associahedra are polytopes which generalize the associahedron, which were discovered independently by Carr and Devadoss [10], Davis, Januszkiewicz, and Scott [14], and Postnikov [33]. There is an intrinsic tiling by associahedra of the Deligne-Knudsen-Mumford compactification of the real moduli space of curves $\overline{M_0^n(\mathbb{R})}$, a space which is related to the Coxeter complex of type A. The motivation for Carr and

Devadoss' work was the desire to generalize this phenomenon to all Coxeter systems.

In order to define graph associahedra, we must introduce the notions of tubes and tubings. We follow the presentation of [10].

Definition 5.3.1. *Let Γ be a graph. A tube is a proper nonempty set of nodes of Γ whose induced graph is a proper, connected subgraph of Γ. There are three ways that two tubes can interact on the graph:*

- *Tubes are nested if $t_1 \subset t_2$.*
- *Tubes intersect if $t_1 \cap t_2 \neq \emptyset$ and $t_1 \not\subset t_2$ and $t_2 \not\subset t_1$.*
- *Tubes are adjacent if $t_1 \cap t_2 = \emptyset$ and $t_1 \cup t_2$ is a tube in Γ.*

Tubes are compatible if they do not intersect and they are not adjacent. A tubing T of Γ is a set of tubes of Γ such that every pair of tubes in T is compatible. A k-tubing is a tubing with k tubes.

Graph-associahedra are defined via a construction which we will now describe.

Definition 5.3.2. *Let Γ be a graph on n nodes, and let Δ_Γ be the $n-1$ simplex in which each facet corresponds to a particular node. Note that each proper subset of nodes of Γ corresponds to a unique face of Δ_Γ, defined by the intersection of the faces associated to those nodes. For a given graph Γ, truncate faces of Δ_Γ which correspond to 1-tubings in increasing order of dimension (i.e. first truncate vertices, then edges, then 2-faces, ...). The resulting polytope $P(\Gamma)$ is the graph associahedron of Carr and Devadoss.*

Figure 5-1 illustrates the construction of the graph associahedron of a Coxeter diagram of type D_4. We start with a simplex, whose four facets correspond to the vertices of the diagram. In the first step, we truncate three of the vertices, to obtain the second polytope shown. We then truncate three of the edges, to obtain the third polytope shown. In the final step, we truncate the four facets which all correspond to tubes. This step is not shown in Figure 5-1, since it does not affect the combinatorial type of the polytope.

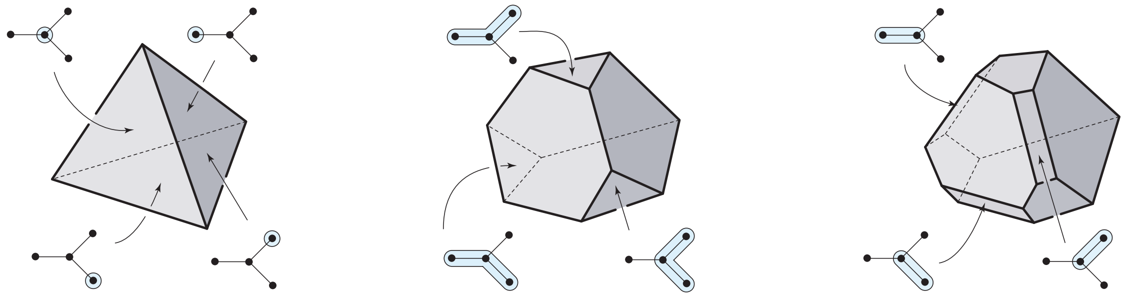


Figure 5-1: $P(D_4)$ ©Satyan Devadoss

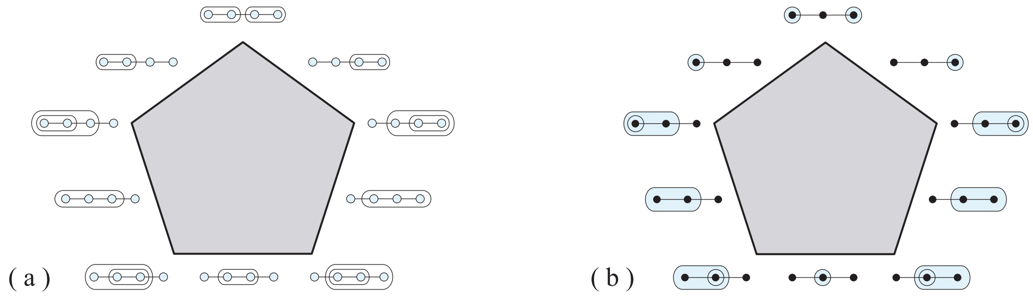


Figure 5-2: A_2 is $P(A_3)$ ©Satyan Devadoss

When the graph Γ is the n-element chain, the polytope $P(\Gamma)$ is the associahedron A_{n-1}. One can see this by considering an easy bijection between valid tubings and parenthesizations of a word of length $n-1$, as illustrated in Figure 5-2.

We thank Satyan Devadoss for allowing us to reproduce in our Figures 5-1 and 5-2, two of his figures from [10].

Carr and Devadoss proved that the face poset of $P(\Gamma)$ can be described in terms of valid tubings.

Theorem 5.3.3. *[10] The face poset of $P(\Gamma)$ is isomorphic to the set of valid tubings of Γ, ordered by reverse containment: $T < T'$ if T is obtained from T' by adding tubes.*

Corollary 5.3.4. *[10] When Γ is a path with $n-1$ nodes, $P(\Gamma)$ is the associahedron A_n of dimension n. When Γ is a cycle with $n-1$ nodes, $P(\Gamma)$ is the cyclohedron W_n.*

5.4 The positive Bergman complex of a Coxeter arrangement

In this section we prove that the positive Bergman complex of a Coxeter arrangement of type Φ is dual to the graph associahedron of type Φ. More precisely, both of these objects are homeomorphic to spheres of the same dimension, and their face posets are dual. We begin by reviewing our conventions about Coxeter systems and the related arrangements and matroids.

A *Coxeter system* is a pair (W, S) consisting of a group W and a set of generators $S \subset W$, subject only to relations of the form

$$(ss')^{m(s,s')} = 1,$$

where $m(s, s) = 1$, $m(s, s') = m(s', s) \geq 2$ for $s \neq s'$ in S. In case no relation occurs for a pair s, s', we make the convention that $m(s, s') = \infty$. We will always assume that S is finite.

Note that to specify a Coxeter system (W, S), it is enough to draw the corresponding *Coxeter diagram* Γ: this is a graph on vertices indexed by elements of S, with vertices s and s' joined by an edge labelled $m(s, s')$ whenever this number (∞ allowed) is at least 3.

Remark 5.4.1. In what follows, the reader should note that nothing will turn out to depend on the edge labels $m(s, s')$ of Γ; the positive Bergman complex, the Bergman complex, or the graph associahedron associated with Γ will depend only upon the undirected graph underlying Γ.

Although an arbitrary Coxeter system (W, S) need not have a faithful representation of W as a group generated by orthogonal reflections with respect to a positive definite inner product, there exists a reasonable substitute, called its *geometric representation* [27, Sec. 5.3, 5.13], which we recall here. Let $V := \mathbb{R}^{|S|}$ with a basis of *simple roots* $\Delta := \{\alpha_s : s \in S\}$. Define an $\mathbb{R}$-valued bilinear form $(\cdot,\cdot)$ on V by

$$(\alpha_s, \alpha_{s'}) := -\cos\left(\frac{\pi}{m(s,s')}\right)$$

and let s act on V by the "reflection" that fixes $\alpha_s^\perp$ and negates α_s:

$$s(v) := v - 2(v, \alpha_s)\alpha_s.$$

This turns out to extend to a faithful representation of W on V, and one defines the *root system* Φ and *positive roots* Φ^+ by

$$\begin{aligned}\Phi &:= \{w(\alpha_s) : w \in W, s \in S\}\\ \Phi^+ &:= \{\alpha \in \Phi : \alpha = \sum_{s\in S} c_s\alpha_s \text{ with } c_s \geq 0\}\end{aligned}$$

It turns out that $\Phi = \Phi^+ \sqcup \Phi^-$ where $\Phi^- := -\Phi^+$. We use M_Φ to denote the matroid represented by Φ^+ in V, which is of finite rank $r = |S|$, but has ground set E of possibly (countably) infinite cardinality. Its lattice of flats L_{M_Φ} may be infinite, although of finite rank r, and is well-known (see e.g. [5]) to be isomorphic to the poset of *parabolic subgroups*

$$\{wW_Jw^{-1} : w \in W, J \subseteq S\}$$

ordered by inclusion. In other words, every flat F is spanned by $w(\Phi_J^+)$ for some standard parabolic subroot system Φ_J^+ and $w \in W$.

Definition 5.4.2. *Given a root $\alpha \in \Phi$, expressed uniquely in terms of the simple roots Δ as $\alpha = \sum_{s\in S} c_s\alpha_s$, define the support of α (written* $\operatorname{supp}\alpha$*) to be the vertex-induced subgraph of the Coxeter diagram Γ on the set of vertices $s \in S$ for which $c_s \neq 0$.*

We will need the following well-known lemma about supports of roots. A proof of its first assertion for the Coxeter systems associated to Kac-Moody Lie algebras can be found in [28, Lemma 1.6]; we will need the assertion in general.

Lemma 5.4.3. *Let (W, S) be an arbitrary Coxeter system with Coxeter graph Γ. Then for any root $\alpha \in \Phi$ the graph* $\operatorname{supp} \alpha$ *is connected, and conversely, every connected subgraph Γ' of Γ occurs as* $\operatorname{supp} \alpha$ *for some positive root α.*

Proof. For the first assertion, let β be a root, which we may assume is positive without loss of generality. It is known (see e.g. [8, Sec. 4.5]) that β can be expressed

$$\beta = s_k s_{k-1} \cdots s_2 s_1(\alpha)$$

for some simple root $\alpha \in \Delta$ and some $s_i \in S$, in such a way that each root $\beta_j := s_j s_{j-1} \cdots s_2 s_1(\alpha)$ is positive. Hence by induction on k, it suffices to show that if $\beta = s(\gamma)$ for positive roots β, γ and s in S, then connectedness of $\operatorname{supp} \gamma$ implies $\operatorname{supp} \beta$ is connected. If not, then the expression

$$\beta = s(\gamma) = \gamma - 2(\gamma, \alpha_s)\alpha_s \tag{5.4}$$

would imply that $\operatorname{supp} \beta = \operatorname{supp} \gamma \cup \{s\}$. If this set is disconnected, one must have $(\gamma, \alpha_s) = 0$, forcing the contradiction that $\beta = \gamma$.

For the second assertion, let Γ' be a connected subgraph of Γ, and we will exhibit a positive root β with $\operatorname{supp} \beta = \Gamma'$ using induction on the number of vertices of Γ'. Let $s \in S$ be a vertex lying in Γ' whose removal leaves a connected subgraph $\Gamma'' = \Gamma - \{s\}$. By induction there exists a positive root γ having $\operatorname{supp} \gamma = \Gamma''$, and we claim that $\beta := s(\gamma)$ has $\operatorname{supp}(\beta) = \Gamma'$. To see this, note that $\gamma = \sum_{t \in \Gamma''} c_t \alpha_t$ with each $c_t > 0$. Hence

$$(\alpha_s, \gamma) = \sum_{t \in \Gamma''} c_t (\alpha_s, \alpha_t) < 0$$

since each (α_s, α_t) is nonpositive, and at least one is negative due to $\Gamma'' \cup \{s\} = \Gamma'$ being connected. Therefore the expression (5.4) for β shows that $\operatorname{supp}(\beta) = \Gamma'$ $\square$

If one wants to think of the oriented matroid M_Φ as the oriented matroid of a hyperplane arrangement (as opposed to the oriented matroid of a collection of vectors), one must work with the contragredient representation V^* [27, 5.13]. Let

$\{\delta_s : s \in S\}$ denote the basis for V^* dual to the basis of simple roots Δ for V. Then the *(closed) fundamental chamber* R is the nonnegative cone spanned by $\{\delta_s : s \in S\}$ inside V^*. The *Tits cone* is the union $\bigcup_{w \in W} w(R)$, a (possibly proper, not necessarily closed nor polyhedral) convex cone inside V^*. Every positive root $\alpha \in \Phi^+$ gives an oriented hyperplane H_α in V^* with nonnegative half-space $\{f \in V^* : f(\alpha) \geq 0\}$. These hyperplanes and half-spaces decompose the Tits cone[1] into (closed) regions that turn out to be simplicial cones which are exactly the images $w(R)$ as w runs through W; the tope (maximal covector) in the oriented matroid M_Φ associated to $w(R)$ will have the sign $+$ on the roots $\Phi^+ \cap w^{-1}(\Phi^+)$ and the sign $-$ on the roots $\Phi^+ \cap w^{-1}(\Phi^-)$.

Remark 5.4.4. We should be somewhat careful when speaking of the Bergman complex and the matroids M_ω when W is infinite, since the ground set $E = \Phi^+$ is infinite. One way around this is to only refer to the matroids $M_{\mathcal{F}}$ associated to flags of flats $\mathcal{F}$ in L_{M_Φ}, viewing the Bergman complex as a coarsening of the ordering complex of L_{M_Φ}. Similarly, there is an issue with interpreting the minimal blow-up of the Coxeter arrangement when W is infinite. In [10] this problem is avoided by assuming finiteness of W_J for proper parabolic subsystems (W_J, J) with $J \subsetneq S$, so that the arrangement of hyperplanes H_α cuts out a Coxeter complex which is locally finite. However, if one is not so concerned with the blow-ups themselves, but rather with the truncations of the fundamental simplex R which tile the blow-up, these polytopes are well-defined in any case.

We will now collect some facts about the matroid M_Φ. But first let us recall the notion of positive flats.

Definition 5.4.5. *Let M be an acyclic oriented matroid on the ground set $[n]$. We say that a covector $v \in \{+, -, 0\}^n$ of M is* positive *if each of its entries is $+$ or 0. We say that a flat of M is* positive *if it is the 0-set of a positive covector.*

[1]We should point out that when W is infinite, only part of the hyperplane or its nonnegative half-space lies inside the Tits cone, so we only consider their intersection with the Tits cone.

Observation 5.4.6. *If M is the acyclic oriented matroid corresponding to a hyperplane arrangement $\mathcal{A}$ and a specified region R, then the positive flats are in correspondence with the faces of R.*

For example, consider the braid arrangement A_3, consisting of the six hyperplanes $x_i = x_j, 1 \leq i < j \leq 4$ in $\mathbb{R}^4$. Figure 5-3 illustrates this arrangement, when intersected with the hyperplane $x_4 = 0$ and the sphere $x_1^2 + x_2^2 + x_3^2 = 1$. Let R be the region specified by the inequalities $x_1 \geq x_2 \geq x_3 \geq x_4$, and let M_{A_3} be the oriented matroid corresponding to the arrangement A_3 and the region R. Then the positive flats are $\emptyset, 1, 4, 6, 124, 16, 456$ and 123456.

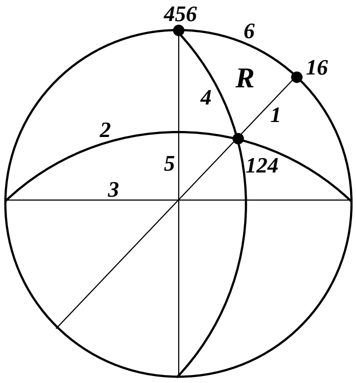


Figure 5-3: The braid arrangement A_3.

Proposition 5.4.7. *Let (W, S) be an arbitrary Coxeter system, with root system Φ and Coxeter diagram Γ.*

(i) Positive flats in the oriented matroid M_Φ correspond to subsets $J \subset S$.

(ii) Connected positive flats in the oriented matroid M_Φ correspond to subsets $J \subset S$ such that the vertex-induced subgraph Γ_J is connected, that is, to tubes in Γ.

(iii) The simple roots Δ form a circuitous base for the matroid M_Φ.

(iv) If $F \subset G$ are flats in M_Φ with G connected, then the matroid quotient G/F is connected.

Proof. (i): The hyperplanes bounding the base region/tope R are $\{H_{\alpha_s} : s \in S\}$, so positive flats are those spanned by sets of the form $\{\alpha_s : s \in J\}$ for subsets $J \subset S$. We denote such a positive flat by $\mathrm{cl}(J)$.

(ii): Let $J \subset S$ with subgraph Γ_J, and consider its associated positive flat $\mathrm{cl}(J)$. The first assertion of Lemma 5.4.3 shows that $\mathrm{cl}(J)$ will not be connected if Γ_J is disconnected. To see this, represent the flat $\mathrm{cl}(J)$ by a matrix in which the rows correspond to simple roots of $\mathrm{cl}(J)$, i.e. vertices of Γ_J, and the columns express each positive root in $\mathrm{cl}(J)$ as a combination of simple roots. By permuting columns, one can obtain a matrix which is a block-direct sum of two smaller matrices, and hence $\mathrm{cl}(J)$ will not be connected.

On the other hand, if Γ_J is connected, then the second assertion of Lemma 5.4.3 shows that there is a positive root α with $\operatorname{supp} \alpha = \Gamma_J$, and consequently $\{\alpha_s : s \in J\} \cup \{\alpha\}$ gives a circuit in M_Φ spanning this flat, so it is connected.

(iii): This follows from the argument in (ii); given $J \subset S$ with Γ_J connected, the basic circuit $\mathrm{circ}(\Delta, \alpha)$ where $\operatorname{supp} \alpha = \Gamma_J$ spans the connected flat corresponding to J.

(iv): Let F, G correspond to the parabolic subgroups uW_Ju^{-1}, vW_Kv^{-1}, or equivalently, assume they are spanned by $u\Phi_J^+, v\Phi_K^+$. One can make the following reductions:

- Translating by v^{-1}, one can assume that v is the identity.
- Since (W_K, K) itself forms a Coxeter system with root system Φ_K, one can assume $M_\Phi = G$ and $K = S$. In particular, M_Φ is connected.
- Replacing the Coxeter system (W, S) by the system (W, uSu^{-1}), one can assume that u is the identity.

In other words, F is the positive flat corresponding to some subgraph Γ_J of Γ, and we must show M_Φ/F is a connected matroid. This is a consequence of (iii) and Lemma 5.2.7. $\square$

We now give our main result.

Theorem 5.1.1. *Let (W, S) be an arbitrary Coxeter system, with root system Φ, Coxeter diagram Γ, and associated oriented matroid M_Φ. Then the face poset of the coarse subdivision of $\mathcal{B}^+(M_\Phi)$ is dual to the face poset of the graph associahedron $P(\Gamma)$.*

Proof. By Theorem 5.3.3, we need to show that the face poset of (the coarse subdivision of) $\mathcal{B}^+(M_\Phi)$ is equal to the poset of tubings of Γ, ordered by containment. We begin by describing a map Ψ from flags of positive flats to tubings of Γ.

By Proposition 5.4.7, positive flats of M_Φ correspond to subsets $J \subset S$ or subgraphs Γ_J of the Coxeter graph Γ. Furthermore, a positive flat is connected if and only if Γ_J is a tube, and hence an arbitrary positive flat corresponds to a disjoint union of compatible tubes, no two of which are nested. Since an inclusion of flats corresponds to an inclusion of the subsets J, a flag $\mathcal{F}$ of positive flats corresponds to a nested chain of such unions of non-nested compatible tubes, that is, to a tubing $\Psi(\mathcal{F})$. Furthermore, in this correspondence, inclusion of flags corresponds to containment of tubings.

We claim that the map from flags to tubings is surjective. Given some tubing of Γ, linearly order its tubes $J_1, \ldots, J_k$ by any linear extension of the inclusion partial ordering, and then the flag $\mathcal{F}$ of positive flats having F_i spanned by $\{\alpha_s : s \in J_1 \cup J_2 \cup \cdots \cup J_i\}$ will map to this tubing.

Lastly, we show that Ψ is actually a well-defined injective map when regarded as a map on cells of the coarse subdivision of $\mathcal{B}^+(M_\Phi)$. To do so, it is enough to show that two flags $\mathcal{F}, \mathcal{F}'$ of positive flats give the same tubing if and only if $M_\mathcal{F}$ and $M_{\mathcal{F}'}$ coincide. By Lemma 5.4.7(iv) and Proposition 5.2.8, we need to show that $\Psi(\mathcal{F})$ and $\Psi(\mathcal{F}')$ coincide if and only if $T_\mathcal{F}$ and $T_{\mathcal{F}'}$ coincide. But this is clear, because by construction, the rooted forest $T_\mathcal{F}$ ignores the ordering within the flag, and only records the data of the tubes which appear, that is, the tubing.

$\square$

Corollary 5.4.8. *The Bergman complex and the positive Bergman complex of a Coxeter arrangement $\mathcal{A}$ are both simplicial. The latter is furthermore a flag simplicial* *sphere.*

Another corollary of our proof is a new realization for the positive Bergman complex of a Coxeter arrangement: we can obtain it from a simplex by a sequence of stellar subdivisions.

5.5 The Bergman complex of a Coxeter arrangement

Nested set complexes are simplicial complexes which are the combinatorial core of De Concini and Procesi's subspace arrangement models [12], and of the resolution of singularities in toric varieties [16]. We now recall the definition of the minimal nested set complex of a meet-semilattice L, which we will simply refer to as **the** nested set complex of L, and denote $\mathbb{N}(L)$.

Say an element y of L is *irreducible* if the lower interval $[\hat{0}, y]$ cannot be decomposed as the product of smaller intervals of the form $[\hat{0}, x]$. The nested set complex $\mathbb{N}(L)$ of L is a simplicial complex whose vertices are the irreducible elements of L. A set X of irreducibles is *nested* if for any antichain $\{x_1, \ldots, x_k\}$ in X, $x_1 \vee \cdots \vee x_k$ is not irreducible. These nested sets are the faces of $\mathbb{N}(L)$.

If M is a matroid and L_M is its lattice of flats, we will also call $\mathbb{N}(L_M)$ the nested set complex of M, and denote it $\mathbb{N}(M)$. (Recall that the irreducible elements of L_M are the connected flats of M.) It turns out that when we are considering the oriented matroid M_Φ of a Coxeter arrangement of type Φ, the Bergman complex $\mathcal{B}(M_\Phi)$ and the nested set complex $\mathbb{N}(M_\Phi)$ are equal.

To prove this theorem, we use a result of Feichtner and Sturmfels [18]. They showed that, for any matroid M, the order complex of $\mathbb{N}(M)$ refines the coarse subdivision of the Bergman complex $\mathcal{B}(M)$ and is refined by its fine subdivision. Moreover, they proved the following theorem.

Theorem 5.5.1. *[18] The nested set complex $\mathbb{N}(M)$ equals the Bergman complex $\mathcal{B}(M)$ if and only if the matroid G/F is connected for every pair of flats $F \subset G$ in* *which G is connected.*

Combining their Theorem 5.5.1 with Proposition 5.4.7(iv) immediately yields the following result.

Theorem 5.1.2. *For any Coxeter system (W, S) and associated root system Φ, the coarse subdivision of the Bergman complex $\mathcal{B}(M_\Phi)$ of the Coxeter arrangement of type Φ is equal to the nested set complex $\mathbb{N}(M_\Phi)$.*

Appendix A

The poset of cells of $Gr^+_{2,4}$

In this section we give two depictions of the poset of cells of $Gr^+_{2,4}$, the first in terms of Ⅎ-diagrams, and the second in terms of decorated permutations. Note that the objects in the two posets have been drawn so as to indicate the natural bijection between Ⅎ-diagrams $(\lambda, D)_{k,n}$ and decorated permutations on 4 letters with 2 weak excedences.

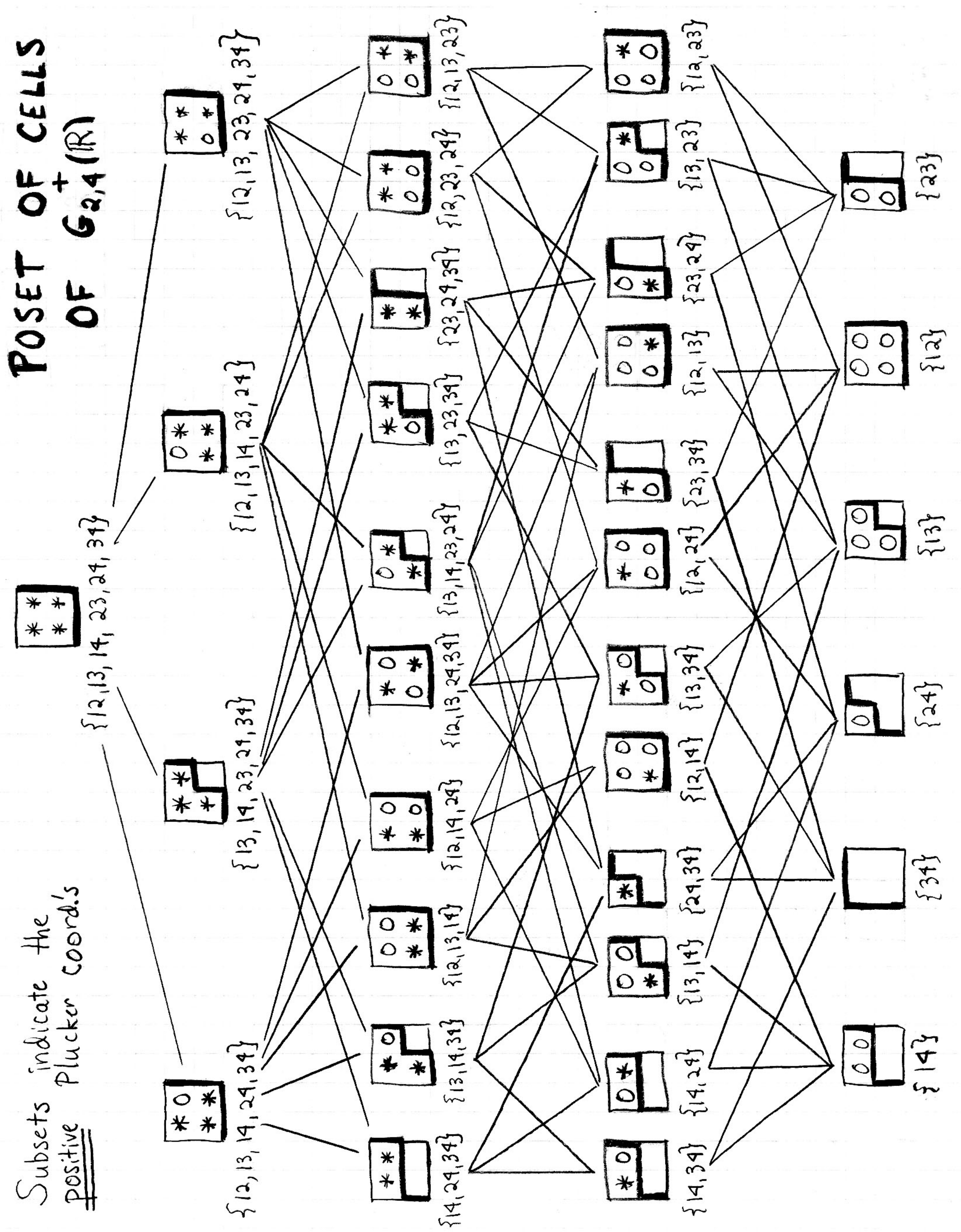


Figure A-1: L-diagrams

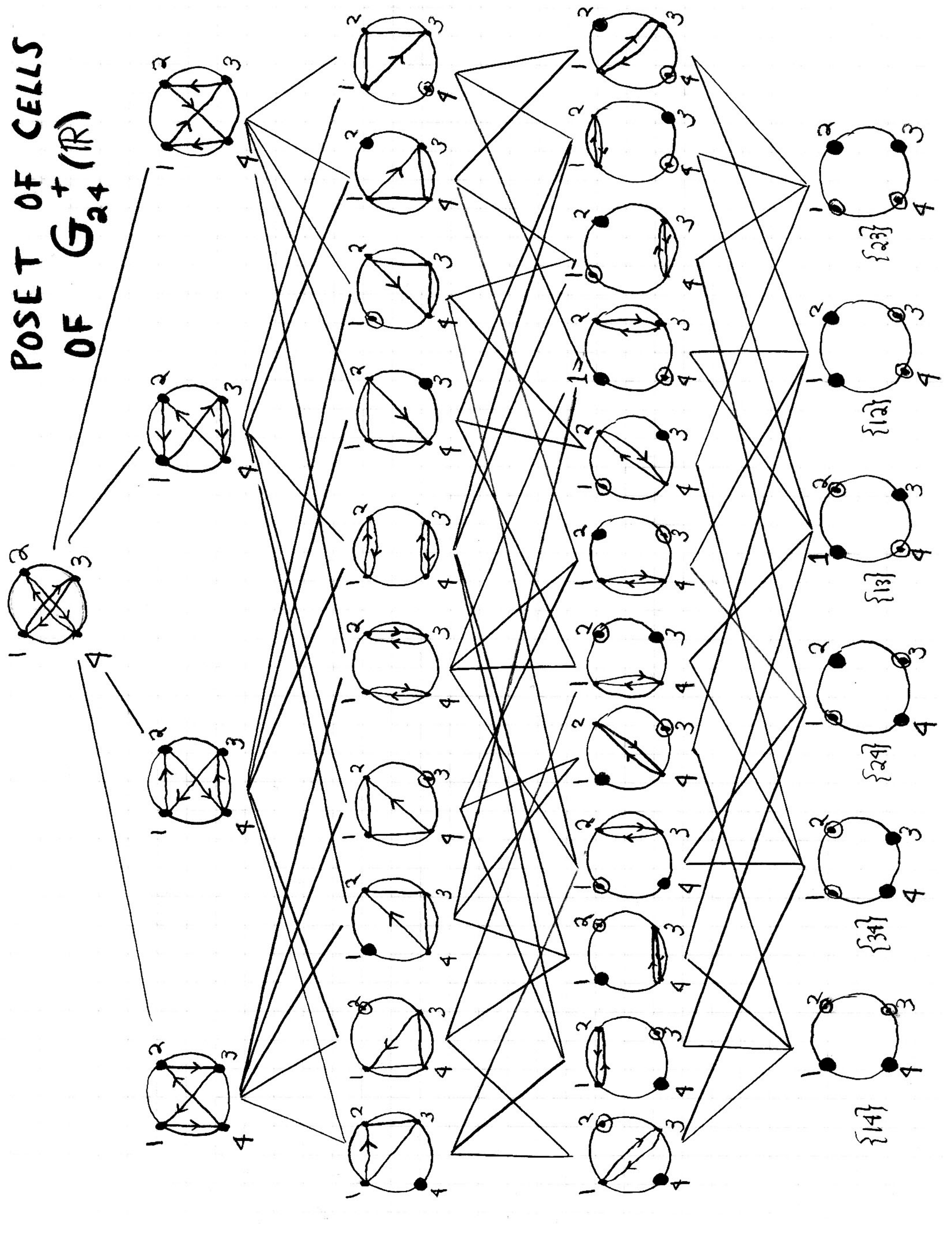


Figure A-2: Decorated permutations

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