# ON LANDAU-GINZBURG MODELS FOR QUADRICS AND FLAT SECTIONS OF DUBROVIN CONNECTIONS

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Abstract. This paper proves a version of mirror symmetry expressing the (small) Dubrovin connection for even-dimensional quadrics in terms of a mirror-dual Landau-Ginzburg model on the complement of an anticanonical divisor in a dual quadric. We then go into greater depth for all quadrics, even and odd, treating them as a series starting with $Q_3$ and $Q_4 = Gr_2(4)$. This turns out to work very naturally after restricting to a particular torus, and leads to a combinatorial model for the superpotential in terms of a quiver, in the vein of those proposed by Batyrev, Ciocan-Fontanine, Kim and van Straten for Grassmannians in the 1990's. The Laurent polynomial superpotentials form a single series, despite the fact that our mirrors of even quadrics are defined on dual quadrics, while the mirror to an odd quadric is naturally defined on a projective space. We use this quiver description to compute explicitly a particular flat section of the Dubrovin connection, and recover the constant term of Givental's $J$-function by a variety of methods.

## Contents



## 1. Introduction

Suppose $X$ is a smooth projective complex Fano variety of dimension $N$. Starting from $X$ as the '$A$-model', Dubrovin constructed a flat connection on a trivial bundle with fiber $H^*(X,\mathbb{C})$, using Gromov-Witten invariants of $X$. The '$B$-models' of Fano varieties were first introduced in [Wit97] and [Giv95]. One of their expected properties is to mirror the Dubrovin connection of the A-model via a Gauss-Manin system.

In our setting $X$ will always have Picard rank 1. In this case the base of the trivial bundle on the $A$-side can be taken to be the two-dimensional complex torus $\mathbb{C}^*_q \times \mathbb{C}^*_\hbar$ with coordinates $q$ and $\hbar$. The Dubrovin connection is flat and therefore defines a $D$-module $M_A$, where $D = \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]\langle \partial_\hbar, \partial_q \rangle$. The $B$-model for $X$ as above is a *Landau-Ginzburg model*, which is a pair $(\check{X}, W_q)$ consisting of an affine algebraic variety $\check{X}$ over $\mathbb{C}$ and a regular function $W_q : \check{X} \to \mathbb{C}$ called the *superpotential*. This data gives rise to a Gauss-Manin system, via a kind of twisted $N$-th (algebraic) de Rham cohomology. Namely one defines the $D$-module

$$M_B = \Omega^N(\check{X}, \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}])/(d + \frac{1}{\hbar} dW_q \wedge \_\ )\Omega^{N-1}(\check{X}, \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}])$$

where $D = \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]\langle \partial_\hbar, \partial_q \rangle$, which is intended to recover the Dubrovin connection of $X$. See Definition 4.1 for the action of $\partial_\hbar$ and $\partial_q$.

One of the central problems of mirror symmetry is how to construct the LG model $(\check{X}, W_q)$ given $X$. In the case of quadrics we consider two direct approaches. One of them is the approach due to Hori and Vafa [HV00], which applies to Fano hypersurfaces in projective space. The other approach is via [Rie08], which applies to homogeneous spaces $G/P$.

When $X$ is a hypersurface in a complex projective space, its conjectured Landau-Ginzburg model, the 'Hori-Vafa mirror', is a torus together with a Laurent polynomial in $N$ variables [HV00]. In the case of an $N$-dimensional quadric $Q_N$ the LG model of Hori and Vafa is $((\mathbb{C}^*)^N, L_q)$, where

$$L_q = Y_1 + Y_2 + \ldots + Y_{N-1} + \frac{(Y_N + q)^2}{Y_1 Y_2 \cdots Y_N}. \tag{1}$$

While this is not the form in which Hori and Vafa originally expressed their mirror, it is equivalent to it (see [Prz09, Rmk. 19]).

On the other hand the smooth quadric $Q_N$ may also be identified with the homogeneous space $\mathrm{Spin}_{N+2}(\mathbb{C})/P$. Here we think of $\mathrm{Spin}_{N+2}(\mathbb{C})$ as the spin group associated to the quadratic form on $\mathbb{C}^{N+2}$ defining $Q_N$ inside $\mathbb{P}^{N+1}$. The mirror construction from [Rie08] applies in this setting and gives a regular function $\mathcal{F}_q$ on an $N$-dimensional affine subvariety $\mathcal{R}$ (generally larger than a torus) of the Langlands dual full flag variety. If $N$ is odd then this Langlands dual full flag variety is $\mathrm{PSp}_{N+1}(\mathbb{C})/B_-$. If $N$ is even then it is $\mathrm{PSO}_{N+2}(\mathbb{C})/B_-$.

One advantage of the mirrors $\mathcal{F}_q$ over the Laurent polynomials $L_q$ is that the former have the expected number of critical points (at fixed generic value of $q$), namely $\dim(H^*(Q_N))$. This is not generally the case for Laurent polynomial mirrors, as was already observed in [EHX97]. In [EHX97] it was suggested to solve this problem using a partial compactification and this was carried out for the first time in the case of $Q_4$, albeit in an ad hoc fashion. Since then a partial compactification of the Hori-Vafa mirror in the case of all odd quadrics was obtained in [GS13], along with a proof of the isomorphism of $D$-modules. This partial compactification was then shown in [PR13a] to be isomorphic to the mirror $\mathcal{F}_q$.

We note that for type $A$ full flag varieties and Grassmannians the mirrors $\mathcal{F}_q$ were shown to be partial compactifications of the Laurent polynomial mirrors of [BCFKvS00, BCFKvS98], see [Rie08, MR13]. A partial comparison in the more general type $A$ case is given in [Rie06].

In this paper we will discuss and compare four different versions of the LG models for quadrics, and prove various identities predicted by mirror symmetry. Here is a summary of our results.

1.1. **A canonical mirror.** Suppose $X$ is a homogeneous space for an adjoint simple complex algebraic group. For cominuscule $X$, such as Grassmannians, Lagrangian Grassmannians, and also quadrics, the Langlands dual group naturally acts on $H^*(X, \mathbb{C})$, by the geometric Satake correspondence [Lus83, MV07, Gin95].

In Section 3, we exploit this to give a very natural formulation of the mirror in the even quadrics case. This point of view first appeared in [MR13, PR13b], compare also [PR13a]. Namely for even dimensional quadrics we prove an isomorphism between the domain $\mathcal{R}$ of $\mathcal{F}_q$ and the complement of an anti-canonical divisor in a 'mirror' quadric $\check{Q}_N$. This mirror quadric is obtained as a closed orbit of the Langlands dual group inside $\mathbb{P}(H^*(Q_N,\mathbb{C})^*)$. Therefore the cohomology classes of $Q_N$ are naturally coordinate functions on the dual quadric $\check{Q}_N$. We then obtain an LG-model $W_q$ on $\check{Q}_N$ by pulling back $\mathcal{F}_q$ and expressing it in the coordinates coming from the Schubert basis of $H^*(Q_N,\mathbb{C})$. We consider this to be the most natural presentation of the LG-model for $Q_N$. For odd $N$ we note that the analogous procedure gives an LG-model on $\mathbb{P}^N$, where $\mathbb{P}^N$ is viewed as a homogeneous space for $Sp_{N+1}(\mathbb{C})$, see [PR13a].

1.2. **Laurent polynomial mirrors analogous to projective space.** By restricting to a natural choice of torus in $\mathcal{R}$, we obtain in Proposition 3.11 a further Laurent polynomial expression for the mirror. Combining this with results from [PR13b] we obtain a series of Laurent polynomial mirrors for all $Q_N$, which resemble the well-known Laurent polynomial mirrors for projective spaces (but differ from the Hori-Vafa mirrors).

1.3. **An isomorphism of $D$-modules.** For even dimensional quadrics we construct in Section 4 an explicit isomorphism from the Dubrovin $D$-module $M_A$ to a natural submodule of the Gauss-Manin $D$-module $M_B$. We conjecture that this submodule is in fact all of $M_B$ so that $M_A$ and $M_B$ are isomorphic. Here we use the new version $W_q$ of the mirror which takes place on a dual quadric. We note that there is a non-trivial cluster algebra structure on the coordinate ring of the mirror, which plays an important role in our proof of the isomorphism.

1.4. **The hypergeometric series of the quadric.** In Section 5, we explicitly construct a flat section of the Dubrovin connection of the quadric $Q_N$ by two different methods. First we obtain the coefficients as residue integrals on the $B$-model side, using the Laurent polynomial formulation from 1.2. On the other hand these coefficients can be interpreted as 1-point descendent Gromov-Witten invariants, and we determine these directly on the $A$-side, using Kontsevich-Manin reconstruction and the usual axioms. The top coefficient is also the coefficient of the fundamental class in Givental's $J$-function, which can be constructed out of the $J$-function of projective space via the quantum Lefschetz formula of Coates and Givental [CG07]. We identify the coefficient of the fundamental class of the $J$-function as a hypergeometric series in Section 7, and obtain the differential equation which it satisfies, which is a 'quantum differential equation' of the quadric.

1.5. **A quiver interpretation of the superpotential.** In Section 6, we interpret our Laurent polynomial version of the mirror from 1.2 in terms of a quiver, in the spirit of [Giv97, BCFKvS98, BCFKvS00]. It is possible to read off the fundamental class coefficient of the $J$-function from the quiver. This is in analogy with the residue formula of [BCFKvS00, Section 5.1] for type $A$ partial flag varieties, which was conjectured to recover the fundamental class coefficient of the $J$-function. This conjecture stems in the case of full flag varieties from the work of Givental [Giv97], and is now proved in the case of Grassmannians in [MR13].

1.6. **Comparison with the Hori-Vafa mirrors.** In Sections 2.2 and 3.7 we show that the Hori-Vafa mirrors are related to our mirror $W_q$ by a birational change of coordinates.

## 2. Landau-Ginzburg models for odd quadrics

The quadrics are cominuscule homogeneous spaces (for the Spin groups). Therefore, in addition to the Hori-Vafa approach [HV00] for constructing LG models, there is another LG model for each quadric on an affine variety (generally larger than a torus), which was defined by the second-named author using a Lie-theoretic construction [Rie08]. Namely for any projective homogeneous space $X = G/P$ of a simple complex algebraic group, [Rie08] constructed a conjectural LG model, which is a regular function on an affine subvariety of the Langlands dual group. It was shown in [Rie08] that this LG model recovers the Peterson variety presentation [Pet97] of the quantum cohomology of $X = G/P$. It therefore defines an LG model whose Jacobi ring has the correct dimension. In this section we will rewrite this LG model in terms of natural projective coordinates on $\mathbb{P}(H^*(Q_N, \mathbb{C})^*)$.

Note that for odd-dimensional quadrics $Q_{2m-1}$ a recent paper [GS13] of Gorbounov and Smirnov constructed directly a partial compactification of the Hori-Vafa mirrors, without making use of [Rie08].

2.1. **The LG model for $Q_{2m-1}$ on a Langlands dual projective space.** LG models for odd-dimensional quadrics with the expected number of critical points have been constructed in [Rie08] (where they appear as a special case), and [GS13], and finally [PR13a]. Here we recall the main results from the paper [PR13a], which contains the formulation for the LG model which we will adopt.

In this section our $A$-model variety $X = X_N = X_{2m-1}$ is the quadric $Q_N = Q_{2m-1}$. Recall that an odd-dimensional quadric has 1-dimensional cohomology groups in even degrees spanned by Schubert classes $\sigma_i \in H^{2i}(Q_{2m-1}, \mathbb{C})$ for $0 \leq i \leq 2m-1$, and no other cohomology. To construct its mirror first consider the projective space $\check{X} = \check{X}_{2m-1} = \mathbb{P}^{2m-1}$ with homogeneous coordinates $(p_0 : p_1 : \cdots : p_{2m-1})$ in one-to-one correspondence with these Schubert classes $\sigma_i$. Inside $\check{X}$ we have the open affine subvariety $\check{X}^\circ \subset \mathbb{P}^{2m-1}$ defined by:

$$\check{X}^\circ = \check{X}^\circ_{2m-1} := \check{X} \setminus D, \tag{2}$$

where $D := D_0 + D_1 + \ldots + D_{m-1} + D_m$, the divisors $D_i$ being given by

$$\begin{aligned} D_0 &:= \{p_0 = 0\}, \\ D_\ell &:= \left\{ \sum_{k=0}^{\ell} (-1)^k p_{\ell-k} p_{2m-1-\ell+k} = 0 \right\} \text{ for } 1 \leq \ell \leq m-1, \\ D_m &:= \{p_{2m-1} = 0\}. \end{aligned}$$

The divisor $D$ is an anti-canonical divisor. Indeed, the index of $\check{X} = \mathbb{P}^{2m-1}$ is $2m$. For simplicity, we will define

$$\delta_\ell = \sum_{k=0}^{\ell} (-1)^k p_{\ell-k} p_{N-\ell+k} \text{ for } 0 \leq \ell \leq m-1. \tag{3}$$

(For odd quadrics, $\mathrm{N} = 2m-1$.) We have:

**Theorem 2.1** ([PR13a, Theorem 1]). *The LG model $\mathcal{F}_q : \mathcal{R} \to \mathbb{C}$ from* [Rie08] *for $X = Q_{2m-1}$ is isomorphic to $W_q : \check{X}^\circ_{2m-1} \to \mathbb{C}$ defined by*

$$W_q = \frac{p_1}{p_0} + \sum_{\ell=1}^{m-1} \frac{p_{\ell+1}p_{2m-1-\ell}}{\delta_\ell} + q\frac{p_1}{p_{2m-1}}. \tag{4}$$

We also have another expression for the superpotential:

**Proposition 2.2** ([PR13a, Proposition 8]). *For $X = Q_{2m-1}$ and $W_q$ as above, there is a torus $(\mathbb{C}^*)^{2m-1} \hookrightarrow \check{X}^\circ_{2m-1}$ to which $W_q$ pulls back giving the Laurent polynomial expression*

$$W_q = a_1 + \cdots + a_{m-1} + c + b_{m-1} + \cdots + b_1 + q\frac{a_1 + b_1}{a_1 \ldots a_{m-1} c b_{m-1} \ldots b_1}. \tag{5}$$

2.2. **Comparison with the Hori-Vafa model for odd quadrics.** Here we check that once restricted to a certain torus, our LG model (4) is isomorphic to the Hori-Vafa LG model. Let us consider the change of coordinates:

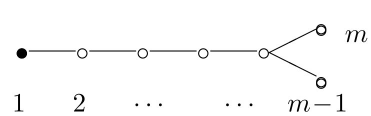
$$Y_i = \begin{cases} \frac{p_i}{p_{i-1}} & \text{for } 1 \le i \le m-1; \\ \frac{p_{2m-1-i}\delta_{2m-3-i}}{p_{2m-2-i}\delta_{2m-2-i}} & \text{for } m \le i \le 2m-3; \\ q\frac{p_1}{p_{2m-1}} & \text{for } i = 2m-2; \\ q\frac{\delta_{m-2}}{\delta_{m-1}} & \text{for } i = 2m-1. \end{cases}$$

This change of coordinates is well-defined on the cluster torus $\{p_i \neq 0 \mid \forall\, 1 \le i \le m-1\}$ inside $\check{X}^\circ$. Moreover, an easy calculation shows that it transforms our LG model (4) into the Hori-Vafa model (1) for odd quadrics. Note that this change of coordinates may also be obtained by combining the isomorphism between (4) and the Gorbounov-Smirnov mirror from [PR13a, Section 6], with the comparison between the Gourbounov-Smirnov mirror and Hori-Vafa's mirror in [GS13].

## 3. Landau-Ginzburg models for even quadrics

We view the quadric $X = X_{2m-2} := Q_{2m-2}$ of dimension $2m-2$ as a homogeneous space for the Spin group $\mathrm{Spin}_{2m}(\mathbb{C})$. In this section we will introduce a natural LG model for $X_{2m-2}$ which will be defined on an open subvariety of a dual quadric $\check{X}_{2m-2} = P\backslash \mathrm{PSO}_{2m}(\mathbb{C})$, see Section 3.2. Note that the projective special orthogonal group $\mathrm{PSO}_{2m}$ is the Langlands dual group to $\mathrm{Spin}_{2m}$, and both groups have the same Dynkin diagram, namely the Dynkin diagram of type $D_m$. The main result of this section, Proposition 3.3, shows that the new LG-model is isomorphic to one defined earlier [Rie08] on a Richardson variety $\mathcal{R}$ inside the full flag variety of $\mathrm{PSO}_{2m}(\mathbb{C})$.

Note that in the following we will denote the group $\mathrm{PSO}_{2m}(\mathbb{C})$ by $G$, since this is the group we will primarily be working with. Then the $A$-model symmetry group is $G^\vee = \mathrm{Spin}_{2m}(\mathbb{C})$, and we have $X_{2m-2} = G^\vee/P^\vee$, where $P^\vee$ is the parabolic subgroup associated to the first node of the Dynkin diagram of type $D_m$.



3.1. **Notations and definitions.** Let $V = \mathbb{C}^{2m}$ with fixed quadratic form

$$Q = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & \iddots & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix}.$$

In other words $Q(v_i, v_j) = (-1)^{\max(i,j)}\delta_{i+j,2m+1}$ where $\{v_i\}$ is the standard basis of $\mathbb{C}^{2m}$. For $G = \mathrm{PSO}(V,Q) = \mathrm{PSO}(V)$ we fix Chevalley generators $(e_i)_{1\le i\le m}$ and $(f_i)_{1\le i\le m}$. To be explicit we embed $\mathfrak{so}(V,Q)$ into $\mathfrak{gl}(V)$ and set

$$e_i = \begin{cases} E_{i,i+1} + E_{2m-i,2m-i+1} & \text{if } 1 \le i \le m-1, \\ E_{m-1,m+1} + E_{m,m+2} & \text{if } i = m, \end{cases}$$

and $f_i := e_i^T$, the transpose matrix, for every $i = 1, \dots, m$. Here $E_{i,j} = (\delta_{i,k}\delta_{l,j})_{k,l}$ is the standard basis of $\mathfrak{gl}(V)$. For elements of the group $\mathrm{PSO}(V)$, we will take matrices to represent their equivalence classes. We have Borel subgroups $B_+ = TU_+$ and $B_- = TU_-$ consisting of upper-triangular and lower-triangular matrices in $\mathrm{PSO}(V)$, respectively. Here $U_+$ and $U_-$ are the unipotent radicals of $B_+$ and $B_-$, respectively, and $T$ is the maximal torus of $\mathrm{PSO}(V)$, consisting of diagonal matrices $(d_{ij})$ with non-zero entries $d_{i,i} = d^{-1}_{2m-i+1,2m-i+1}$. We let $X(T) = \mathrm{Hom}(T, \mathbb{C}^*)$, $R \subset X(T)$ the set of roots, and $R^+$ the positive roots. We denote the set of simple roots by $\Pi = \{\alpha_i \mid 1 \le i \le m\} \subset R^+ \subset R \subset X(T)$, and the set of fundamental weights (which is the dual basis in $X(T)$) by $\{\omega_i \mid 1 \le i \le m\} \subset X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

The parabolic subgroup $P$ of $\mathrm{PSO}(V)$ we are interested in is the one whose Lie algebra $\mathfrak{p}$ is generated by all of the $e_i$ together with $f_2, \dots, f_m$, leaving out $f_1$. Let $x_i(a) := \exp(ae_i)$ and $y_i(a) := \exp(af_i)$. The Weyl group $W$ of $\mathrm{PSO}(V)$ is generated by simple reflections $s_i$ for which we choose representatives

$$\dot{s}_i = y_i(-1)x_i(1)y_i(-1). \tag{6}$$

We let $W_P$ denote the parabolic subgroup of the Weyl group $W$, namely $W_P = \langle s_2, \dots, s_m\rangle$. The length of a Weyl group element $w$ is denoted by $\ell(w)$. The longest element in $W_P$ is denoted by $w_P$. We also let $w_0$ be the longest element in $W$. Next $W^P$ is defined to be the set of minimal length coset representatives for $W/W_P$. The minimal length coset representative for $w_0$ is denoted by $w^P$.

We introduce the following notation for the elements of $W^P$. Namely, $W^P = \{e, w_1, \dots, w_{m-1}, w'_{m-1}, w_m, w_{m+1}, \dots w_{2m-2}\}$, where

$$w_k = \begin{cases} s_k s_{k-1} \dots s_1 & \text{if } 1 \le k \le m-2, \\ s_{m-1}s_{m-2}\dots s_1 & \text{if } k = m-1, \\ s_m s_{m-1} s_{m-2} \dots s_1 & \text{if } k = m, \\ s_{2m-1-k}\dots s_{m-2}s_m s_{m-1}s_{m-2}\dots s_1 & \text{if } m+1 \le k \le 2m-2. \end{cases}$$

and $w'_{m-1} = s_m s_{m-2} \dots s_1$.

For any $w \in W$ let $\dot{w}$ denote the representative of $w$ in $G$ obtained by setting $\dot{w} = \dot{s}_{i_1} \cdots \dot{s}_{i_r}$, where $w = s_{i_1} \cdots s_{i_r}$ is a reduced expression and $\dot{s}_i$ is as in (6). Each $\dot{w}_k \in \mathrm{PSO}(V)$ can be represented by a matrix $[w_k] \in \mathrm{SO}(V)$ such that

$$[w_k] \cdot v_{2m} = \begin{cases} v_{2m-k} & 1 \le k < m-1, \\ v_{2m-k-1} & m-1 < k \le 2m-2, \end{cases} \tag{7}$$

and $[w'_{m-1}] \cdot v_{2m} = v_m$ and $[w_{m-1}] \cdot v_{2m} = v_{m+1}$.

3.2. **The dual quadric and its Plücker coordinates.** Consider the homogeneous space $\check{X}_{2m-2} = P\backslash \mathrm{PSO}(V)$. It is canonically identified with the isotropic Grassmannian of lines in $V^*$, when this Grassmannian is viewed as a homogeneous space via the action of $\mathrm{PSO}(V)$ from the right. Moreover the isotropic Grassmannian of lines is also a $(2m-2)$-dimensional quadric $\check{X}_{2m-2} =: \check{Q}_{2m-2}$, now in $\mathbb{P}(V^*)$. So in this case, the varieties $X$ and $\check{X}$ are (non-canonically) isomorphic. The reason for this isomorphism of varieties is that the group $G^\vee$ is of simply-laced type. However Lie-theoretically we still think of $X_{2m-2}$ and $\check{X}_{2m-2}$ as being very different homogeneous spaces, with $X_{2m-2} = \mathrm{Spin}_{2m}(\mathbb{C})/P^\vee$ and $\check{X}_{2m-2} = P\backslash \mathrm{PSO}_{2m}(\mathbb{C})$.

**Definition 3.1** (Plücker coordinates)**.** The Plücker coordinates for $\check{X}_{2m-2} = P\backslash \mathrm{PSO}(V)$ are the homogeneous coordinates coming from the embedding of $\check{X}_{2m-2}$ into $\mathbb{P}(V^*)$ as the (right) $G$-orbit of the line $\mathbb{C}v^*_{2m}$:

$$\check{X}_{2m-2} = P\backslash \mathrm{PSO}(V) \to \mathbb{P}(V^*) : Pg \mapsto (\mathbb{C}v^*_{2m}) \cdot g.$$

We think of the Plücker coordinates as corresponding to the elements of $W^P$. Let $v^-_{\omega_i}$ (respectively $v^+_{\omega_i}$) denote lowest and highest weight vectors in the highest weight representation $V_{\omega_i}$. Then the Plücker coordinates may be defined by:

$$\begin{aligned} p_0(g) &= \langle v^*_{2m} \cdot [g], v_{2m} \rangle, \\ p_k(g) &= \langle v^*_{2m} \cdot [g], [w_k] \cdot v_{2m} \rangle \text{ for } 1 \leq k \leq 2m-2, \\ p'_{m-1}(g) &= \langle v^*_{2m} \cdot [g], [w'_{m-1}] \cdot v_{2m} \rangle, \end{aligned}$$

where $[g] \in \mathrm{SO}(V)$ is any fixed matrix representing $g \in \mathrm{PSO}(V)$. The homogeneous coordinates of $Pg$ are then given by

$$(p_0(g) : \ldots : p_{m-2}(g) : p_{m-1}(g) : p'_{m-1}(g) : p_m(g) : \ldots \ : p_{2m-2}(g)).$$

These are simply the bottom row entries of $[g]$ read from right to left, keeping in mind (7).

We may now write down the equation of the quadric $\check{X}_{2m-2}$ in terms of Plücker coordinates:

$$p_{m-1}p'_{m-1} - p_{m-2}p_m + p_{m-3}p_{m+1} - \cdots + (-1)^{m-1} p_0 p_{2m-2} = 0. \tag{8}$$

We note that as in the case of the odd quadric these Plücker coordinates are to be thought of as $B$-model incarnations of the Schubert classes of $Q_{2m-2}$. Namely recall that $H^*(Q_{2m-2}, \mathbb{C})$ has a Schubert basis indexed by $W^P$. We will use the notation $\sigma_i = \sigma_{w_i}$ and $\sigma'_{m-1} = \sigma_{w'_{m-1}}$ and $\sigma_0 = \sigma_e$. As a special case of the geometric Satake correspondence [Lus83, Gin95, MV07] we have that the (defining) projective representation $V$ of $PSO_{2m}(V)$ is identified with the cohomology of $Q_{2m-2}$,

$$V = H^*(Q_{2m-2}, \mathbb{C}),$$

and the standard basis $v_i$ agrees with the Schubert basis via $v_{2m} = \sigma_0$ and

$$[w_i] \cdot v_{2m} = \sigma_i, \quad [w'_{m-1}] \cdot v_{2m} = \sigma'_{m-1}. \tag{9}$$

The Schubert classes $\sigma_w$ are in this way naturally identified with the Plücker coordinates.

3.3. **The superpotential for $Q_{2m-2}$ on a dual quadric.** In this section we state our theorem describing a superpotential for $Q_{2m-2}$ in terms of Plücker coordinates on the dual quadric $\check{X}_{2m-2} = \check{Q}_{2m-2}$. Consider

$$\check{X}^\circ = \check{X}^\circ_{2m-2} := \check{X} \setminus D, \tag{10}$$

where $D := D_0 + D_1 + \ldots + D_{m-2} + D_{m-1} + D'_{m-1}$, the $D_i$'s being given by

$$\begin{aligned}
&D_0 := \{p_0 = 0\}, \\
&D_\ell := \left\{ \sum_{k=0}^{\ell} (-1)^k p_{\ell-k} p_{2m-2-\ell+k} = 0 \right\} \text{ for } 1 \le \ell \le m-3, \\
&D_{m-2} := \{p_{2m-2} = 0\}, \\
&D_{m-1} := \{p_{m-1} = 0\}, \\
&D'_{m-1} := \{p'_{m-1} = 0\}.
\end{aligned}$$

The divisor $D$ is an anti-canonical divisor in $\check{X}$. For simplicity, we will define

$$\delta_\ell = \sum_{k=0}^{\ell} (-1)^k p_{\ell-k} p_{N-\ell+k} \text{ for } 0 \le \ell \le m-3. \tag{11}$$

(For even quadrics, $N = 2m-2$.)

Our first result is the following theorem.

**Theorem 3.2.** *The LG model for $Q_{2m-2} = \mathrm{Spin}_{2m}/P^\vee$ from* [Rie08] *is isomorphic to $(\check{X}^\circ_{2m-2}, W_q)$, where $W_q : \check{X}^\circ_{2m-2} \to \mathbb{C}$ is defined by*

$$W_q = \frac{p_1}{p_0} + \sum_{\ell=1}^{m-3} \frac{p_{\ell+1} p_{2m-2-\ell}}{\delta_\ell} + \frac{p_m}{p_{m-1}} + \frac{p_m}{p'_{m-1}} + q \frac{p_1}{p_{2m-2}}. \tag{12}$$

This isomorphism is defined in Section 3.5. Before we begin the proof we need to recall the definition of the LG-model from [Rie08].

3.4. **The superpotential for $Q_{2m-2}$ on a Richardson variety.** Following [Rie08] consider the (open) Richardson variety $\mathcal{R} := R_{w_P, w_0} \subset G/B_-$, namely

$$\mathcal{R} := R_{w_P, w_0} = (B_+ \dot{w}_P B_- \cap B_- \dot{w}_0 B_-)/B_-.$$

This Richardson variety $\mathcal{R}$ is irreducible of dimension $2m-2$, and its closure is the Schubert variety $\overline{B_+ \dot{w}_P B_-/B_-}$. Let $T^{W_P}$ be the $W_P$-fixed part of the maximal torus $T$. Note that since we are in the setting of Section 3.1 we have that $T^{W_P} \cong \mathbb{C}^*$ with isomorphism given by $\alpha_1$. The inverse isomorphism is $\omega_1^\vee : \mathbb{C}^* \to T^{W_P}$. We fix a $d \in T^{W_P}$. Then one can define

$$Z_d := B_- \dot{w}_0 \cap U_+ d \dot{w}_P U_- \subset G, \tag{13}$$

and the map

$$\pi_R : Z_d \to \mathcal{R} : g \mapsto gB_-. \tag{14}$$

is an isomorphism from $Z_d$ to the open Richardson variety [Rie08, Section 4.1].

Let $q$ be the non-vanishing coordinate on the 1-dimensional torus $T^{W_P}$ given by $\alpha_1 : T^{W_P} \to \mathbb{C}^*$. The mirror LG model is a regular function on $\mathcal{R}$ depending also on $q$, and hence a regular function on $\mathcal{R} \times T^{W_P}$. It is defined as follows [Rie08]:

$$\mathcal{F} : (u_1 \dot{w}_P B_-, d) \;\mapsto\; g = u_1 d \dot{w}_P \bar{u}_2 \in Z_d \;\mapsto\; \sum e_i^*(u_1) + \sum f_i^*(\bar{u}_2), \tag{15}$$

where $u_1 \in U_+, \bar{u}_2 \in U_-$, and where $\bar{u}_2$ is determined by $u_1$ and the property that $u_1 d\dot{w}_P \bar{u}_2 \in Z_d$.

The corresponding map from $\mathcal{R}$, when the coordinate $q$ is fixed, is denoted

$$\mathcal{F}_q : \mathcal{R} \to \mathbb{C} : u_1 \dot{w}_P B_- \mapsto \mathcal{F}(u_1 \dot{w}_P B_-, \omega_1^\vee(q)).$$

*Remark* 1. Note that if $g = u_1 d\dot{w}_P \bar{u}_2 \in Z_d$, then we have a simple identity concerning the Plücker coordinates:

$$(p_0(g) : \ldots : p_{2m-2}(g)) = (p_0(\bar{u}_2) : \ldots : p_{2m-2}(\bar{u}_2)).$$

The remainder of Section 3 will be devoted to proving Theorem 3.2, which now says that there is an isomorphism $\check{X}^\circ_{2m-2} \xrightarrow{\sim} \mathcal{R}$ under which $W_q$ is identified with $\mathcal{F}_q$.

3.5. **Isomorphism with the Richardson variety.** To prove Theorem 3.2, the first step is to construct an isomorphism between $\check{X}^\circ_{2m-2}$ and the open Richardson variety $\mathcal{R}$. We define the following maps:

$$\begin{array}{ccccc} \check{X} = P\backslash G & \xleftarrow{\pi_L} & Z_d = B_- \dot{w}_0 \cap U_+ d\dot{w}_P U_- & \xrightarrow{\pi_R} & \mathcal{R}, \\ Pg & \leftarrow\!\shortmid & g & \mapsto & gB_-, \end{array}$$

given by taking left and right cosets, respectively. Note that $g$ is equal to $b_- \dot{w}_0$ in our previous notation and factorizes (a priori non-uniquely) as

$$g = u_1 d\dot{w}_P \bar{u}_2.$$

Moreover $\pi_R$ is an isomorphism, so we have $\pi := \pi_L \circ \pi_R^{-1} : \mathcal{R} \to \check{X}_{2m-2}$. Our next goal is to prove:

**Proposition 3.3.** *$\pi_L$ defines an isomorphism from $Z_d$ to $\check{X}^\circ_{2m-2}$. As a consequence, $\pi$ defines an isomorphism from $\mathcal{R}$ to $\check{X}^\circ_{2m-2}$.*

Our proof uses a presentation of the coordinate ring of the unipotent cell

$$U_-^P := U_- \cap B_+ (\dot{w}^P)^{-1} B_+ \tag{16}$$

due to [GLS11]. The strategy of the proof of Proposition 3.3 is as follows.

- The first step is to show that the natural map $\pi_L : Z_d \to \check{X}$ factorizes as $\phi \circ \theta$ where $\phi : U_-^P \to \check{X}$ with $\phi(\bar{u}) = P\bar{u}$ and $\theta : Z_d \to U_-^P$ is an isomorphism which will be constructed in Lemma 3.4.
- We then use the presentation of the coordinate ring of $U_-^P$ to show that the image of the map $\phi$ lands in $\check{X}^\circ_{2m-2}$ and not just $\check{X}_{2m-2}$. That is, the Plücker coordinates $p_0$, $p_{2m-2}$, $p_{m-1}$, $p'_{m-1}$ and the functions $\delta_\ell$ (defined in (3)) do not vanish. Finally, we show that $\phi$ is an isomorphism from $U_-^P$ to $\check{X}^\circ_{2m-2}$. The main step is to find a pre-image for each of the functions generating $\mathbb{C}[U_-^P]$.

**Lemma 3.4.** *There exists an isomorphism $\theta : Z_d \to U_-^P$ such that for $b\dot{w}_0 \in Z_d$,*

$$Pb\dot{w}_0 = P\bar{u}_2, \tag{17}$$

*where $\bar{u}_2 := \theta(b\dot{w}_0)$.*

To prove Lemma 3.4 we use an isomorphism introduced by Berenstein and Zelevinsky in [BZ97] (and joint with Fomin in type $A$ [BFZ96]) which is sometimes called the BZ twist (or BFZ twist).

**Theorem 3.5.** [BZ97, Theorem 1.2] *Let $y \in U_- \cap B_+\dot{w}^{-1}B_+$. There exists a unique $x \in U_+ \cap B_-\dot{w}B_-$ such that $U_+ \cap B_-\dot{w}y = \{x\}$. The resulting map $\tilde{\eta}_w : U_- \cap B_+\dot{w}^{-1}B_+ \to U_+ \cap B_-\dot{w}B_-$ sending $y$ to $x$ is an isomorphism. In particular we have an inverse isomorphism*

$$\varepsilon_w : U_+ \cap B_-\dot{w}B_- \to U_- \cap B_+\dot{w}^{-1}B_+.$$

*Remark* 2. We note that the original twist map of Berenstein and Zelevinsky is an automorphism $\eta_w : U_+ \cap B_-\dot{w}B_- \to U_+ \cap B_-\dot{w}B_-$. Our map $\tilde{\eta}_w$ is related to $\eta_w$ by

$$\tilde{\eta}_w(y) = \eta_w(y^T),$$

where $y^T$ denotes the transpose of $y$. We have

$$\tilde{\eta}_w(y) = x \quad \iff \quad B_-wy = B_-x.$$

Here we may write $B_-w$ for $B_-\dot{w}$, as the coset doesn't depend on the representative of $w$.

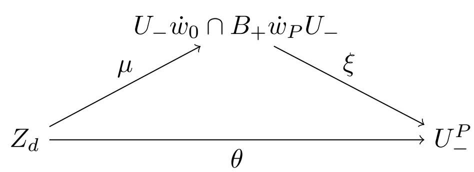
*Proof of Lemma 3.4.* The idea is to consider the two birational maps

$$\begin{array}{llllll} \Psi_1 : & U_-^P & \to & P\backslash G\,, & \bar{u}_2 & \mapsto & P\bar{u}_2, \\ \pi_L : & Z_d & \to & P\backslash G\,, & b_-\dot{w}_0 = u_1 d\dot{w}_P\bar{u}_2 & \mapsto & Pb_-\dot{w}_0, \end{array}$$

and to show that the composition

$$\theta := \Psi_1^{-1} \circ \pi_L : Z_d \to U_-^P. \tag{18}$$

is an isomorphism. We construct a commutative triangle of maps as follows.




Here $\mu : Z_d \to U_-\dot{w}_0 \cap B_+\dot{w}_P U_-$ is an isomorphism defined by $b_-\dot{w}_0 \mapsto [b_-]_0^{-1}b_-\dot{w}_0$, where $[b_-]_0$ is the torus part of $b_-$. The inverse isomorphism $\mu^{-1}$ is given by $b_+\dot{w}_P u_- \mapsto d[b_+]_0^{-1}b_+\dot{w}_P u_-$. Note that clearly $Pz = P\mu(z)$ for all $z \in Z_d$.

We now define a composition $\xi$ of isomorphisms as follows,

$$U_-\dot{w}_0 \cap B_+\dot{w}_P U_- \xrightarrow{\ell_{\dot{w}_0^{-1}}} U_+ \cap B_-w^P B_- \xrightarrow{\varepsilon_{w^P}} U_- \cap B_+(\dot{w}^P)^{-1}B_+,$$

where $\ell_{\dot{w}_0^{-1}}$ is the left multiplication by $\dot{w}_0^{-1}$ map. Hence we obtain an isomorphism

$$\xi : U_-\dot{w}_0 \cap B_+\dot{w}_P U_- \longrightarrow U_-^P.$$

Suppose $u_-\dot{w}_0 \in U_-\dot{w}_0 \cap B_+\dot{w}_P U_-$. To prove the identity (17) it remains to check that $Pu_-\dot{w}_0 = P\bar{u}_2$ where $\bar{u}_2 = \xi(u_-\dot{w}_0)$. This follows from the defining property of $\varepsilon_{w^P}$. Namely if $u_-\dot{w}_0 \in U_-^P\dot{w}_0$ then if $y = \varepsilon_{s^P}(\dot{w}_0^{-1}w\dot{w}_0)$, we have

$$B_-\dot{w}_0^{-1}u_-\dot{w}_0 = B_-w^P\bar{u}_2 = B_-w_0w_P\bar{u}_2.$$

Therefore $B_+u_-\dot{w}_0 = B_+w_P\bar{u}_2$. □

For the second step of the proof of Proposition 3.3 we use a result of [GLS11] to describe the coordinate ring of the unipotent cell $U_-^P$. In Lemma 3.10 we then explicitly relate the coordinates on $U_-^P$ to the coordinates on $\check{X}^\circ_{2m-2}$, which are the Plücker coordinates from Definition 3.1. In this way we show that the map

$$\phi : U_-^P \to \check{X}, \quad \bar{u}_2 \mapsto P\bar{u}_2$$

restricts to an isomorphism onto its image, and that this image is $\check{X}^\circ_{2m-2}$.

We must first define the generalized minors involved in the presentation due to [GLS11]. Let $G^{sc}$ be the simply-connected covering group of $G = \mathrm{PSO}(V)$, with Borel subgroup $B_-^{sc}$ and unipotent radical $U_-^{sc}$ projecting to $B_-$ and $U_-$ in $G$. Here $G^{sc} = \mathrm{Spin}(V)$. Since $U_-^{sc} \cong U_-$ via this projection, we may use representations of $G^{sc}$ to define generalized minors of elements of $U_-$. For $u \in U_-$ we denote by $u^{sc}$ its lift to $U_-^{sc}$, and similarly for elements of $U_+$.

Let $w \in W$ have reduced expression $w = s_{i_1} s_{i_2} \dots s_{i_r}$. Write

$$\bar{s}_j = y_j^{sc}(1) x_j^{sc}(-1) y_j^{sc}(1)$$

and $\bar{w} = \bar{s}_{i_1} \bar{s}_{i_2} \dots \bar{s}_{i_r}$.

**Definition 3.6.** Let $w \in W$ and $\omega_j$ be a fundamental weight of $G^{sc}$. Let $V_{\omega_j}$ be the irreducible representation of $G^{sc}$ with highest weight $\omega_j$ and $v^+_{\omega_j}$ be a fixed highest weight vector. Define for any $u \in U_-$:

$$\Delta_{\omega_j, w\cdot\omega_j}(u) = \langle u^{sc} \cdot v^+_{\omega_j}, \bar{w} \cdot v^+_{\omega_j} \rangle.$$

Here $\langle u^{sc} \cdot v^+_{\omega_j}, \bar{w} \cdot v^+_{\omega_j} \rangle = \langle \bar{w}^{-1} u^{sc} \cdot v^+_{\omega_j}, \cdot v^+_{\omega_j} \rangle$ denotes the highest weight vector coefficient of $\bar{w}^{-1} u^{sc} \cdot v^+_{\omega_j}$ in terms of the weight space decomposition.

Note that the smallest representative $w^P$ in $W$ of $[w_0] \in W/W_P$ has the following reduced expression:

$$w^P = s_1 \dots s_{m-2} s_{m-1} s_m s_{m-2} \dots s_1. \tag{19}$$

Here we state the result from [GLS11] applied to our particular setting.

**Theorem 3.7** ([GLS11, Section 8])**.** *Consider the reduced expression $s_{i_1} \dots s_{i_{2m-2}} = s_1 \dots s_{m-2} s_m s_{m-1} s_{m-2} \dots s_1$ for $(\dot{w}^P)^{-1}$ coming from (19). The coordinate ring of the unipotent cell $U_-^P := U_- \cap B_+ (\dot{w}^P)^{-1} B_+$ inside $\mathrm{PSO}_{2m}$ is*

$$\mathbb{C}\left[U_-^P\right] = \mathbb{C}\left[\Delta_{\omega_{i_r}, (\dot{w}^P)^{-1}_{\le r} \cdot \omega_{i_r}}, \Delta^{-1}_{\omega_{2m-2-s}, (\dot{w}^P)^{-1}_{\le s} \cdot \omega_{2m-2-s}}\right]$$

*where*

- $1 \le r \le 2m-2$*;* $m-1 \le s \le 2m-2$ *;*
- $(\dot{w}^P)^{-1}_{\le r} := s_{i_1} \dots s_{i_r}$*.*

If $j < m$ then $\Delta_{\omega_j, w\cdot\omega_j}(u)$ is a minor in the usual sense for the unique matrix $u^{\mathrm{SO}_{2m}}$ in $U_-^{\mathrm{SO}_{2m}}$ representing $u$. We denote the minor of $u^{\mathrm{SO}_{2m}}$ with row set $\{i_1, \dots, i_p\}$ and column set $\{j_1, \dots, j_p\}$ by $D^{i_1,\dots,i_p}_{j_1,\dots,j_p}(u)$. We now reformulate Theorem 3.7 as follows.

**Corollary 3.8.** *The coordinate ring $\mathbb{C}\left[U_-^P\right]$ is generated by the minors*

$$D^{2,\dots,r,r+1}_{1,2,\dots,r}, \quad 1 \le r \le m-2;$$

$$D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}, \quad m+1 \le s \le 2m-3, \ \textit{and}\ D^{2m}_1;$$

*the functions*

$$\Delta_{\omega_m,\frac{1}{2}[-\epsilon_1+\epsilon_2+\cdots+\epsilon_{m-1}-\epsilon_m]} \text{ and } \Delta_{\omega_{m-1},\frac{1}{2}[-\epsilon_1+\epsilon_2+\cdots+\epsilon_m]},$$

*which are Pfaffians; the inverses of minors*

$$\left(D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}\right)^{-1}, \quad m+1\le s\le 2m-3, \text{ and } \left(D^{2m}_1\right)^{-1};$$

*and the inverses of Pfaffians*

$$\Delta^{-1}_{\omega_m,\frac{1}{2}[-\epsilon_1+\epsilon_2+\cdots+\epsilon_{m-1}-\epsilon_m]} \text{ and } \Delta^{-1}_{\omega_{m-1},\frac{1}{2}[-\epsilon_1+\epsilon_2+\cdots+\epsilon_m]}.$$

To relate the minors and Pfaffians of Corollary 3.8 to the Plücker coordinates we will need to use a specific factorisation of generic elements of $U^P_-$. By an application of Bruhat's lemma [Lus94], a generic element in $U^P_-$ can be assumed to have a particular factorisation:

$$\bar u_2 = y_1(a_1)\dots y_{m-2}(a_{m-2})y_m(d)y_{m-1}(c)y_{m-2}(b_{m-2})\dots y_1(b_1), \tag{20}$$

where $a_i, c, d, b_j \neq 0$.

We have the following standard expression for the $p_k$ on factorized elements, which is a simple consequence of their definition.

**Lemma 3.9.** *Fix $0\le k\le 2m-2$ an integer. Then if $\bar u_2$ is of the form (20) we have*

$$p_k(\bar u_2) = \begin{cases} 1 & \text{if } k=0,\\ a_1\dots a_{k-1}(a_k+b_k) & \text{if } 1\le k\le m-2,\\ a_1\dots a_{m-2}c & \text{if } k=m-1,\\ a_1\dots a_{m-2}cd & \text{if } k=m,\\ a_1\dots a_{m-2}cdb_{m-2}\dots b_{2m-1-k} & \text{otherwise.}\end{cases}$$

*and*

$$p'_{m-1}(\bar u_2) = a_1\dots a_{m-2}d.$$ □

We can now prove the lemma we need.

**Lemma 3.10.** *We have the following equalities of generalised minors and Plücker coordinates evaluated on $\bar u_2\in U^P_-$:*

$$D^{2m}_1(\bar u_2) = p_{2m-2}(\bar u_2), \tag{21}$$

$$\Delta_{\omega_{m-1},\frac{1}{2}[-\epsilon_1+\epsilon_2+\cdots+\epsilon_m]}(\bar u_2) = p_{m-1}(\bar u_2), \tag{22}$$

$$\Delta_{\omega_m,\frac{1}{2}[-\epsilon_1+\epsilon_2+\cdots+\epsilon_{m-1}-\epsilon_m]}(\bar u_2) = p'_{m-1}(\bar u_2), \tag{23}$$

$$D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}(\bar u_2) = \delta_{s-m}(\bar u_2), \text{ for } m+1\le s\le 2m-3, \tag{24}$$

*where we recall that $\delta_{s-m} = \sum_{k=s}^m (-1)^{s-k}p_{k-m}p_{3m-2-k}$.*

*Proof.* The identity (21) follows immediately from the definition of the Plücker coordinates. For the identity (22), write

$$\Delta_{\omega_{m-1},\frac{1}{2}[-\epsilon_1+\epsilon_2+\cdots+\epsilon_m]}(\bar u_2) = (D^{2,\dots,m,2m}_{1,\dots,m-1,m+1}(\bar u_2))^{\frac{1}{2}}.$$

Note that in the definition of $\Delta_{\omega_j,w\cdot\omega_j}$ we have chosen the representative $\bar w$ in such a way that evaluated on a factorized $\bar u_2$ the generalized minors will be nonnegative for any positive choice of the coordinates $a_i, b_i, c, d$ (i.e. on 'totally positive' $\bar u_2$).

This determines the choice of square root. Then developing $D^{2,\dots,m,2m}_{1,\dots,m-1,m+1}(\bar u_2)$ with respect to the last column, we get

$$D^{2,\dots,m,2m}_{1,\dots,m-1,m+1}(\bar u_2) = D^{2,\dots,m}_{1,\dots,m-1}(\bar u_2) D^{2m}_{m+1}(\bar u_2) = p_{m-1}(\bar u_2) D^{2,\dots,m}_{1,\dots,m-1}(\bar u_2)$$

using the definition of $p_{m-1}(\bar u_2)$. Finally, since the matrix is $\bar u_2$ orthogonal:

$$D^{2,\dots,m}_{1,\dots,m-1}(\bar u_2) = D^{1,\dots,m,2m}_{1,\dots,m+1}(\bar u_2).$$

Developing again with respect to the last column, we obtain

$$D^{1,\dots,m,2m}_{1,\dots,m+1}(\bar u_2) = D^{1,\dots,m}_{1,\dots,m}(\bar u_2) D^{2m}_{m+1}(\bar u_2) = p_{m-1}(\bar u_2),$$

using the definition of $p_{m-1}(\bar u_2)$ and the fact that $\bar u_2$ is lower unipotent. The identity (22) then follows. The proof of the identity (23) is similar.

Let us now prove the identity (24). Developing $D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}(\bar u_2)$ with respect to the $(2m-1-s)$-th column, we see that it is equal to

$$D^{m+1}_{2m-1-s}(\bar u_2) D^{2,\dots,2m-1-s}_{1,2,\dots,2m-2-s}(\bar u_2) - D^{2,\dots,2m-2-s,m+1}_{1,\dots,2m-2-s}(\bar u_2).$$

Since $\bar u_2$ is orthogonal for $Q$, we have

$$D^{2,\dots,2m-1-s}_{1,2,\dots,2m-2-s}(\bar u_2) = D^{1,\dots,s+1,2m}_{1,\dots,s+2}(\bar u_2),$$

and since $\bar u_2$ is in $U_-$,

$$D^{1,\dots,s+1,2m}_{1,\dots,s+2}(\bar u_2) = D^{2m}_{s+2}(\bar u_2) = p_{2m-2-s}(\bar u_2).$$

Finally

$$D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}(\bar u_2) = D^{m+1}_{2m-1-s}(\bar u_2) p_{2m-2-s}(\bar u_2) - D^{2,\dots,2m-2-s,m+1}_{1,\dots,2m-2-s}(\bar u_2),$$

hence

$$D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}(\bar u_2) = \sum_{k=s}^{2m-2} (-1)^{s-k} D^{m+1}_{2m-1-s}(\bar u_2) p_{2m-2-s}(\bar u_2).$$

We also have $D^{m+1}_{2m-1-s}(\bar u_2) = d b_{2m-2} \dots b_{2m-1-s}$ for $m+1 \le s \le 2m-2$. Indeed, by definition

$$D^{m+1}_{2m-1-s}(\bar u_2) = \langle v^*_{m+1} \cdot \bar u_2, v_{2m-1-s} \rangle = d b_{2m-2} \dots b_{2m-1-s}.$$

Hence

$$\begin{aligned} D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}(\bar u_2) &= \sum_{k=s}^{2m-2} (-1)^{s-k} d b_{2m-2} \dots b_{2m-1-s} p_{2m-2-s} \\ &= \sum_{k=s}^{m} (-1)^{s-k} p_{k-m}(\bar u_2) p_{3m-2-k}(\bar u_2). \end{aligned}$$ □

*Proof of Proposition 3.3.* Recall that $\pi_L = \phi \circ \theta$ where $\theta$ is the isomorphism constructed in Lemma 3.4 and $\phi : U^P_- \to \check X$ is the natural map $\bar u_2 \mapsto P \bar u_2$. It remains to prove that $\phi$ is an isomorphism onto $\check X^\circ_{2m-2}$. We start by proving that the image of $\phi$ is contained in $\check X^\circ_{2m-2}$. Indeed, if $\bar u_2 \in U^P_-$, then by Corollary 3.8 the minors $D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}(\bar u_2)$ and $D^{2m}_1(\bar u_2)$ and the Pfaffians $\Delta_{\omega_m, \frac12[-\epsilon_1+\epsilon_2+\dots+\epsilon_{m-1}-\epsilon_m)]}(\bar u_2)$ and $\Delta_{\omega_{m-1}, \frac12[-\epsilon_1+\epsilon_2+\dots+\epsilon_m)]}(\bar u_2)$ do not vanish. Since we have proved in Lemma 3.10 that those correspond precisely to the divisors involved in defining $\check X^\circ_{2m-2}$, it follows that $P \bar u_2 \in \check X^\circ_{2m-2}$. We may now prove that $\phi$ is an isomorphism between $U^P_-$ and $\check X^\circ_{2m-2}$.

Injectivity of the pullback map $\phi^* : \mathbb{C}[\check{X}^\circ_{2m-2}] \to \mathbb{C}[U^P_-]$ is a simple consequence of the fact that the map $U^P_- \to \check{X}_{2m-2}$ is dominant. We now prove that $\phi^*$ is surjective by observing that each of the functions generating $\mathbb{C}[U^P_-]$ (as in Corollary 3.8) has a preimage.

We have already seen that the inverses of minors and Pfaffians correspond to the inverses of denominators of $W_q$. Let us now consider the minors $D^{2,\ldots,r,r+1}_{1,2,\ldots,r}$ for $1 \leq r \leq m-2$ and $D^{2,\ldots,2m-1-s,m+1}_{1,2,\ldots,2m-1-s}$ for $m+1 \leq s \leq 2m-3$. In Lemma 3.10, we proved that

$$D^{2,\ldots,2m-1-s,m+1}_{1,2,\ldots,2m-1-s} = \phi^*(\delta_{s-m})$$

and

$$D^{2,\ldots,r+1}_{1,\ldots,r} = D^{1,\ldots,2m-1-r}_{1,\ldots,2m-r} = D^{2m}_{2m-r} = \phi^*(p_r).$$

Finally, $D^{2m}_1 = \phi^*(p_{2m-2})$, and the Pfaffians

$$\Delta_{\omega_m, \frac{1}{2}[-\epsilon_1+\epsilon_2+\cdots+\epsilon_{m-1}-\epsilon_m)]} \text{ and } \Delta_{\omega_{m-1}, \frac{1}{2}[-\epsilon_1+\epsilon_2+\cdots+\epsilon_m)]}.$$

are pullbacks of the Plücker coordinates $p'_{m-1}$ and $p_{m-1}$, by Lemma 3.10. This concludes the proof. $\square$

3.6. **Comparison of the superpotentials.** In this section we will prove Theorem 3.2. We saw in the previous section that $\pi = \pi_L \circ \pi_R^{-1} : \mathcal{R} \to \check{X}^\circ_{2m-2}$ is an isomorphism. Note that we have a commutative diagram

$$\begin{array}{ccc} Z_d & \xrightarrow[\sim]{\pi_R} & \mathcal{R} \\ & {\scriptstyle F_q}\searrow \quad \swarrow{\scriptstyle \mathcal{F}_q} & \\ & \mathbb{C} & \end{array}$$

Therefore

$$(\pi^{-1})^*(\mathcal{F}_q) = (\pi_L^{-1})^*(F_q).$$

This gives a regular function on $\check{X}^\circ_{2m-2}$ which we denote by $\widetilde{W}_q$. The statement of Theorem 3.2 says that $\widetilde{W}_q$ and $W_q$ agree. We will prove this by expressing both functions in terms of coordinates introduced earlier. Namely we consider the set of factorized elements $P\bar{u}_2$ with $\bar{u}_2$ as in (20) with nonzero coordinates $a_i, b_i, c, d$ as defining an open dense subvariety inside $\check{X}^\circ_{2m-2}$ which is isomorphic to a torus. We call this subvariety $\mathcal{X}^\circ_{2m-2}$. To finish the proof we will show that the restrictions of $\widetilde{W}_q$ and of $W_q$ to $\mathcal{X}^\circ_{2m-2}$ agree. This will additionally give an interesting Laurent polynomial formula for the superpotential, which we will use in Section 7 to describe a flat section of the Dubrovin connection.

**Proposition 3.11.** *$\widetilde{W}_q$ restricted to $\mathcal{X}^\circ_{2m-2}$ has the following Laurent polynomial expression*

$$a_1 + \cdots + a_{m-2} + c + d + b_{m-2} + \cdots + b_1 + q\frac{a_1 + b_1}{a_1 \ldots a_{m-2}cdb_{m-2}\ldots b_1}. \tag{25}$$

To prove Proposition 3.11 we will need the following:

**Lemma 3.12.** *If* $u_1 \in U_+$, $\bar{u}_2 \in U_-$, $u_1 d\dot{w}_P \bar{u}_2 \in Z_d$, *and* $\bar{u}_2$ *can be written as in* (20), *then we have the following identities:*

$$f_i^*(\bar{u}_2) = \begin{cases} a_i + b_i & \text{if } 1 \le i \le m-2, \\ c & \text{if } i = m-1, \\ d & \text{if } i = m. \end{cases} \tag{26}$$

$$e_i^*(u_1) = \begin{cases} 0 & \text{if } 2 \le i \le m, \\ q\frac{a_1+b_1}{a_1 \dots a_{m-1} c d b_{m-1} \dots b_1} & \text{if } i = 1. \end{cases} \tag{27}$$

*Proof.* Equation (26) is obtained immediately from the definition of $\bar{u}_2$. For Equation (27), notice that

$$\begin{aligned} e_i^*(u_1) &= \frac{\langle u_1^{-1} \cdot v_{\omega_i}^-, e_i \cdot v_{\omega_i}^- \rangle}{\langle u_1^{-1} \cdot v_{\omega_i}^-, v_{\omega_i}^- \rangle} \\ &= \frac{\langle d\dot{w}_P \bar{u}_2 \cdot v_{\omega_i}^+, e_i \cdot v_{\omega_i}^- \rangle}{\langle d\dot{w}_P \bar{u}_2 \cdot v_{\omega_i}^+, v_{\omega_i}^- \rangle}. \end{aligned}$$

Assume $2 \le i \le m$. Then $e_i^*(u_1) = 0$ if and only if $\langle \bar{u}_2 \cdot v_{\omega_i}^+, \dot{w}_P^{-1} e_i \cdot v_{\omega_i}^- \rangle = 0$. Now the vector $w_P^{-1} e_i \cdot v_{\omega_i}^-$ is in the $\mu$-weight space of the $i$-th fundamental representation, where $\mu = w_P^{-1} s_i(-\omega_i)$. Moreover, $\bar{u}_2 \in B_+(\dot{w}^P)^{-1}B_+$, hence $\bar{u}_2 \cdot v_{\omega_i}^+$ can have non-zero components only down to the weight space of weight $(w^P)^{-1}(\omega_i) = w_P^{-1}(-\omega_i)$. Since $l(w_P^{-1}s_i) > l(w_P^{-1})$ for $2 \le i \le m$, this is higher than $\mu$, which proves that $e_i^*(u_1) = 0$.

Now assume $i = 1$. We have

$$\begin{aligned} e_1^*(u_1) &= \frac{\langle d\dot{w}_P \bar{u}_2 \cdot v_{\omega_1}^+, e_1 \cdot v_{\omega_1}^- \rangle}{\langle d\dot{w}_P \bar{u}_2 \cdot v_{\omega_1}^+, v_{\omega_1}^- \rangle} \\ &= (\omega_1 + \alpha_1 - \omega_1)(d) \frac{\langle \bar{u}_2 \cdot v_{\omega_1}^+, \dot{w}_P^{-1} e_1 \cdot v_{\omega_1}^- \rangle}{\langle \bar{u}_2 \cdot v_{\omega_1}^+, \dot{w}_P v_{\omega_1}^- \rangle} \\ &= q\frac{\langle \bar{u}_2 \cdot v_{\omega_1}^+, \dot{w}_P^{-1} e_1 \cdot v_{\omega_1}^- \rangle}{\langle \bar{u}_2 \cdot v_{\omega_1}^+, v_{\omega_1}^- \rangle}. \end{aligned}$$

First look at the denominator. The only way to go from the highest weight vector $v_{\omega_1}^+$ of the first fundamental representation to the lowest weight vector $v_{\omega_1}^-$ is to apply $g \in B_+ w B_+$ for $w \ge (w^P)^{-1}$. Since $\bar{u}_2 \in B_+(\dot{w}^P)^{-1}B_+$, it follows that we need to take all factors of $\bar{u}_2$, and normalising $v_{\omega_1}^-$ appropriately, we get

$$\langle \bar{u}_2 \cdot v_{\omega_1}^+, v_{\omega_1}^- \rangle = a_1 \dots a_{m-1} c d b_{m-1} \dots b_1.$$

Finally, we look at the numerator $\langle \bar{u}_2 \cdot v_{\omega_1}^+, \dot{w}_P^{-1} e_1 \cdot v_{\omega_1}^- \rangle$. The vector $\dot{w}_P^{-1} e_1 \cdot v_{\omega_1}^-$ has weight

$$\mu' = \dot{w}_P^{-1} s_1(-\omega_1) = \dot{w}_P^{-1}(-\epsilon_2) = \epsilon_2.$$

Write $w_P^{-1}s_1$ as a prefix $w' = s_1 s_2 \dots s_{m-2} s_m s_{m-1} s_{m-2} \dots s_2$ of $(w^P)^{-1}$. We have $w' s_1 = (w^P)^{-1}$, hence the way from $v_{\omega_1}^+$ to $w' \cdot v_{\omega_1}^-$ is through $s_1$. From the factorization of $\bar{u}_2$ in (20), it follows that $\langle \bar{u}_2 \cdot v_{\omega_1}^+, \dot{w}_P^{-1} e_1 \cdot v_{\omega_1}^- \rangle = a_1 + b_1$. □

*Proof of Proposition 3.11.* Using the expression (15) of the superpotential from [Rie08], we immediately deduce expression for $\widetilde{W}_q$ as a Laurent polynomial from Lemma 3.12. □

Next, using Lemma 3.9 and Proposition 3.11, we express $\widetilde{W}_q$ in terms of Plücker coordinates and deduce the theorem.

*Proof of Theorem 3.2.* From Lemma 3.9, it follows that for $\bar{u}_2$ as in (20)

$$p_{\ell+1}(\bar{u}_2)p_{2m-2-\ell}(\bar{u}_2) = (a_{\ell+1}+b_{\ell+1})(a_1\dots a_\ell)^2 a_{\ell+1}\dots a_{m-2}cdb_{m-2}\dots b_{\ell+1}$$

for $0 \le \ell \le m-3$. We also get that $p_k(\bar{u}_2)p_{2m-2-k}(\bar{u}_2)$ is equal to

$$\begin{cases} a_1\dots a_{m-2}cdb_{m-2}\dots b_1 & \text{if } k=0;\\ (a_1+b_1)a_1\dots a_{m-2}cdb_{m-2}\dots b_2 & \text{if } k=1;\\ (a_k+b_k)(a_1\dots a_{k-1})^2a_k\dots a_{m-2}cdb_{m-2}\dots b_{k+1} & \text{if } 2\le k\le m-3. \end{cases} \tag{28}$$

Using (28), we find that most terms in $\delta_\ell(\bar{u}_2) = \sum_{k=0}^{\ell}(-1)^k p_{\ell-k}(\bar{u}_2)p_{2m-2+k-\ell}(\bar{u}_2)$ cancel, and

$$\delta_\ell(\bar{u}_2) = (a_1\dots a_\ell)^2 a_{\ell+1}\dots a_{m-2}cdb_{m-2}\dots b_{\ell+1}.$$

This proves that

$$\frac{p_{\ell+1}p_{2m-2-\ell}}{\delta_\ell}(\bar{u}_2) = a_{\ell+1}+b_{\ell+1}$$

for $0 \le \ell \le m-3$. Moreover:

$$\frac{p_m}{p_{m-1}}(\bar{u}_2) = \frac{a_1\dots a_{m-2}cd}{a_1\dots a_{m-2}c} = d$$

and

$$\frac{p_m}{p'_{m-1}}(\bar{u}_2) = \frac{a_1\dots a_{m-2}cd}{a_1\dots a_{m-2}d} = c.$$

For the first and last terms, we obtain

$$\frac{p_1}{p_0}(\bar{u}_2) = a_1+b_1$$

and

$$\frac{p_1}{p_{2m-2}}(\bar{u}_2) = \frac{a_1+b_1}{a_1\dots a_{m-1}cdb_{m-1}\dots b_1}$$

as easy consequences of Lemma 3.9. Using Proposition 3.11, this proves that $\widetilde{W}_q$ coincides with the definition of $W_q$ from Equation (12)

$$W_q = \frac{p_1}{p_0} + \sum_{\ell=1}^{m-3}\frac{p_{\ell+1}p_{2m-2-\ell}}{\delta_\ell} + \frac{p_m}{p_{m-1}} + \frac{p_m}{p'_{m-1}} + q\frac{p_1}{p_{2m-2}}. \qquad \square$$

3.7. **Comparison with the Hori-Vafa model for even quadrics.** Here we check that once restricted to a particular intersection $\tilde{T}$ of two cluster tori our LG model embeds into the Hori-Vafa LG model. Let

$$\tilde{T} := \{x \in \check{X}^\circ \mid p_i(x) \neq 0 \text{ for all } 0 \le i \le m-2 \text{ and } p_m(x) \neq 0\}.$$

We consider the change of coordinates:

$$Y_i = \begin{cases} \frac{p_i}{p_{i-1}} & \text{for } 1 \le i \le m-2;\\ \frac{p_{2m-3-i}\delta_{2m-5-i}}{p_{2m-4-i}\delta_{2m-4-i}} & \text{for } m-1 \le i \le 2m-5;\\ \frac{p_m}{p_{m-1}} & \text{for } i = 2m-4;\\ \frac{p_m}{p'_{m-1}} & \text{for } i = 2m-3;\\ q\frac{\delta_{m-3}}{\delta_{m-2}} & \text{for } i = 2m-2. \end{cases}$$

An easy calculation shows that it transforms our superpotential (12) into the Hori-Vafa superpotential (1) for even quadrics.

## 4. The A-model and B-model connections

Our expression for the LG-model $W$ in terms of homogeneous coordinates coming from $\check{X}^\circ \subset \mathbb{P}(H^*(X,\mathbb{C})^*)$ makes it possible to compare the (small) Dubrovin connection on the A side and the Gauss-Manin connection on the B side. We recall the relevant definitions on the $A$-side.

Let $X = Q_N$. Consider $H^*(X, \mathbb{C}[\hbar, q])$ as a space of sections on a trivial bundle with fiber $H^*(X,\mathbb{C})$, over the base $\mathbb{C}_\hbar \times \mathbb{C}_q$, where the $\hbar$ and $q$ are the coordinates. Let Gr be the operator on sections defined on the fibres as the 'grading operator' $H^*(X,\mathbb{C}) \to H^*(X,\mathbb{C})$ which multiplies $\sigma \in H^{2k}(X,\mathbb{C})$ by $k$. We define the Dubrovin connection by

$$ {}^A\nabla_{q\partial_q} S \quad := \quad q\frac{\partial S}{\partial q} + \frac{1}{\hbar}\sigma_1 \star_q S \tag{29} $$

$$ {}^A\nabla_{\hbar\partial_\hbar} S \quad := \quad \hbar\frac{\partial S}{\partial \hbar} - \frac{1}{\hbar}c_1(TX) \star_q S + \mathrm{Gr}(S), \tag{30} $$

following the conventions of Iritani [Iri09], where $\star_q$ denotes the quantum cup product in the quantum cohomology. This defines a meromorphic flat connection, see also [Dub96, Giv96, CK99]. Moreover it therefore turns $H^*(X, \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}])$ into a $D$-module for $\mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]\langle \partial_\hbar, \partial_q\rangle$, sometimes called the *quantum cohomology $D$-module*. This is the connection or $D$-module we consider on the $A$-model side.

For the $B$-model, recall that $\check{X}^\circ$ is the complement of an anti-canonical divisor in $\check{X}$ (see [KLS14, Lemma 5.4]). Therefore there is an up to scalar unique non-vanishing holomorphic $N$-form on $\check{X}^\circ$ which we will fix and call $\omega$. This is the same form as the one appearing in [GHK11, Lemma 5.14], and also agrees with the one from [Rie08] after the isomorphism of $\check{X}^\circ$ with $\mathcal{R}$. Let $\Omega^k(\check{X}^\circ)$ denote the space of all algebraic $k$-forms.

**Definition 4.1.** Define the $\mathbb{C}[\hbar, q]$-module

$$ G_0^{W_q} := \Omega^n(\check{X}^\circ)[\hbar, q]/(\hbar d + dW_q \wedge -)\Omega^{n-1}(\check{X}^\circ)[\hbar, q]. $$

It has a meromorphic (Gauss-Manin) connection given by

$$ {}^B\nabla_{q\partial_q}[\alpha] \quad = \quad q\frac{\partial}{\partial q}[\alpha] + \frac{1}{\hbar}\left[q\frac{\partial W_q}{\partial q}\alpha\right], \tag{31} $$

$$ {}^B\nabla_{\hbar\partial_\hbar}[\alpha] \quad = \quad \hbar\frac{\partial}{\partial \hbar}[\alpha] - \frac{1}{\hbar}[W_q\alpha]. \tag{32} $$

Let $G^{W_q} = G_0^{W_q} \otimes_{\mathbb{C}[\hbar,q]} \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]$. We view $G^{W_q}$ as a $\mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]\langle \partial_\hbar, \partial_q\rangle$-module with $q\partial_q$ acting by ${}^B\nabla_{q\partial_q}$ and $\hbar\partial_\hbar$ acting by ${}^B\nabla_{\hbar\partial_\hbar}$.

4.1. **The case of odd-dimensional quadrics.** For odd-dimensional quadrics, an isomorphism between the connections (or $D$-modules) on the two sides has been proved by Gorbounov and Smirnov in [GS13], for their LG model constructed there. Moreover, the two first-named authors have established in [PR13a] that the Gorbounov-Smirnov LG model is isomorphic to the one obtained from the general construction of [Rie08] for homogeneous spaces. Hence we obtain the following result.

**Theorem 4.2.** *For $X = Q_{2m-1}$ with its mirror LG-model $(\check{X}^\circ_{2m-1}, W_q)$ from Theorem 2.1, the map*

$$\begin{array}{ccc} H^*(Q_{2m-1}, \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]) & \to & G^{W_q} \\ \sigma_i & \mapsto & [p_i \omega] \end{array}$$

*defines an injective homomorphism of D-modules. Here the D-module on the left-hand side is the one defined in terms of the (small) Dubrovin connection in the A-model, and the D-module on the right-hand side is the one defined via the B-model Gauss-Manin connection.*

4.2. **The case of even-dimensional quadrics.** We want to prove a similar result to Theorem 4.2 for even quadrics $Q_{2m-2}$. To do this we will consider a cluster algebra structure on our mirror $\check{X}^\circ_{2m-2}$, see below. Cluster algebras were introduced by Fomin and Zelevinsky in the seminal paper [FZ02], which was the first of the series [FZ03, BFZ05, FZ07]. Our goal is to use the cluster algebra structure of the mirror to prove the following theorem.

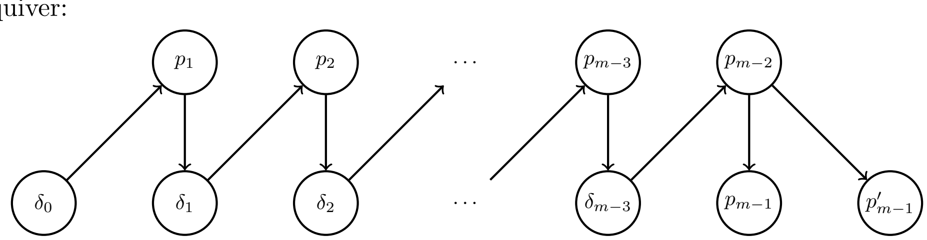
**Theorem 4.3.** *For $X = Q_{2m-2}$ with its mirror LG-model $(\check{X}^\circ_{2m-2}, W_q)$ from Theorem 3.2, the map*

$$\begin{array}{ccc} \Psi : H^*(Q_{2m-2}, \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]) & \to & G^{W_q} \\ \sigma_i & \mapsto & [p_i \omega] \\ \sigma'_{m-1} & \mapsto & [p'_{m-1} \omega] \end{array}$$

*defines an injective homomorphism of D-modules.*

*Remark* 3. In the odd quadrics case, [GS13] (with Nemethi and Sabbah) prove an additional property, cohomological tameness, for the superpotential, which implies that the dimension of $G^{W_q}$ agrees with the number of critical points of $W_q$. It is an interesting question whether this proof could be adapted to give a proof of cohomological tameness in the even case. This would imply that the injective homomorphism in Theorem 4.3 is an isomorphism.

By the isomorphism of $\check{X}^\circ_{2m-2}$ with the Richardson variety, or indeed the unipotent cell $U^P_-$, from Section 3.5 and a result of [GLS11], we obtain a cluster structure on $\mathbb{C}[\check{X}^\circ_{2m-2}]$ which can be described as follows. Consider the following initial quiver:




Here the initial cluster variables correspond to the vertices in the top row of the quiver, while the frozen variables (or coefficients) correspond to the vertices in the bottom row. Recall that the $p_i$ are Plücker coordinates, and the $\delta_i$ are defined as in (11). We see from this description that the coordinate ring of $\check{X}^\circ_{2m-2}$ has a cluster structure of type $A_1^{m-2}$. In particular, it is of finite type, and there are $2^{m-2}$ different clusters, consisting of

- the cluster variables $q_1, \dots, q_{m-2}$, where $q_i \in \{p_i, p_{2m-2-i}\}$;
- the frozen variables (or coefficients) $\delta_0, \dots, \delta_{m-3}$, $p_{m-1}$ and $p'_{m-1}$.

The exchange relations are

$$p_i p_{2m-2-i} = \begin{cases} \delta_{i-1} + \delta_i & \text{for } 1 \le i \le m-3; \\ \delta_{m-3} + p_{m-1}p'_{m-1} & \text{for } i = m-2. \end{cases} \tag{33}$$

Note that the exchange relation for $i = m-2$ is a Plücker relation: it is the equation of the dual quadric (8).

*Remark* 4. In the case of $\check{X}^\circ_{2m-3}$ the isomorphism with the Richardson variety combined with [GLS11] also gives a cluster algebra structure of type $A_1^{m-2}$, with a similar quiver to the one shown on page 18 but where the frozen vertices labelled $p_{m-1}$ and $p'_{m-1}$ are identified.

*Proof of Theorem 4.3.* For the injectivity of $\Psi$ we refer to [MR13, Lemma 5.1].

It remains to prove that $\Psi$ preserves the $D$-module structure. We use a method of [Rie14] to reduce the problem to checking only the action of $q\partial_q$. The method involves a change of coordinates from $(p_i, q, \hbar)$ to $(\mathbf{p}_i, \mathbf{q}, \hbar)$ by

$$\mathbf{p}_i = \hbar^{-i} p_i, \qquad \mathbf{p}'_{m-1} = \hbar^{1-m} p'_{m-1}, \qquad \mathbf{q} = \hbar^{-N} q, \qquad \hbar = \hbar$$

and showing that the Gauss-Manin system for $\frac{1}{\hbar}W$, written in these coordinates, no longer involves the $\hbar$.

Now we check that the map $\Psi$ preserves the action of $q\partial_q$. We consider the following identities in $QH^*(Q_{2m-2}, \mathbb{C})$, which are a special case of results in [FW04]:

$$\sigma_1 \star_q \sigma_i = \begin{cases} \sigma_{i+1} & \text{for } 0 \le i \le m-3 \text{ or } m-1 \le i \le 2m-4; \\ \sigma_{m-1} + \sigma'_{m-1} & \text{for } i = m-2; \\ \sigma_{2m-2} + q\sigma_0 & \text{for } i = 2m-3; \\ q\sigma_1 & \text{for } i = 2m-2, \end{cases} \tag{34}$$

$$\sigma_1 \star_q \sigma'_{m-1} = \sigma_m. \tag{35}$$

We need to prove that there are similar identities on the $B$ side:

(36)

$$\left[q\frac{\partial W_q}{\partial q} p_i \omega\right] = \begin{cases} [p_{i+1}\omega] & \text{for } 0 \le i \le m-3 \text{ or } m-1 \le i \le 2m-4; \\ [(p_{m-1} + p'_{m-1})\omega] & \text{for } i = m-2; \\ [(p_{2m-2} + q)\omega] & \text{for } i = 2m-3; \\ [qp_1\omega] & \text{for } i = 2m-2, \end{cases}$$

$$\left[q\frac{\partial W_q}{\partial q} p'_{m-1}\omega\right] = [p_m\omega], \tag{37}$$

where $\omega$ is the canonical $(2m-2)$-form on $\check{X}^\circ$.

The proof of these identities in $G^{W_q}$ proceeds by constructing closed $(2m-3)$-forms $\nu_i$ and $\nu'_{m-1}$ such that the relation corresponding to $p_i$ will follow from the fact that

$$[dW_q \wedge \nu_i] = [(\hbar d + dW_q \wedge -)\nu_i] = 0$$

and similarly for $p'_{m-1}$. (The first equality above comes from the fact that $\nu_i$ is closed, and the second comes from the definition of $G^{W_q}$.)

Concretely, we will pick a cluster $\mathcal{C}$ containing a particular Plücker coordinate, say $p_i$, and use the following Ansatz for constructing $\nu_i$. We define a vector field

$$\xi_i = p_i \left( \sum_{c \in \mathcal{C} \setminus \{p_i\}} m_c c \partial_c \right) \tag{38}$$

and define an associated $(2m-3)$-form by insertion $\nu_i = \iota_{\xi_i}\omega$, and analogously for $\nu'_{m-1} = \iota_{\xi'_{m-1}}\omega$. Here the $m_c$'s are constants and $\iota$ is the interior product.

To see that these $(2m-3)$-forms are closed, write $\omega = \bigwedge_{p \in \mathcal{C}} \frac{dp}{p}$. For $c \in \mathcal{C}$, we have $\iota_{c\partial_c}\omega = \bigwedge_{p \in \mathcal{C}\setminus\{c\}} \frac{dp}{p}$, and so $\nu_i$ is a $\mathbb{C}$-linear combination of terms of the form $p_i \bigwedge_{p \in \mathcal{C}\setminus\{c\}} \frac{dp}{p}$ for $c \neq p_i$. Such a term is closed, because $p_i$ lies in $\mathcal{C} \setminus \{c\}$.

Using the fact that $dW_q \wedge \omega = 0$, we get $dW_q \wedge \nu_i = \pm dW_q(\xi_i)\omega$. It follows that

$$dW_q \wedge \nu_i = p_i \left( \sum_{c \in \mathcal{C}\setminus\{p_i\}} m_c c \frac{\partial W_q}{\partial c} \right) \omega. \tag{39}$$

Therefore e.g. in order to prove that $\left[q \frac{\partial W_q}{\partial q} p_i \omega\right] - [p_{i+1}\omega] = 0$, we will show that $q\frac{\partial W_q}{\partial q} p_i - p_{i+1}$ has the form $p_i \left( \sum_{c \in \mathcal{C}\setminus\{p_i\}} m_c c \frac{\partial W_q}{\partial c}\right)$, for some choice of coefficients $m_c$.

To prove these identities, we will work with two clusters:

- the initial cluster $\mathcal{C}_1 = \{p_1, \ldots, p_{m-2}, \delta_0, \ldots, \delta_{m-3}, p_{m-1}, p'_{m-1}\}$;
- the cluster $\mathcal{C}_2 = \{p_{2m-3}, \ldots, p_m, \delta_0, \ldots, \delta_{m-3}, p_{m-1}, p'_{m-1}\}$.

Let us first start with $\mathcal{C}_1$ and express $W_q$ in terms of it using the exchange relations (33), having set $p_0 = 1$ :

$$\begin{aligned} W_q = p_1 + \sum_{\ell=1}^{m-3} \left( \frac{p_{\ell+1}\delta_{\ell-1}}{p_\ell \delta_\ell} + \frac{p_{\ell+1}}{p_\ell} \right) + \frac{\delta_{m-3}}{p_{m-2}p_{m-1}} + \frac{\delta_{m-3}}{p_{m-2}p'_{m-1}} \\ + \frac{p_{m-1}}{p_{m-2}} + \frac{p'_{m-1}}{p_{m-2}} + q\frac{p_1}{\delta_0}. \end{aligned}$$

The partial derivatives of $W_q$ are:

$$
\begin{aligned}
q\frac{\partial W_q}{\partial q} &= q\frac{p_1}{\delta_0},\\
p_1\frac{\partial W_q}{\partial p_1} &= p_1 - \frac{p_2\delta_0}{p_1\delta_1} - \frac{p_2}{p_1} + q\frac{p_1}{\delta_0},\\
p_i\frac{\partial W_q}{\partial p_i} &= \frac{p_i\delta_{i-2}}{p_{i-1}\delta_{i-1}} + \frac{p_i}{p_{i-1}} - \frac{p_{i+1}\delta_{i-1}}{p_i\delta_i} - \frac{p_{i+1}}{p_i} \text{ for } 2 \leq i \leq m-3,\\
p_{m-2}\frac{\partial W_q}{\partial p_{m-2}} &= \frac{p_{m-2}\delta_{m-4}}{p_{m-3}\delta_{m-3}} + \frac{p_{m-2}}{p_{m-3}} - \frac{\delta_{m-3}}{p_{m-2}p_{m-1}} - \frac{\delta_{m-3}}{p_{m-2}p'_{m-1}} - \frac{p_{m-1}}{p_{m-2}} - \frac{p'_{m-1}}{p_{m-2}},\\
\delta_0\frac{\partial W_q}{\partial \delta_0} &= \frac{p_2\delta_0}{p_1\delta_1} - q\frac{p_1}{\delta_0},\\
\delta_i\frac{\partial W_q}{\partial \delta_i} &= -\frac{p_{i+1}\delta_{i-1}}{p_i\delta_i} + \frac{p_{i+2}\delta_i}{p_{i+1}\delta_{i+1}} \text{ for } 1 \leq i \leq m-4,\\
\delta_{m-3}\frac{\partial W_q}{\partial \delta_{m-3}} &= -\frac{p_{m-2}\delta_{m-4}}{p_{m-3}\delta_{m-3}} + \frac{\delta_{m-3}}{p_{m-2}p_{m-1}} + \frac{\delta_{m-3}}{p_{m-2}p'_{m-1}},\\
p_{m-1}\frac{\partial W_q}{\partial p_{m-1}} &= -\frac{\delta_{m-3}}{p_{m-2}p_{m-1}} - \frac{\delta_{m-3}}{p_{m-2}p'_{m-1}} + \frac{p_{m-1}}{p_{m-2}}, \text{ and}\\
p'_{m-1}\frac{\partial W_q}{\partial p'_{m-1}} &= -\frac{\delta_{m-3}}{p_{m-2}p_{m-1}} - \frac{\delta_{m-3}}{p_{m-2}p'_{m-1}} + \frac{p'_{m-1}}{p_{m-2}}.
\end{aligned}
$$

Hence

$$
q\frac{\partial W_q}{\partial q}p_i - p_{i+1} = -p_i\left(\sum_{j=i+1}^{m-1} p_j\frac{\partial W_q}{\partial p_j} + p'_{m-1}\frac{\partial W_q}{\partial p'_{m-1}} + \sum_{j=0}^{m-3}\delta_j\frac{\partial W_q}{\partial \delta_j} + \sum_{j=i}^{m-3}\delta_j\frac{\partial W_q}{\partial \delta_j}\right)
$$

for $0 \leq i \leq m-3$, and

$$
q\frac{\partial W_q}{\partial q}p_{m-2} - (p_{m-1}+p'_{m-1}) = -p_{m-2}\left(p_{m-1}\frac{\partial W_q}{\partial p_{m-1}} + p'_{m-1}\frac{\partial W_q}{\partial p'_{m-1}} + \sum_{j=0}^{m-3}\delta_j\frac{\partial W_q}{\partial \delta_j}\right).
$$

Since the right-hand sides of the above equations have the form $p_i\left(\sum_{c\in\mathcal{C}\setminus\{p_i\}} m_c c\partial_c W_q\right)$, this proves identity (36) for $0 \leq i \leq m-2$.

To prove the remaining identities, we use the cluster $\mathcal{C}_2$. In this cluster chart, $W_q$ takes the following form:

$$
\begin{aligned}
W_q =& \frac{\delta_0}{p_{2m-3}} + \frac{\delta_1}{p_{2m-3}} + \sum_{\ell=1}^{m-4}\left(\frac{p_{2m-2-\ell}}{p_{2m-3-\ell}} + \frac{p_{2m-2-\ell}\delta_{\ell+1}}{p_{2m-3-\ell}\delta_\ell}\right) + \frac{p_m}{p_{m-1}}\\
&+ \frac{p_m}{p'_{m-1}} + \frac{p_{m+1}}{p_m} + \frac{p_{m-1}p'_{m-1}p_{m+1}}{p_m\delta_{m-3}} + \frac{q}{p_{2m-3}} + q\frac{\delta_1}{p_{2m-3}\delta_0}.
\end{aligned}
$$

Working out the partial derivatives of $W_q$ as before, we get

$$q\frac{\partial W_q}{\partial q}p_{m-1} - p_m = p_{m-1}\left(p'_{m-1}\frac{\partial W_q}{\partial p'_{m-1}} + \sum_{j=m}^{2m-3} p_j\frac{\partial W_q}{\partial p_j} + \sum_{j=0}^{m-3}\delta_j\frac{\partial W_q}{\partial \delta_j}\right) \tag{40}$$

$$q\frac{\partial W_q}{\partial q}p'_{m-1} - p_m = p'_{m-1}\left(p_{m-1}\frac{\partial W_q}{\partial p_{m-1}} + \sum_{j=m}^{2m-3} p_j\frac{\partial W_q}{\partial p_j} + \sum_{j=0}^{m-3}\delta_j\frac{\partial W_q}{\partial \delta_j}\right) \tag{41}$$

$$q\frac{\partial W_q}{\partial q}p_i - p_{i+1} = p_i\left(-\sum_{j=i+1}^{2m-3} p_j\frac{\partial W_q}{\partial p_j} - \sum_{j=0}^{2m-3-i}\delta_j\frac{\partial W_q}{\partial \delta_j}\right) \tag{42}$$

$$\text{for } m \le i \le 2m-4,$$

Recall that $\delta_0$ is $p_{2m-2}$. The final two relations are

$$q\frac{\partial W_q}{\partial q}p_{2m-3} - (p_{2m-2}+q) = -p_{2m-3}\delta_0\frac{\partial W_q}{\partial \delta_0} \quad \text{and} \tag{43}$$

$$q\frac{\partial W_q}{\partial q}p_{2m-2} - qp_1 = 0 \tag{44}$$

This gives us the identities (36) for $m-1 \le i \le 2m-2$, as well as (37).

□

## 5. The hypergeometric series of $Q_N$

Givental in [Giv96] constructed solutions $S_j$ to the flat section equation of the (dual) $A$-model connection ${}^A\nabla^\vee$ in terms of the $A$-model. By comparison of Givental's solution $S_N$ with its corresponding $B$-model counterpart $\mathbb{S}_{\Gamma_0}$, constructed out of $W$ as in (53), we see $W$ encoding directly descendent Gromov-Witten invariants coming up in Givental's formula. In this section we compute all the components of this flat section and the resulting invariants directly and explicitly in two different ways. The first component we consider is also a particular component of the $J$-function.

5.1. **The dual Dubrovin connection and the $J$-function.** In this section we define Givental's $J$-function and the quantum differential operators. Consider the dual connection to ${}^A\nabla$ with respect to the pairing

$$\langle \sigma, \tau\rangle = \frac{1}{(2\pi i\hbar)^N}\int_X \sigma\cup\tau.$$

Here $\sigma\cup\tau$ is the usual cup product of $\sigma$ and $\tau$, which we will subsequently also denote by $\sigma\tau$. Explicitly, the dual connection is given by the formulas:

$$ {}^A\nabla^\vee_{q\partial_q}S \;:=\; q\frac{\partial S}{\partial q} - \frac{1}{\hbar}\sigma_1 \star_q S \tag{45}$$

$$ {}^A\nabla^\vee_{\hbar\partial_\hbar}S \;:=\; \hbar\frac{\partial S}{\partial \hbar} + \frac{1}{\hbar}c_1(TX)\star_q S + \mathrm{Gr}(S). \tag{46}$$

For the purposes of the $J$-function we ignore the ${}^A\nabla^\vee_{\hbar\partial_\hbar}$ part of the covariant derivative and consider ${}^A\nabla^\vee_{q\partial_q}$ as a family of connections (in the parameter $\hbar$). Formal flat sections indexed by the cohomology basis were written down by Givental [Giv96] in terms of descendent Gromov-Witten invariants. We denote these sections by

$S_0, \ldots, S_{2m-1}$ in the case of $Q_{2m-1}$, and by $S_0, \ldots, S_{m-1}, S'_{m-1}, S_m, \cdots, S_{2m-2}$ for $Q_{2m-2}$, in keeping with the notation from (9) for Schubert classes. See [CK99, (10.14)] for a precise definition of the sections $S_i$.

We also consider the *quantum differential operators*, see for example [CK99, Definition 10.3.2], as the differential operators $P$ which are formal power series in

$$\hbar q\partial_q, q, \hbar$$

and which annihilate the top coefficients of Givental's flat sections, for example, $P \cdot \langle S_j, \sigma_0 \rangle = 0$ for the flat section $S_j$. These operators are quantisations of the quantum cohomology relations in the sense that if $P(\hbar q\frac{\partial}{\partial q}, q, \hbar)A_N = 0$, then $P(\sigma_1, q, 0) = 0$ in $QH^*(Q_N, \mathbb{C})$.

**Definition 5.1.** We define Givental's $J$-function in our setting as

$$J = (2\pi i\hbar)^N \sum \langle S_j, \sigma_0 \rangle \sigma_{PD(j)}$$

where the sum is over all the Schubert classes, including $\sigma'_{m-1}$ in the even case, and where $\sigma_{PD(j)}$ stands for the Poincaré dual cohomology class to $\sigma_j$.

In the case of a quadric the $J$-function is computed explicitly by the Coates-Givental quantum Lefschetz theorem [CG07] from the $J$-function of projective space. Namely

$$J^{Q_N} = e^{\frac{\ln(q)\sigma_1}{\hbar}} \sum_{d \geq 0} \frac{\prod_{j=1}^{2d}(\sigma_1 + j\hbar)}{\prod_{j=1}^{d}(\sigma_1 + j\hbar)^N} q^d. \tag{47}$$

5.2. **The hypergeometric term of the $J$-function.** A special role is played by the term $(2\pi i\hbar)^N \langle S_N, \sigma_0 \rangle$, appearing as the coefficient of the fundamental class in the definition of $J$-function. This term is special in that it is a power series in $\mathbf{q} = \hbar^{-N}q$. We define it as in [BCFKvS98]:

**Definition 5.2.** The *hypergeometric series* $A_X$ of $X$ is the unique power series of the form $A_X(q) = 1 + \sum_{k=1}^{\infty} a_k q^k$, for which $P(q\partial_q, q, 1)A_X = 0$ for all quantum differential operators $P(\hbar q\partial_q, q, \hbar)$ specialized to $\hbar = 1$. We denote the hypergeometric series $A_{Q_N}$ of the quadric $Q_N$ by $A_N$.

The hypergeometric series $A_N$ of the quadric $Q_N$ may be obtained by setting $\hbar$ to 1 in $(2\pi i\hbar)^N \langle S_N, \sigma_0 \rangle$. Or in our example $\langle S_N, 1 \rangle = A_N(\hbar^{-N}q)$.

We recall the geometric interpretation of the coefficients of $A_X$ as follows. The flat sections $S_i$ and in particular the $J$-function encode certain descendent Gromov-Witten invariants. Let

$$I_k(\psi_1^{a_1}\gamma_1, \ldots, \psi_r^{a_r}\gamma_r) \tag{48}$$

denote the degree $k$ descendant Gromov-Witten invariant associated to the cohomology classes $\gamma_1, \ldots, \gamma_r$, where the $\psi$-class $\psi_i$ denotes the first Chern class of the $i$th cotangent bundle of the moduli space of degree $k$ genus 0 stable maps with $r$ marked points, see [CK99, Section 10.1]. Let $\psi$ stand for $\psi_1$. If we write

$$J^{Q_N} = (2\pi i\hbar)^N \sum J_i^{Q_N} \sigma_{\mathrm{PD}(i)},$$

we have

$$J_N^{Q_N} = \langle S_N, \sigma_0 \rangle = 1 + \sum_{k=1}^{\infty} q^k I_k \left( \frac{\sigma_N e^{\frac{\ln(q)\sigma_1}{\hbar}}}{\hbar - \psi}, \sigma_0 \right)$$
$$= 1 + \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \frac{q^k}{\hbar} I_k \left( \sigma_N \left( \frac{\ln(q)\sigma_1}{\hbar} \right)^j \frac{1}{j!} \left( \frac{\psi}{\hbar} \right)^i, \sigma_0 \right)$$

The cup-product $\sigma_N \left( \frac{\ln(q)\sigma_1}{\hbar} \right)^j$ is nonzero if and only if $j = 0$. Therefore we have

$$J_N^{Q_N} = 1 + \sum_{k=1}^{\infty}\sum_{i=0}^{\infty} \frac{q^k}{\hbar} I_k \left( \sigma_N \left( \frac{\psi}{\hbar} \right)^i, \sigma_0 \right).$$

Now the dimension of the moduli space of stable maps $\overline{\mathcal{M}}_{0,2}(Q_N, k)$ is equal to $(k+1)N - 1$, hence

$$J_N^{Q_N} = 1 + \sum_{k=1}^{\infty} \frac{q^k}{\hbar} I_k \left( \sigma_N \left( \frac{\psi}{\hbar} \right)^{kN-1}, \sigma_0 \right).$$

Next we use the fundamental class axiom to get

$$J_N^{Q_N} = 1 + \sum_{k=1}^{\infty} \left( \frac{q}{\hbar^N} \right)^k I_k \left( \sigma_N \psi^{kN-2} \right).$$

Therefore $J_N^{Q_N} = A_N(\hbar^{-N} q)$. Or alternatively, if we set $\hbar = 1$ in $J_N^{Q_N}$, this gives exactly the hypergeometric series of the quadric. Hence we obtain the following geometric interpretation of the coefficient $a_k$ of $q^k$ in $A_N(q)$:

$$a_k = I_k \left( \sigma_N \psi^{kN-2} \right). \tag{49}$$

5.3. **Two computations of a flat section of the Dubrovin connection of $Q_N$.** In this Section, as an illustration of the mirror theorem, we compute explicitly the coefficients of the hypergeometric flat section of the Dubrovin connection, once using the $A$-model and once using the $B$-model. The main result of the computations is the following.

**Theorem 5.3.** *The constant term coefficient of $p_\ell \exp(\frac{1}{\hbar} W_q)$ for the quadric $Q_N$ is given by:*

$$\sum_{k \geq 0} \frac{k^\ell}{\hbar^{kN-\ell}(k!)^N} \cdot \binom{2k}{k} \cdot q^k \qquad \text{if } 0 \leq \ell \leq \lfloor \frac{N-1}{2} \rfloor,$$
$$\sum_{k \geq 0} \frac{k^\ell}{2\hbar^{kN-\ell}(k!)^N} \cdot \binom{2k}{k} \cdot q^k \qquad \text{if } \lfloor \frac{N+1}{2} \rfloor \leq \ell \leq N-1,$$
$$\sum_{k \geq 0} \frac{1}{\hbar^{(k-1)N}(k-1)!^N} \cdot \frac{k-1}{k} \cdot \binom{2k-2}{k-1} \cdot q^k \qquad \text{if } \ell = N.$$

*Moreover, when $N = 2m-2$ is even, the constant term coefficient of $p'_{m-1} \exp(\frac{1}{\hbar} W_q)$ is given by:*

$$\sum_{k \geq 0} \frac{k^{m-1}}{2\hbar^{kN+1-m}(k!)^N} \cdot \binom{2k}{k} \cdot q^k.$$

The series in Theorem 5.3 also have an interpretation in terms of descendent Gromov-Witten invariants. For example the coefficient of $\frac{1}{\hbar^{kN-\ell}}q^k$ of the $\ell$-th series is given by the descendent Gromov-Witten invariant defined in (52) below.

The $\ell = 0$ special case of Theorem 5.3 gives the following.

**Corollary 5.4.** *The hypergeometric series of the quadric $Q_N$ is*

$$A_N(q) = 1 + \sum_{k\geq 1} \frac{1}{(k!)^N}\binom{2k}{k} q^k. \tag{50}$$

*The Gromov-Witten invariant $I_k(\sigma_N \psi^{Nk-2})$ is given by*

$$I_k(\sigma_N \psi^{Nk-2}) = \frac{1}{(k!)^N}\binom{2k}{k}. \tag{51}$$

Here the formula (50) also follows from the formula (47) for the $J$-function of $Q_N$. The formula (51) follows from equation (50) and equation (49).

We now give a direct A-model proof of Theorem 5.3.

*A-model proof.* Our A-model proof works by recovering Theorem 5.3 from the recurrence relations of Kontsevich-Manin for Gromov-Witten invariants [KM98]. Define

$$\beta_{\ell,k} = I_k(\psi^{Nk-1-\ell}\sigma_N, \sigma_\ell). \tag{52}$$

Let us first assume that $N = 2m - 1$. Using the divisor axiom and topological recursion, we get:

$$k\beta_{\ell,k} = I_k(\psi^{Nk-1-\ell}\sigma_N, \sigma_\ell, \sigma_1) = \begin{cases} \beta_{\ell+1,k} & \text{if } \ell \notin \{m-1, N-1, N\}, \\ 2\beta_{m,k} & \text{if } \ell = m-1, \\ \beta_{N,k} + \beta_{0,k-1} & \text{if } \ell = N-1, \\ \beta_{1,k-1} & \text{if } \ell = N. \end{cases}$$

A straightforward computation then gives

$$\frac{\beta_{\ell,k+1}}{\beta_{\ell,k}} = \begin{cases} \frac{2(2k+1)}{k^\ell (k+1)^{N+1-\ell}} & \text{if } 0 \leq \ell \leq N-1, \\ \frac{2(2k-1)}{(k-1)k^{N-1}(k+1)} & \text{if } \ell = N, \end{cases}$$

and $\beta_{1,1} = 2$, which yields Theorem 5.3. □

Similarly, in the case where $N = 2m - 2$:

$$k\beta_{\ell,k} = I_k(\psi^{Nk-1-\ell}\sigma_N, \sigma_\ell, \sigma_1) = \begin{cases} \beta_{\ell+1,k} & \text{if } \ell \notin \{m-2, N-1, N\}, \\ \beta_{m-1,k} + \beta'_{m-1,k} & \text{if } \ell = m-2, \\ \beta_{N,k} + \beta_{0,k-1} & \text{if } \ell = N-1, \\ \beta_{1,k-1} & \text{if } \ell = N, \end{cases}$$

and

$$k\beta'_{m-1,k} = \beta_{m,k}.$$

Theorem 5.3 is then easily checked. □

*B-model proof.* We consider the distinguished flat section of the Dubrovin connection whose coefficients are expressed in terms of the $B$-model as residue integrals, see [MR13, Theorem 4.2]. Namely let $\Gamma_0 \cong (S^1)^N$ be a compact cycle inside $\check{X}^\circ$ such that $\int_{\Gamma_0} \omega = 1$. The integral formula

$$\mathbb{S}_{\Gamma_0}(\hbar, q) := \frac{1}{(2\pi i\hbar)^N} \sum \left( \int_{\Gamma_0} e^{\frac{1}{\hbar}W_q} p_i \omega \right) \sigma_{N-i} \tag{53}$$

defines a flat section of the Dubrovin connection in the $N = 2m-1$ case, and with $(\int_{\Gamma_0} e^{\frac{1}{\hbar}W_q} p_{m-1}\omega)\sigma_{m-1}$ replaced by $(\int_{\Gamma_0} e^{\frac{1}{\hbar}W_q} p'_{m-1}\omega)\sigma_{m-1} + (\int_{\Gamma_0} e^{\frac{1}{\hbar}W_q} p_{m-1}\omega)\sigma'_{m-1}$ in the $N = 2m-2$ case.

We will prove the formula in Theorem 5.3 in one representative case, but omit the other cases, which are extremely similar.

Let us consider the case that $N = 2m-2$, and $m \le \ell \le 2m-3$. In this case recall that $p_\ell = a_1 \dots a_{m-2} c d b_{m-2} \dots b_{2m-1-\ell}$, and recall from (25) that the superpotential $W_q$ equals

$$a_1 + \dots + a_{m-2} + c + d + b_{m-2} + \dots + b_1 + \frac{q}{a_2 \dots a_{m-2} c d b_{m-2} \dots b_1} + \frac{q}{a_1 \dots a_{m-2} c d b_{m-2} \dots b_2}.$$

To compute the constant term of $p_\ell \exp(\frac{1}{\hbar}W_q)$, we consider

$$p_\ell \left( 1 + \frac{1}{\hbar} W_q + \frac{1}{\hbar^2}\frac{W_q^2}{2!} + \frac{1}{\hbar^3}\frac{W_q^3}{3!} + \dots \right),$$

and we pick out from each $p_\ell \frac{W_q^i}{\hbar^i i!}$ every term which has the form $\lambda q^j$ where $\lambda \in \mathbb{Q}[\frac{1}{\hbar}]$. Here we just need to look at each $\frac{W_q^{kN-\ell}}{\hbar^{kN-\ell}(kN-\ell)!}$ for $k = 1, 2, \dots$, because the expansion of $p_\ell \frac{W_q^i}{\hbar^i i!}$ for $i$ not of the form $kN - \ell$ will contain no terms of the form $\lambda q^j$ for $\lambda \in \mathbb{Q}[\frac{1}{\hbar}]$.

Now let us analyze $p_\ell \frac{W_q^{kN-\ell}}{\hbar^{kN-\ell}(kN-\ell)!}$ for $N = 2m-2$. A (Laurent) monomial in the expansion of $p_\ell W_q^{k(2m-2)-\ell}$ is obtained by choosing one term in each of the $k(2m-2)-\ell$ factors. Some of the monomials in the expansion will be pure in the variable $q$ alone – in which case they will equal $q^k$. We need to show that the number of such monomials divided by $(k(2m-2)-\ell)!$ equals $\frac{1}{2}\binom{2k}{k} k^\ell / (k!)^{k(2m-2)}$. To count the number of such monomials, we need to pick one term in each of the $k(2m-2)-\ell$ factors so that we:

- choose $i$ terms which are $\frac{q}{a_2 \dots a_{m-2} c d b_{m-2} \dots b_1}$ for some $0 \le i \le k$;
- choose $k-i$ terms which are $\frac{q}{a_1 \dots a_{m-2} c d b_{m-2} \dots b_2}$;
- choose $k-1$ terms which are $c$;
- choose $k-1$ terms which are $d$;
- choose $i$ terms which are $b_1$;
- choose $k-i-1$ terms which are $a_1$;
- for each $j$ such that $2 \le j \le m-2$, choose $k-1$ terms which are $a_j$;
- for each $j$ such that $2 \le j \le 2m-2-\ell$, choose $k$ terms which are $b_j$.
- for each $j$ such that $2m-2-\ell < j \le m-2$, choose $k-1$ terms which are $b_j$.

The number of ways to do this is the sum of multinomial coefficients

$$\sum_{i=0}^{k} \binom{k(2m-2)-\ell}{i, i, k-i, k-i-1, k \dots k, k-1 \dots k-1}, \tag{54}$$

where the number of $k$'s in the string $k \dots k$ above is $2m-2-\ell-1$, and the number of $k-1$'s in the string $k-1 \dots k-1$ above is $\ell-1$. When we simplify (54) and divide by $(k(2m-2)-\ell)!$, we obtain $\frac{1}{2}\binom{2k}{k}k^{\ell}/(k!)^{k(2m-2)}$, as desired. □

## 6. A quiver description of the Laurent polynomial mirrors

In this section we will explain how our Laurent polynomial superpotential for $Q_N$ can be read off from a certain quiver. This is analogous to the complete flag variety case [Giv97], and the Grassmannian case [BCFKvS98], where one can also read off the Laurent polynomial superpotentials from quivers.

We begin by explaining the [BCFKvS98] formula for the Grassmannian $Gr_2(4)$. Note that since $Gr_2(4)$ is defined by a single (quadratic) Plücker relation, it is isomorphic to the quadric $Q_4$.

For $Gr_2(4)$ the quiver from [BCFKvS98] is shown in Figure 1. The Laurent polynomial superpotential can be read off easily. There are two versions. In the left hand picture the coordinates $t_{ij}$ of the torus $(\mathbb{C}^*)^4$ are in bijection with vertices of the quiver. To each arrow we associate a Laurent monomial by taking the coordinate at the head of the arrow divided by the coordinate at the tail. The Laurent polynomial corresponding to the quiver is the sum of all of the Laurent monomials associated to the arrows.

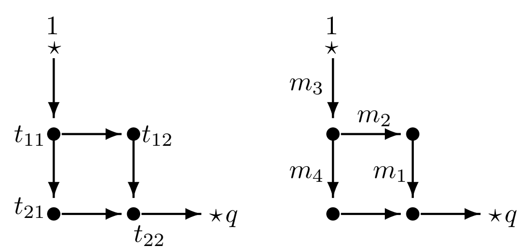



Figure 1. The quiver for $Gr_2(4)$ and two choices of coordinates.

The labels $m_i$ of the arrows in the right hand version are another natural choice of coordinates on the same torus. Indeed these are coordinates related to factorizations into one-parameter subgroups of Lie theoretic mirrors, compare [MR13]. We suppose the remaining arrows are labelled in such a way that the square commutes and any path leading from 1 to $q$ has labels whose product equals $q$. These are Laurent monomials in the variables $m_i$ (depending on $q$). Then the Laurent polynomial superpotential is obtained in [BCFKvS98] as the sum of the labels of all of the arrows of the quiver. In the case of $Gr_2(4)$ it is

$$m_1 + m_2 + m_3 + m_4 + \frac{m_1 m_2}{m_4} + q\frac{1}{m_1 m_2 m_3}. \tag{55}$$

Since $Gr_2(4)$ is isomorphic to $Q_4$, this suggests it should be related to the superpotential (25) for $Q_4$,

$$a_1 + c + d + b_1 + q\frac{a_1 + b_1}{a_1 b_1 c d}. \tag{56}$$

There is indeed a toric change of coordinates turning Equation (55) into Equation (56):

$$m_1 \mapsto \frac{q}{a_1 c d};\ m_2 \mapsto a_1;\ m_3 \mapsto c;\ m_4 \mapsto b_1.$$

Note that the torus of the Hori-Vafa model for $Q_4$ is a different one, as seen in Section 3.7.

The superpotential (56) also comes from a quiver, see Figure 2.

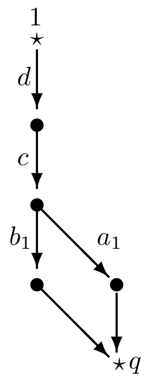



FIGURE 2. The quiver for $Q_4$.

This generalises to all quadrics $Q_N$. Indeed our Laurent polynomial superpotentials (5) and (25) for $Q_N$ can be described using quivers as in Figure 3. The factorisation of $\bar{u}_2$ from (20) can also be naturally read off the quiver (compare with [MR13, Section 5.3]). This goes as follows.

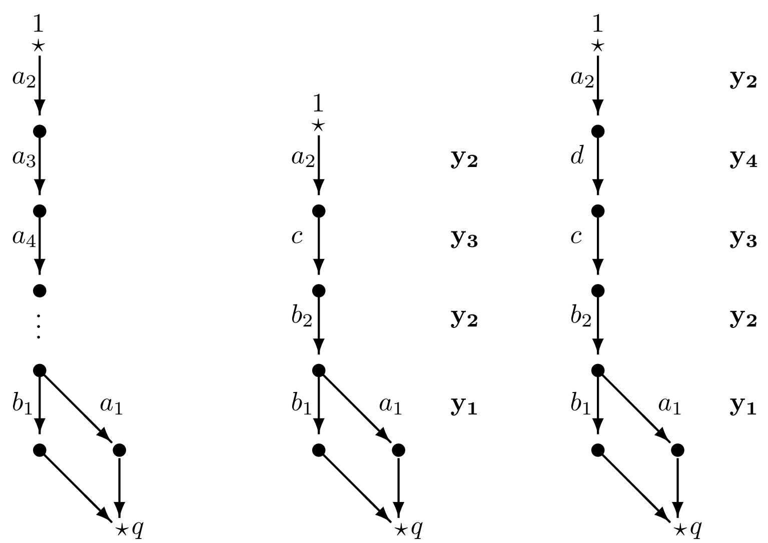



FIGURE 3. The quiver for $Q_N$, and the labelled quivers for $Q_5$ and $Q_6$.

Let the $N-2$ vertical arrows on the left-hand edge be labelled from top to bottom by $a_2, a_3, \dots, a_{m-1}, c, b_{m-1}, \dots, b_2$ for odd quadrics $Q_{2m-1}$, and by $a_2, a_3, \dots, a_{m-2}$, $d, c, b_{m-2}, \dots, b_2$ for even quadrics $Q_{2m-2}$. The diagonal arrow with the same tail as $b_1$ is labelled by $a_1$. The arrows below are not labelled. The labelled arrows can be organized into 'levels' starting with $a_1, b_1$ at the bottom level. The levels are associated to the one-parameter subgroups $y_i$ (of $\mathrm{PSO}_{2m}$ for $X = Q_{2m-2}$, respectively of $\mathrm{PSp}_{2m}$ for $X = Q_{2m-1}$) as shown in the $Q_5$ and $Q_6$ examples. Reading off column by column from right to left and from top to bottom we recover the factorization (20).

*Remark* 5. It is interesting to note that our quivers (restricted to the vertices which are not labelled by $q$) are orientations of type $D$ Dynkin diagrams with a special vertex added at either end. So we have three ways to associate a Dynkin diagram to a quadric: the type of its symmetry group, the type of the cluster algebra associated to the coordinate ring of its mirror, and the type of the quiver defining its superpotential. See Table 1.

| Quadric | Type of symmetry group | Cluster type | Superpotential Quiver |
|---|---|---|---|
| $Q_3$ | $B_2$ | $A_1$ | $D_3$ |
| $Q_4$ | $D_3$ | $A_1$ | $D_4$ |
| $Q_5$ | $B_3$ | $A_1^2$ | $D_5$ |
| $Q_6$ | $D_4$ | $A_1^2$ | $D_6$ |
| $Q_7$ | $B_4$ | $A_1^3$ | $D_7$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Table 1. Dynkin diagrams associated to quadrics

## 7. The hypergeometric equation of a quadric

Justifying its name, the hypergeometric series of the quadric computed in Corollary 5.4 is a generalised hypergeometric series; indeed, the general $k$-th coefficient of the series is a rational function of $k$. Following standard notation we will denote by

$$ {}_pF_r\left(a_1, \dots, a_p; b_1, \dots, b_r; z\right) $$

the series whose general term $\beta_k$ is such that

$$ \frac{\beta_{k+1}}{\beta_k} = \frac{(a_1 + k)\dots(a_p + k)}{(b_1 + k)\dots(b_r + k)(1 + k)} $$

and $\beta_0 = 1$. We immediately get that

$$ A_N(q, \hbar) = {}_1F_N\left(\frac{1}{2}; 1, \dots, 1; \frac{4}{\hbar^N} q\right). $$

It is well-known that the hypergeometric series $w = {}_pF_r\left(a_1, \dots, a_p; b_1, \dots, b_r; z\right)$ satisfies the differential equation

$$ z \prod_{n=1}^{p} (z\frac{\partial}{\partial z} + a_n)w = z\frac{\partial}{\partial z} \prod_{n=1}^{r} (z\frac{\partial}{\partial z} + b_n - 1)w, $$

see for example [AOD10, (16.8.3)]. As a consequence, we obtain a differential equation satisfied by the hypergeometric series of the quadric.

**Proposition 7.1.** *The hypergeometric series of the $N$-dimensional quadric $Q_N$ satisfies the following differential equation :*

$$\left[\left(\hbar q\frac{\partial}{\partial q}\right)^{N+1} - q\left(4\hbar q\frac{\partial}{\partial q} + 2\hbar\right)\right] A_N(q,\hbar) = 0.$$

As mentioned in Section 5, the quantum differential equation gives rise to a relation in quantum cohomology. Namely we get

$$\sigma_1^{N+1} - 4q\sigma_1 = 0,$$

which indeed holds in $QH^*(Q_N, \mathbb{C})$, for example by an application of the quantum Chevalley formula, see [FW04].

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