DECOMPOSITIONS OF AMPLITUHEDRA

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Abstract. The *(tree) amplituhedron* $\mathcal{A}_{n,k,m}$ is the image in the Grassmannian $\mathrm{Gr}_{k,k+m}$ of the totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$, under a (map induced by a) linear map which is totally positive. It was introduced by Arkani-Hamed and Trnka in 2013 in order to give a geometric basis for the computation of scattering amplitudes in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. In the case relevant to physics ($m = 4$), there is a collection of recursively-defined $4k$-dimensional *BCFW cells* in $\mathrm{Gr}_{k,n}^{\geq 0}$, whose images conjecturally "triangulate" the amplituhedron—that is, their images are disjoint and cover a dense subset of $\mathcal{A}_{n,k,4}$. In this paper, we approach this problem by first giving an explicit (as opposed to recursive) description of the BCFW cells. We then develop sign-variational tools which we use to prove that when $k = 2$, the images of these cells are disjoint in $\mathcal{A}_{n,k,4}$. We also conjecture that for arbitrary even m, there is a decomposition of the amplituhedron $\mathcal{A}_{n,k,m}$ involving precisely $M(k, n-k-m, \frac{m}{2})$ top-dimensional cells (of dimension km), where $M(a, b, c)$ is the number of plane partitions contained in an $a \times b \times c$ box. This agrees with the fact that when $m = 4$, the number of BCFW cells is the Narayana number $N_{n-3,k+1} = \frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$.

Contents

1. Introduction

The totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ is the subset of the real Grassmannian $\mathrm{Gr}_{k,n}$ where all Plücker coordinates are nonnegative. Following seminal work of Lusztig [Lus94], as well as by Fomin and Zelevinsky [FZ99], Postnikov initiated the combinatorial study of

$\mathrm{Gr}_{k,n}^{\geq 0}$ and its cell decomposition [Pos]. In particular, Postnikov showed how the cells in the cell decomposition are naturally indexed by combinatorial objects including *decorated permutations*, Ⅎ-*diagrams*, and equivalence classes of *plabic graphs*. Since then the totally nonnegative Grassmannian has found applications in diverse contexts such as mirror symmetry [MR], soliton solutions to the KP equation [KW14], and scattering amplitudes for planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [AHBC$^+$16].

Building on [AHBC$^+$16], Arkani-Hamed and Trnka [AHT14] recently introduced a beautiful new mathematical object called the *(tree) amplituhedron*, which is the image of the totally nonnegative Grassmannian under a particular map.

Definition 1.1. For $a \leq b$, define $\mathrm{Mat}_{a,b}^{>0}$ as the set of real $a \times b$ matrices whose $a \times a$ minors are all positive. Let $Z \in \mathrm{Mat}_{k+m,n}^{>0}$, where $m \geq 0$ is fixed with $k + m \leq n$. Then Z induces a map

$$\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$$

defined by

$$\tilde{Z}(\langle v_1, \dots, v_k \rangle) := \langle Z(v_1), \dots, Z(v_k) \rangle,$$

where $\langle v_1, \dots, v_k \rangle$ is an element of $\mathrm{Gr}_{k,n}^{\geq 0}$ written as the span of k basis vectors.[1] The *(tree) amplituhedron* $\mathcal{A}_{n,k,m}(Z)$ is defined to be the image $\tilde{Z}(\mathrm{Gr}_{k,n}^{\geq 0})$ inside $\mathrm{Gr}_{k,k+m}$.

In special cases the amplituhedron recovers familiar objects. If Z is a square matrix, i.e. $k + m = n$, then $\mathcal{A}_{n,k,m}(Z)$ is isomorphic to the totally nonnegative Grassmannian. If $k = 1$, then it follows from [Stu88] that $\mathcal{A}_{n,1,m}(Z)$ is a *cyclic polytope* in projective space $\mathbb{P}^m$.

While the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is an interesting mathematical object for any m, the case relevant to physics is $m = 4$. In this case, it provides a geometric basis for the computation of *scattering amplitudes* in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. These amplitudes are complex numbers related to the probability of observing a certain scattering process of n particles. It is expected that the tree-level (i.e. leading-order) term of such amplitudes equals a canonical differential form on the amplituhedron $\mathcal{A}_{n,k,4}(Z)$. This statement is closely related to the following conjecture of Arkani-Hamed and Trnka [AHT14].

Conjecture 1.2. *Let $Z \in \mathrm{Mat}_{k+4,n}^{>0}$, and let $\mathcal{C}_{n,k,4}$ be the collection of* BCFW cells *in $\mathrm{Gr}_{k,n}^{\geq 0}$. Then the images under $\tilde{Z}$ of the cells $\mathcal{C}_{n,k,4}$ "triangulate" the $m = 4$ amplituhedron, i.e. they are pairwise disjoint, and together they cover a dense subset of the amplituhedron $\mathcal{A}_{n,k,4}(Z)$.*

More specifically, the BCFW recurrence [BCF05, BCFW05], of Britto, Cachazo, Feng, and Witten, provides one way to compute scattering amplitudes. Translated into the Grassmannian formulation of [AHBC$^+$16], the terms in the BCFW recurrence can be identified with a collection of $4k$-dimensional cells in $\mathrm{Gr}_{k,n}^{\geq 0}$ which we refer to as the *BCFW cells* $\mathcal{C}_{n,k,4}$. If the images of these BCFW cells in $\mathcal{A}_{n,k,4}(Z)$ fit together in a nice way, then we can combine the canonical form coming from each term to obtain the canonical form on $\mathcal{A}_{n,k,4}(Z)$.

In this paper, we make a first step towards understanding Conjecture 1.2. The BCFW cells are defined recursively in terms of plabic graphs (see Section 5.1), and moreover there is

[1] The fact that Z has positive maximal minors ensures that $\tilde{Z}$ is well defined [AHT14]. See [Kar17, Theorem 4.2] for a characterization of when a matrix Z gives rise to a well-defined map $\tilde{Z}$.

a 'shift by 2' applied at the end of the recursion (see Definition 5.4). Hence proving anything about how the images of the BCFW cells fit together is not at all straightforward from the definitions. To approach Conjecture 1.2, we start by giving an explicit, non-recursive description of the BCFW cells. Namely, we index the BCFW cells in $\mathrm{Gr}_{k,n}^{\geq 0}$ by pairs of noncrossing lattice paths inside a $k \times (n-k-4)$ rectangle, and associate a Ꮣ-diagram to each pair of lattice paths, from which we can read off the corresponding cell (see Theorem 6.3, proved in Section 7). We then use these Ꮣ-diagrams to understand the case $k = 2$: we derive an elegant description of basis vectors for elements of each BCFW cell in terms of 'dominoes' (Theorem 10.3), and show that the images of distinct BCFW cells are disjoint in the amplituhedron $\mathcal{A}_{n,2,4}(Z)$ (Theorem 11.1). The proof uses classical results about sign variation, along with some new tools particularly suited to our problem.

We expect that our techniques may be helpful in understanding the case of arbitrary k, and we make a step in this direction in an appendix to this paper, which is joint with Hugh Thomas. We use a bijection between BCFW cells and Dyck paths to associate a conjectural 'domino basis' to each element of a BCFW cell (Conjecture A.7). We leave the proof of the conjecture and the analysis of how general BCFW cells fit together to future work.

As a warmup to our study of BCFW cells in the case $m = 4$, in Section 4 we develop an analogous story in the case $m = 2$. Namely, we give a BCFW-style recursion on plabic graphs, describe the resulting cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ using lattice paths and domino bases, and prove disjointness of the images of these cells inside the $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}(Z)$.

It was observed (e.g. in [AHBC+16]) that the number of BCFW cells $|\mathcal{C}_{n,k,4}|$ is the Narayana number $\frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$. Motivated by this fact, as well as known results about decompositions of amplituhedra when $m = 2$ or $k = 1$, we conjecture that when m is even, there is a decomposition of $\mathcal{A}_{n,k,m}(Z)$ which involves precisely $M(k, n-k-m, \frac{m}{2})$ top-dimensional cells (see Conjecture 8.1). Here $M(a, b, c)$ denotes the number of collections of c noncrossing lattice paths inside an $a \times b$ rectangle, or equivalently, the number of plane partitions which fit inside an $a \times b \times c$ box. See Section 8.1 for other combinatorial interpretations of $M(a, b, c)$, as well as Remark 8.2 for a possible extension of Conjecture 8.1 to odd m.

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2. Background on the totally nonnegative Grassmannian

The *(real) Grassmannian* $\mathrm{Gr}_{k,n}$ (for $0 \leq k \leq n$) is the space of all k-dimensional subspaces of $\mathbb{R}^n$. An element of $\mathrm{Gr}_{k,n}$ can be viewed as a $k \times n$ matrix of rank k modulo invertible row operations, whose rows give a basis for the k-dimensional subspace.

Let $[n]$ denote $\{1, \dots, n\}$, and $\binom{[n]}{k}$ denote the set of all k-element subsets of $[n]$. Given $V \in \mathrm{Gr}_{k,n}$ represented by a $k \times n$ matrix A, for $I \in \binom{[n]}{k}$ we let $\Delta_I(V)$ be the $k \times k$ minor of A using the columns I. The $\Delta_I(V)$ do not depend on our choice of matrix A (up to simultaneous rescaling by a nonzero constant), and are called the *Plücker coordinates* of V.

Definition 2.1 ([Pos, Section 3])**.** We say that $V \in \mathrm{Gr}_{k,n}$ is *totally nonnegative* if $\Delta_I(V) \geq 0$ for all $I \in \binom{[n]}{k}$, and *totally positive* if $\Delta_I(V) > 0$ for all $I \in \binom{[n]}{k}$. The set of all totally nonnegative $V \in \mathrm{Gr}_{k,n}$ is the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$, and the set of all totally positive V is the *totally positive Grassmannian* $\mathrm{Gr}_{k,n}^{>0}$. For $M \subseteq \binom{[n]}{k}$, let S_M be the set of $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ with the prescribed collection of Plücker coordinates strictly positive (i.e. $\Delta_I(V) > 0$ for all $I \in M$), and the remaining Plücker coordinates equal to zero (i.e. $\Delta_J(V) = 0$ for all $J \in \binom{[n]}{k} \setminus M$). If $S_M \neq \emptyset$, we call M a *positroid* and S_M its *positroid cell.*

Each positroid cell S_M is indeed a topological cell [Pos, Theorem 6.5], and moreover, the positroid cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ glue together to form a CW complex [PSW09].

2.1. **Combinatorial objects parameterizing cells.** In [Pos], Postnikov defined several families of combinatorial objects which are in bijection with cells of the totally nonnegative Grassmannian, including *decorated permutations*, *⅃-diagrams*, and equivalence classes of *reduced plabic graphs*. In this section, we introduce these objects, and give bijections between them. This will give us a canonical way to label each positroid by a decorated permutation, a ⅃-diagram, and an equivalence class of plabic graphs.

Definition 2.2. A *decorated permutation* of $[n]$ is a bijection $\pi : [n] \to [n]$ whose fixed points are each colored either black or white. We denote a black fixed point i by $\pi(i) = \underline{i}$, and a white fixed point i by $\pi(i) = \overline{i}$. An *anti-excedance* of the decorated permutation π is an element $i \in [n]$ such that either $\pi^{-1}(i) > i$ or $\pi(i) = \overline{i}$.

Postnikov showed that the positroids for $\mathrm{Gr}_{k,n}^{\geq 0}$ are indexed by decorated permutations of $[n]$ with exactly k anti-excedances [Pos, Section 16].

Now we introduce certain fillings of Young diagrams with the symbols 0 and $+$, called *⊕-diagrams*, and associate a decorated permutation to each such diagram. Postnikov [Pos, Section 20] showed that a special subset of these diagrams, called *⅃-diagrams*, are in bijection with decorated permutations. We introduce the more general ⊕-diagrams here, since in Section 5 we will use a distinguished subset of them to index the BCFW cells of $\mathrm{Gr}_{k,n}^{\geq 0}$.

Definition 2.3. Fix $0 \leq k \leq n$. Given a partition λ, we let Y_λ denote the Young diagram of λ. A *⊕-diagram of type (k,n)* is a filling D of a Young diagram Y_λ fitting inside a $k \times (n-k)$ rectangle with the symbols 0 and $+$ (such that each box of Y is filled with exactly one symbol). We call λ the *shape* of D. (See Figure 1.)

We associate a decorated permutation π of $[n]$ to D as follows (see [Pos, Section 19]).

1. Replace each $+$ in D with an elbow, and each 0 in D with a cross.
2. View the southeast border of Y_λ as a lattice path of n steps from the northeast corner to the southwest corner of the $k \times (n-k)$ rectangle, and label its edges by $1, \dots, n$.
3. Label each edge of the northwest border of Y_λ with the label of its opposite edge on the southeast border. This gives a *pipe dream* P associated to D.
4. Read off the decorated permutation π from P by following the 'pipes' from the southeast border to the northwest border Y_λ. If the pipe originating at i ends at j, we set $\pi(i) := j$. If $\pi(i) = i$, then either i labels two horizontal edges or two vertical edges of P. In the former case, we set $\pi(i) := \underline{i}$, and in the latter case, we set $\pi(i) := \overline{i}$.

Figure 1 illustrates this procedure. We denote the pipe dream P by $P(D)$, and the decorated permutation π by π_D. Note that the anti-excedances of π correspond to the vertical steps of the southeast border of Y_λ, so π has exactly k anti-excedances. We denote the corresponding positroid cell of $\mathrm{Gr}_{k,n}^{\geq 0}$ (see [Pos, Section 16], and also Theorem 2.11) by S_π or S_D.

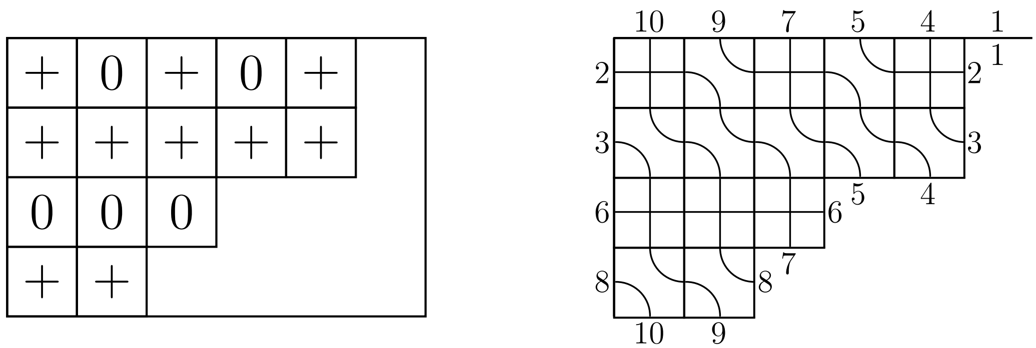


FIGURE 1. A ⅃-diagram D of type $(4, 10)$ with shape $\lambda = (5, 5, 3, 2)$, and its corresponding pipe dream P with decorated permutation $\pi_D = (\underline{1}, 5, 4, 9, 7, \overline{6}, 2, 10, 3, 8)$.

Definition 2.4. Let D be a $\oplus$-diagram, and P its associated pipe dream from Definition 2.3.
(i) We say that D is *reduced* if no two pipes of P cross twice.
(ii) We say that D is a *⅃-diagram* (or Le-diagram) if it avoids the ⅃-configuration, i.e. it has no 0 with both a + above it in the same column and a + to its left in the same row. Equivalently (see [Pos, Lemma 19.3]), any pair of pipes which cross do not later touch or cross, when read from southeast to northwest. (Two pipes *cross* if they form a cross $+$, and *touch* if they form an elbow.) So, ⅃-diagrams are reduced.

Postnikov showed that ⅃-diagrams correspond to decorated permutations. Lam and Williams later showed how to transform any reduced $\oplus$-diagram into a ⅃-diagram by using certain moves. We state these results.

Lemma 2.5 ([Pos, Corollary 20.1 and Theorem 6.5])**.** *The map $D \mapsto \pi_D$ from Definition 2.3 is a bijection from the set of ⅃-diagrams of type (k, n) to the set of decorated permutations of $[n]$ with k anti-excedances. Therefore, ⅃-diagrams of type (k, n) index the cells of $\mathrm{Gr}_{k,n}^{\geq 0}$. The dimension of the positroid cell S_D indexed by D is the number of +'s in D.*

Lemma 2.6 ([LW08])**.** *Let D be a $\oplus$-diagram. Then D is reduced if and only if D can be transformed into a ⅃-diagram D' by a sequence of ⅃-moves:*

$$\begin{matrix} \boxed{+} & \cdots & \boxed{+} \\ \vdots & \ddots & \vdots \\ \boxed{+} & \cdots & \boxed{0} \end{matrix} \quad \mapsto \quad \begin{matrix} \boxed{0} & \cdots & \boxed{+} \\ \vdots & \ddots & \vdots \\ \boxed{+} & \cdots & \boxed{+} \end{matrix} .$$

In this picture, the four boxes on each side denote the corners of a rectangle whose height and width are both at least 2, and whose other boxes (aside from the corners) all contain 0's.

⅃-moves preserve the decorated permutation of a $\oplus$-diagram. Hence D indexes the cell $S_{D'}$ with decorated permutation $\pi_D = \pi_{D'}$, and the dimension of $S_{D'}$ is the number of $+$'s in D.

For example, here is a sequence of ⅃-moves which transforms a reduced $\oplus$-diagram into a ⅃-diagram:

$$\begin{array}{|c|c|c|c|c|c|}\hline + & 0 & + & + & 0 & + \\ \hline + & 0 & 0 & 0 & + & \multicolumn{1}{c}{} \\ \cline{1-5}\end{array} \mapsto \begin{array}{|c|c|c|c|c|c|}\hline 0 & 0 & + & + & 0 & + \\ \hline + & 0 & + & 0 & + & \multicolumn{1}{c}{} \\ \cline{1-5}\end{array} \mapsto \begin{array}{|c|c|c|c|c|c|}\hline 0 & 0 & 0 & + & 0 & + \\ \hline + & 0 & + & + & + & \multicolumn{1}{c}{} \\ \cline{1-5}\end{array}.$$

Proof. We explain how to deduce this result from the work of Lam and Williams. They showed [LW08, Section 5] that *any* $\oplus$-diagram can be transformed into a ⅃-diagram using the ⅃-moves above, as well as the *uncrossing moves*

$$\begin{array}{ccc} \boxed{0} & \cdots & \boxed{+} \\ \vdots & \ddots & \vdots \\ \boxed{+} & \cdots & \boxed{0} \end{array} \quad \mapsto \quad \begin{array}{ccc} \boxed{+} & \cdots & \boxed{+} \\ \vdots & \ddots & \vdots \\ \boxed{+} & \cdots & \boxed{+} \end{array}.$$

Note that a ⅃-move in the statement of the lemma takes a reduced $\oplus$-diagram to a reduced $\oplus$-diagram, and an uncrossing move above can only be performed on a non-reduced $\oplus$-diagram. These observations imply the result. ■

Definition 2.7. A *plabic graph*[2] is an undirected planar graph G drawn inside a disk (considered modulo homotopy) with n *boundary vertices* on the boundary of the disk, labeled $1, \dots, n$ in clockwise order, as well as some *internal vertices*. Each boundary vertex is incident to a single edge, and each internal vertex is colored either black or white. If a boundary vertex is incident to a leaf (a vertex of degree 1), we refer to that leaf as a *lollipop*.

The following construction of Postnikov associates a hook diagram, network, and plabic graph to any ⅃-diagram.

Definition 2.8 ([Pos, Sections 6 and 20])**.** Let D be a ⅃-diagram of type (k, n). We define the *hook diagram* $H(D)$ of D, the *network* $N(D)$ of D, and the *plabic graph* $G(D)$ of D. (See Figure 3 for examples.)

To construct $H(D)$, we delete the 0's of D, and replace each $+$ with a vertex. From each vertex we construct a hook which goes east and south, to the border of the Young diagram of D. We label the edges of the southeast border of D by $1, \dots, n$ from northeast to southwest.

To construct $N(D)$, we direct the edges of $H(D)$ west and south. Let E be the set of horizontal edges of $N(D)$. To each such edge $e \in E$, we associate a variable a_e.

To construct $G(D)$ from $H(D)$, we place boundary vertices $1, \dots, n$ along the southeast border. Then we replace the local region around each internal vertex as in Figure 2, and add a black (respectively, white) lollipop for each black (respectively, white) fixed point of the decorated permutation π_D.

More generally, each ⅃-diagram D is associated with a family of *reduced plabic graphs* consisting of $G(D)$ together with other plabic graphs which can be obtained from $G(D)$ by certain *moves*; see [Pos, Section 12].

[2]"Plabic" stands for *planar bi-colored*.

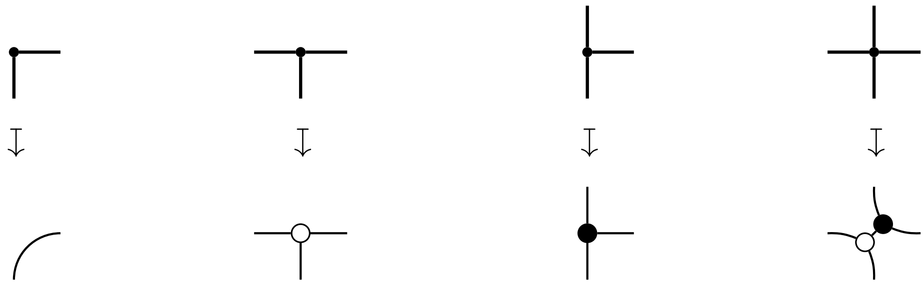
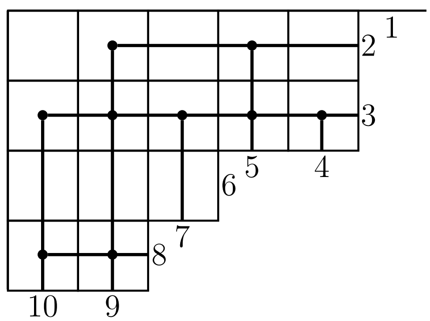
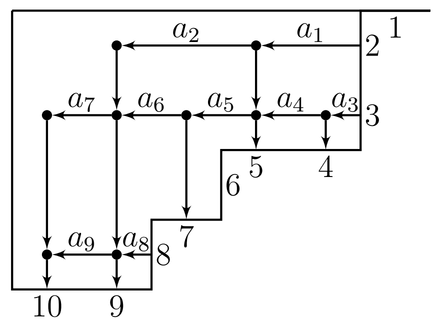


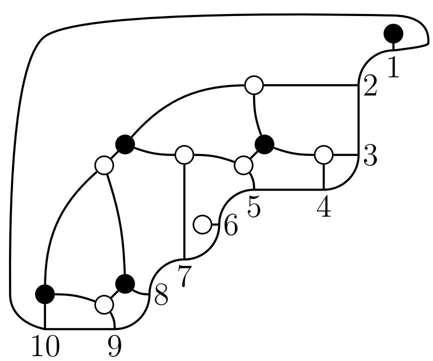
FIGURE 2. Local substitutions for going from the hook diagram $H(D)$ to the plabic graph $G(D)$.



(A) The hook diagram $H(D)$.



(B) The network $N(D)$.



(C) The plabic graph $G(D)$.

FIGURE 3. The hook diagram, network, and plabic graph associated to the Ⅎ-diagram D from Figure 1.

From the plabic graph constructed in Definition 2.8 (and more generally from a reduced plabic graph G), we can read off the corresponding decorated permutation π_G as follows.

Definition 2.9. Let G be a reduced plabic graph as above with boundary vertices $1, \ldots, n$. For each boundary vertex $i \in [n]$, we follow a path along the edges of G starting at i, turning (maximally) right at every internal black vertex, and (maximally) left at every internal white vertex. This path ends at some boundary vertex $\pi(i)$. By [Pos, Section 13], the fact that

G is reduced implies that each fixed point of π is attached to a lollipop; we color each fixed point by the color of its lollipop. In this way we obtain the *decorated permutation* $\pi_G = \pi$ of G. We say that G is of *type* (k,n), where k is the number of anti-excedances of π_G. By [Pos, Definition 11.5][3], we have

$$k = \#\text{edges} - \#\text{black vertices} - \sum_{\text{white vertices } v} (\deg(v) - 1). \tag{2.10}$$

We invite the reader to verify that the decorated permutation of the plabic graph G of Figure 3 is $\pi_G = (\underline{1}, 5, 4, 9, 7, \overline{6}, 2, 10, 3, 8)$. By (2.10), we calculate $k = 21 - 5 - 12 = 4$.

We now explain how to parameterize elements of the cell S_D associated to a ⅃-diagram D from its network $N(D)$.

Theorem 2.11 ([Pos, Section 6])**.** *Let D be a ⅃-diagram of type (k,n), with associated cell $S_D \subseteq \mathrm{Gr}_{k,n}^{\geq 0}$ and network $N(D)$ from Definition 2.8. Let E denote the set of horizontal edges of $N(D)$. We obtain a parameterization $\mathbb{R}_{>0}^E \to S_D$ of S_D, as follows. Let $s_1 < \cdots < s_k$ be the labels of the vertical edges of the southeast border of $N(D)$. Also, to any directed path p of $N(D)$, we define its weight w_p to be the product of the edge variables a_e over all horizontal edges e in p. We let A be the $k \times n$ matrix with rows indexed by $\{s_1, \dots, s_k\}$ whose (s_i, j)-entry equals*

$$(-1)^{|\{i' \in [k] : s_i < s_{i'} < j\}|} \sum_{p : s_i \to j} w_p,$$

where the sum is over all directed paths p from s_i to j. Then the map sending $(a_e)_{e \in E} \in \mathbb{R}_{>0}^E$ to the element of $\mathrm{Gr}_{k,n}^{\geq 0}$ represented by A is a homeomorphism from $\mathbb{R}_{>0}^E$ to S_D.

For example, the network in Figure 3 gives the parameterization

$$\begin{array}{c} \\ 2 \\ 3 \\ 6 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & -a_1 & 0 & a_1a_5 & 0 & -a_1(a_2+a_5a_6) & -a_1(a_2+a_5a_6)(a_7+a_9) \\ 0 & 0 & 1 & a_3 & a_3a_4 & 0 & -a_3a_4a_5 & 0 & a_3a_4a_5a_6 & a_3a_4a_5a_6(a_7+a_9) \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a_8 & a_8a_9 \end{bmatrix} \end{array}$$

of $S_D \subseteq \mathrm{Gr}_{4,10}^{\geq 0}$, where D is the ⅃-diagram in Figure 1.

3. Background on sign variation

In this section we provide some background on sign variation, and state Lemma 3.4, which will be useful later for proving that two cells have disjoint image in the amplituhedron.

Definition 3.1. A *sign vector* is a vector with entries in $\{0, +, -\}$. For a vector $v \in \mathbb{R}^n$, we let $\mathrm{sign}(v) \in \{0, +, -\}^n$ be the sign vector naturally associated to v. For example, $\mathrm{sign}(1, 0, -4, 2) = (+, 0, -, +)$.

[3]There is a typo in Postnikov's formula: $k + (n-k)$ should read $k - (n-k)$. Our formula looks different than his, but is equivalent.

Definition 3.2. Given $v \in \mathbb{R}^n$, we let $\mathrm{var}(v)$ be the number of times v changes sign, when viewed as a sequence of n numbers and ignoring any zeros. Also let

$$\overline{\mathrm{var}}(v) := \max\{\mathrm{var}(w) : w \in \mathbb{R}^n \text{ such that } w_i = v_i \text{ for all } i \in [n] \text{ with } v_i \neq 0\},$$

i.e. $\overline{\mathrm{var}}(v)$ is the maximum number of times v changes sign after we choose a sign for each zero coordinate. Note that we can apply var and $\overline{\mathrm{var}}$ to sign vectors. For example, $\mathrm{var}(+, 0, 0, +, -) = 1$ and $\overline{\mathrm{var}}(+, 0, 0, +, -) = 3$.

The following result of Gantmakher and Krein characterizes total positivity in $\mathrm{Gr}_{k,n}$ using sign variation.

Theorem 3.3 ([GK50, Theorems V.3, V.7, V.1, V.6])**.** *Let $V \in \mathrm{Gr}_{k,n}$.*
(i) $V \in \mathrm{Gr}_{k,n}^{\geq 0} \iff \mathrm{var}(v) \leq k-1$ for all $v \in V \setminus \{0\} \iff \overline{\mathrm{var}}(w) \geq k$ for all $w \in V^{\perp} \setminus \{0\}$.
(ii) $V \in \mathrm{Gr}_{k,n}^{> 0} \iff \overline{\mathrm{var}}(v) \leq k-1$ for all $v \in V \setminus \{0\} \iff \mathrm{var}(w) \geq k$ for all $w \in V^{\perp} \setminus \{0\}$.

The following lemma gives a sign-variational characterization for when two elements of $\mathrm{Gr}_{k,n}^{\geq 0}$ correspond to the same point of the amplituhedron $\mathcal{A}_{n,k,m}(Z)$. We will apply it repeatedly in Section 11 to show that the images of the BCFW cells are disjoint in the $k = 2, m = 4$ amplituhedron $\mathcal{A}_{n,2,4}(Z)$.

Lemma 3.4. *Let $Z \in \mathrm{Mat}_{k+m,n}^{>0}$, where $k, m, n \geq 0$ satisfy $k + m \leq n$, and $V, V' \in \mathrm{Gr}_{k,n}^{\geq 0}$. Then $\tilde{Z}(V) = \tilde{Z}(V')$ if and only if for all $v \in V$, there exists a unique $v' \in V'$ such that $Z(v) = Z(v')$. We call v' the* matching vector *for v. Note that by Theorem 3.3(ii), we either have $v = v'$ or $\mathrm{var}(v - v') \geq k + m$.*

Proof. ($\Rightarrow$): Suppose that $\tilde{Z}(V) = \tilde{Z}(V')$, i.e. $\{Z(v) : v \in V\} = \{Z(v') : v' \in V'\}$. Then for any $v \in V$, there exists $v' \in V'$ with $Z(v) = Z(v')$. Since $\dim(\tilde{Z}(V')) = k$ (see Definition 1.1), we have $\ker(Z) \cap V' = \{0\}$, which implies that v' is unique.

($\Leftarrow$): Suppose that for all $v \in V$, there exists a unique $v' \in V'$ such that $Z(v) = Z(v')$. Then $\tilde{Z}(V) \subseteq \tilde{Z}(V')$, and since $\dim(\tilde{Z}(V)) = \dim(\tilde{Z}(V')) = k$, we get $\tilde{Z}(V) = \tilde{Z}(V')$. ■

4. Warmup: the $m = 2$ amplituhedron

In this section we focus on the $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}(Z)$, as a warmup to our study of the $m = 4$ amplituhedron in the following sections. In analogy to the $m = 4$ case, the outline of this section is as follows.

1. In Section 4.1, we give a recurrence on plabic graphs, which (after a 'shift by 1') produces a collection $\mathcal{C}_{n,k,2}$ of $2k$-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$, whose images should triangulate $\mathcal{A}_{n,k,2}(Z)$. (This is analogous to the $m = 4$ BCFW recursion; see Section 5.)
2. In Section 4.2, we describe the Ꞁ-diagrams of these $2k$-dimensional cells, indexed by lattice paths inside a $k \times (n - k - 2)$ rectangle. (This is analogous to our description for $m = 4$ in terms of pairs of noncrossing lattice paths inside a $k \times (n - k - 4)$ rectangle; see Section 6 and Section 7.)
3. In Section 4.3, we find a nice basis of any subspace V coming from our cells $\mathcal{C}_{n,k,2}$. (This is analogous to our 'domino bases' for $m = 4$; see Section 10 for the case $k = 2$, and Conjecture A.7 for a conjectural generalization to all k.)

4. In Section 4.4, we prove that the images of the cells $\mathcal{C}_{n,k,2}$ in $\mathcal{A}_{n,k,2}(Z)$ are disjoint, using sign variation arguments. (In Section 11, we will prove disjointness for $m = 4$ BCFW cells when $k = 2$, using more intricate arguments along the same lines.)

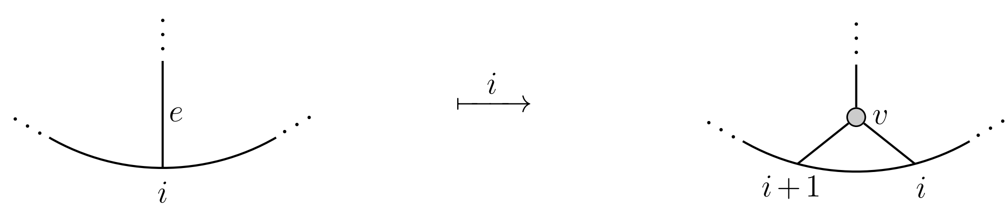
While we will not need the results of this section to handle the $m = 4$ case, the ideas and techniques used here will hopefully give the reader a flavor of our arguments for $m = 4$. We will also cite the $m = 2$ case as evidence for our conjecture in Section 8.

Remark 4.1. Arkani-Hamed, Thomas, and Trnka [AHTT, Section 7] consider the same collection of cells $\mathcal{C}_{n,k,2}$, up to a cyclic shift. They define their cells in terms of bases as in item 3 above, and do not consider a recurrence on plabic graphs or the Ⅎ-diagrams. They show both that the images of these cells in $\mathcal{A}_{n,k,2}(Z)$ are disjoint, and that their union covers a dense subset of $\mathcal{A}_{n,k,2}(Z)$ (we will not prove the latter fact here). We remark that their arguments also employ sign variation.

4.1. **A BCFW-style recursion for $m = 2$.** We start by giving a recursion on plabic graphs which produces, for each pair (k, n) with $1 \le k \le n-1$, a collection $\mathcal{C}_{n,k,2}$ of $2k$-dimensional positroid cells of $\mathrm{Gr}_{k,n}^{\ge 0}$. This recursion is an analogue of the well-known *BCFW recursion*, which produces the BCFW cells $\mathcal{C}_{n,k,4}$ for the $m = 4$ amplituhedron (see Section 5).

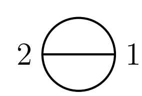
Definition 4.2. We define a local operation on plabic graphs which we call *splitting*. Let G be a plabic graph of type (k, n) (see Definition 2.9), and $i \in [n]$ a boundary vertex incident to a unique edge e. We *split e at i* by locally replacing e by a vertex v with three incident edges as follows, while leaving the rest of the graph unchanged:

(4.3)



(The color of v may be either black or white.) This produces a plabic graph G' of type $(k', n+1)$, whose boundary vertices are labeled by $[n+1]$, so that i and $i+1$ are labeled as above (i.e. labels $1, \dots, i-1$ from G remain unchanged, and labels $i+1, \dots, n$ from G get increased by 1). Note that by (2.10), $k' = k$ if v is white, while $k' = k+1$ if v is black.

Definition 4.4 (Recursion for cells when $m = 2$)**.** For positive integers k and n such that $1 \le k \le n-1$, we recursively define a set $\tilde{\mathcal{G}}_{n,k,2}$ of plabic graphs of type (k, n) as follows.[4]

(1) If $n = 2$ and $k = 1$, then $\tilde{\mathcal{G}}_{n,k,2}$ contains a unique element, the plabic graph 
 .

(2) For $n \ge 3$, $\tilde{\mathcal{G}}_{n,k,2}$ is the set of all plabic graphs obtained either by splitting a plabic graph in $\tilde{\mathcal{G}}_{n-1,k,2}$ at $n-1$ with v white, or by splitting a plabic graph in $\tilde{\mathcal{G}}_{n-1,k-1,2}$ at $n-1$ with v black. (See Figure 4.)

[4] In our notation $\tilde{\mathcal{G}}_{n,k,2}$, the subscript 2 is to remind us that these graphs correspond to the $m = 2$ amplituhedron, and the tilde is to remind us that these graphs do **not** directly label the cells $\mathcal{C}_{n,k,2}$. Rather, we must first 'shift by 1'; see Definition 4.5.

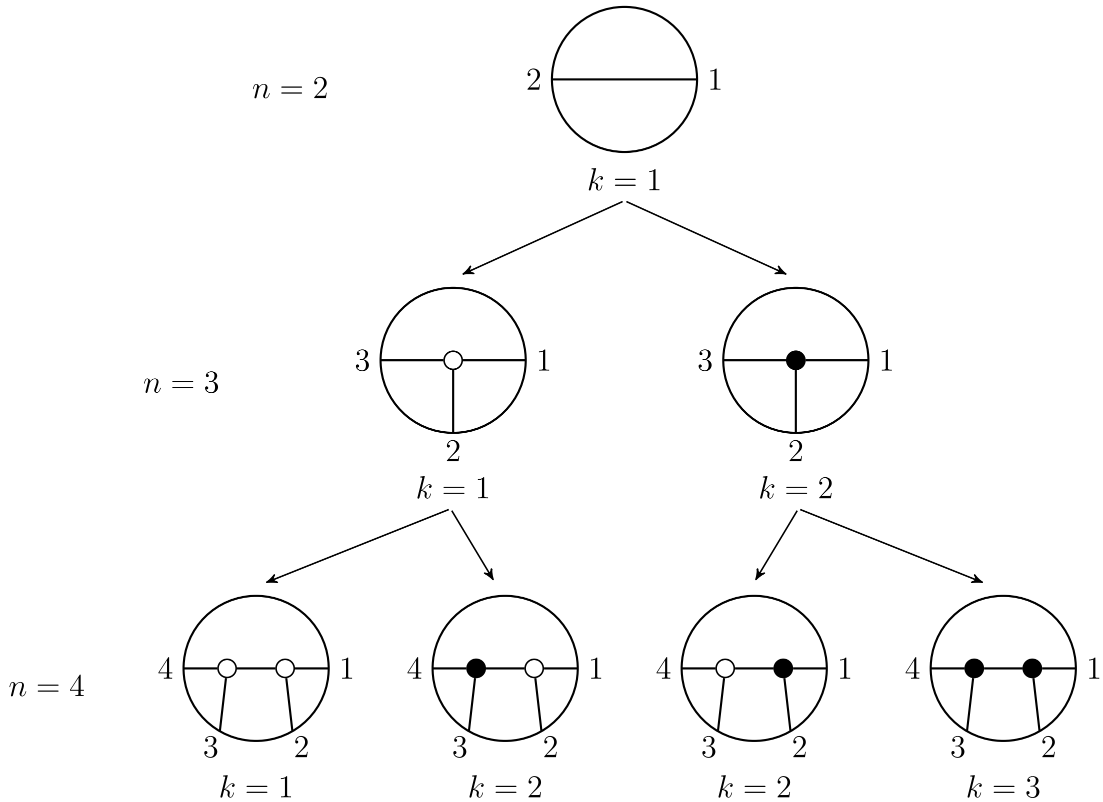


FIGURE 4. The first two steps of the $m=2$ recursion from Definition 4.4, giving the graphs $\tilde{\mathcal{G}}_{n,k,2}$ for $n \le 4$.

Alternatively, the plabic graphs in $\tilde{\mathcal{G}}_{n,k,2}$ are precisely those shown in Figure 5, where $k-1$ of the vertices $v_1, \ldots, v_{n-2}$ are black, and the rest are white. Therefore $|\tilde{\mathcal{G}}_{n,k,2}| = \binom{n-2}{k-1}$.

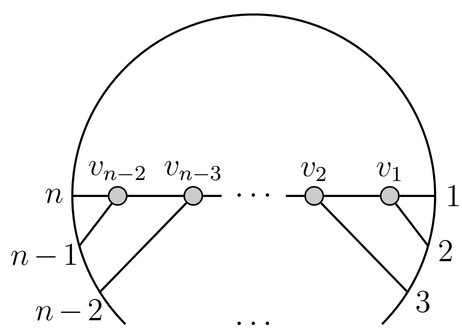


FIGURE 5. An arbitrary element of $\tilde{\mathcal{G}}_{n,k,2}$, where precisely $k-1$ of the vertices $v_1, \ldots, v_{n-2}$ are black.

Definition 4.5. Let $k, n \geq 0$ satisfy $k \leq n-2$, and $c_n := (n \;\; n{-}1 \;\; \cdots \;\; 2 \;\; 1)$ be the long cycle in the symmetric group on $[n]$. We define a collection $\Pi_{n,k,2}$ of decorated permutations of type (k, n) by

$$\Pi_{n,k,2} := \{c_n \pi_G : G \in \tilde{\mathcal{G}}_{n,k+1,2}\},$$

where above we color any fixed points of $c_n\pi_G$ black. (Note that if G is a plabic graph coming from the recursion in Definition 4.4, then π_G has no fixed points, so indeed multiplying π_G by c_n on the left decreases the number of anti-excedances by 1.) We let $\mathcal{C}_{n,k,2}$ denote the collection of cells $S_\pi \subseteq \mathrm{Gr}_{k,n}^{\geq 0}$ corresponding to permutations $\pi \in \Pi_{n,k,2}$.

4.2. ⅃-diagrams for $m = 2$.

Definition 4.6. Let $\mathcal{D}_{n,k,2}$ denote the set of all ⅃-diagrams D of type (k, n) such that:

- each of the k rows of D contains at least 2 boxes;
- the leftmost and rightmost entry in each row of D is $+$, and all other entries are 0.

(See Figure 6.) In particular, D has precisely $2k$ $+$'s, and hence indexes a cell S_D of $\mathrm{Gr}_{k,n}^{\geq 0}$ of dimension $2k$. Also note that the elements of $\mathcal{D}_{n,k,2}$ are naturally indexed by lattice paths inside a $k \times (n-k-2)$ rectangle[5]; the lattice path corresponding to D is the southeast border of the Young diagram obtained by deleting the two leftmost columns of D.

We can verify that in fact $\mathcal{D}_{n,k,2}$ indexes the cells $\mathcal{C}_{n,k,2}$, via the following bijection.

Proposition 4.7. *Given $G \in \tilde{\mathcal{G}}_{n,k+1,2}$ as in Figure 5, let W by the lattice path inside a $k \times (n-k-2)$ rectangle given by reading the vertices $v_1, \dots, v_{n-2}$ of G, moving west if v_i is white and south if v_i is black. Let $D \in \mathcal{D}_{n,k,2}$ be the ⅃-diagram corresponding to W, as in Definition 4.6. (See Figure 6.) Then*

$$\pi_D = c_n \pi_G,$$

where $c_n := (n \;\; n{-}1 \;\; \cdots \;\; 2 \;\; 1)$, π_D is defined in Definition 2.3, and all fixed points of $c_n\pi_G$ are colored black. In particular, $\mathcal{D}_{n,k,2}$ indexes the cells $\mathcal{C}_{n,k,2}$.

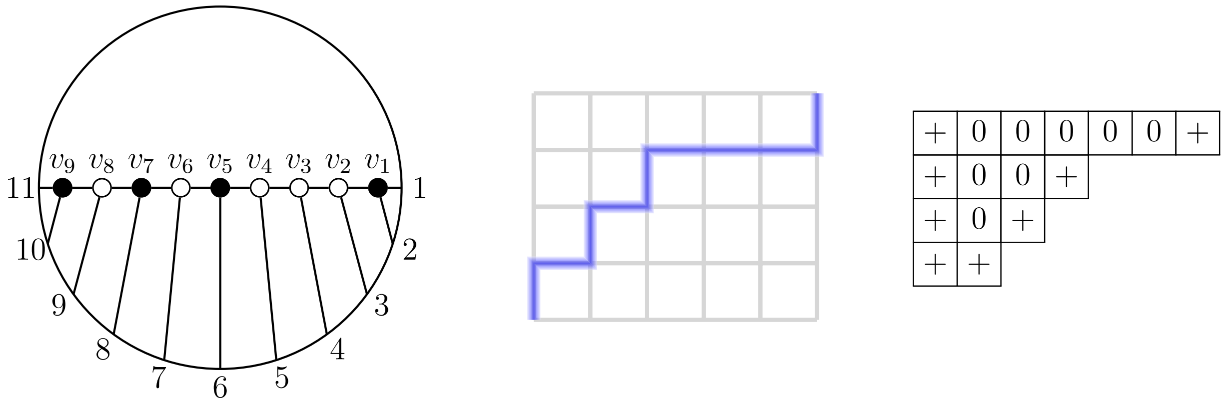
We leave the proof as an exercise to the reader, as a warmup to Theorem 7.4.

Remark 4.8. Let $\mathcal{D}_{n,k,1}$ be a set of ⅃-diagrams of type (k, n) in bijection with $\mathcal{D}_{n+1,k,2}$, where the element of $\mathcal{D}_{n,k,1}$ corresponding to $D \in \mathcal{D}_{n+1,k,2}$ is formed by deleting the leftmost column of D. For example,

$$\begin{array}{|c|c|c|c|c|c|c|} \hline + & 0 & 0 & 0 & 0 & 0 & + \\ \hline + & 0 & 0 & + \\ \cline{1-4} + & 0 & + \\ \cline{1-3} + & + \\ \cline{1-2} \end{array} \in \mathcal{D}_{11,4,2} \qquad \longleftrightarrow \qquad \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 & + \\ \hline 0 & 0 & + \\ \cline{1-3} 0 & + \\ \cline{1-2} + \\ \cline{1-1} \end{array} \in \mathcal{D}_{10,4,1}\,.$$

Let $\mathcal{C}_{n,k,1}$ be the cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ indexed by $\mathcal{D}_{n,k,1}$. In [KW17], two of us showed that the images of $\mathcal{C}_{n,k,1}$ inside the $m = 1$ amplituhedron $\mathcal{A}_{n,k,1}(Z)$ induce a triangulation (in fact, a cell decomposition) of $\mathcal{A}_{n,k,1}(Z)$. In fact, one of our motivations for studying this cell decomposition came from discovering the $m = 2$ BCFW-like recursion and the corresponding

[5] i.e. lattice paths moving from northeast to southwest by unit steps west and south; see Definition 6.1.



G W D

FIGURE 6. A plabic graph $G \in \tilde{\mathcal{G}}_{11,5,2}$, the corresponding lattice path W inside a 4×5 rectangle, and the corresponding Ⅎ-diagram $D \in \mathcal{D}_{11,4,2}$. We have $\pi_D = (2, 11, \underline{3}, \underline{4}, 6, 1, 8, 5, 10, 7, 9) = c_{11}\pi_G$.

diagrams $\mathcal{D}_{n,k,2}$. In light of the bijection $\mathcal{D}_{n+1,k,2} \to \mathcal{D}_{n,k,1}$, it is natural to ask whether the diagrams corresponding to the BCFW cells for $m = 4$ can be similarly truncated to give diagrams inducing a triangulation of the $m = 3$ amplituhedron. In Section 12 we show that this fails, at least for the most obvious such truncation.

4.3. **Domino bases for $m = 2$.** We now describe a nice basis of any subspace coming from a cell of $\mathcal{C}_{n,k,2}$. We remark that this is the same basis appearing in [AHTT, (7.5)], up to a cyclic shift (which preserves the positivity of $\mathrm{Gr}_{k,n}$ and Z). Therefore, our collection of cells $\mathcal{C}_{n,k,2}$ is the same as Arkani-Hamed, Trnka, and Thomas' up to a cyclic shift.

Definition 4.9. We say that $d \in \mathbb{R}^n \setminus \{0\}$ is a *domino* if either d has exactly two nonzero components, which are adjacent and have the same sign, or d has exactly one nonzero component, which is component n.

For example, the vectors $(0, 0, 0, 6, 1, 0)$ and $(0, 0, 0, 0, 0, -2)$ in $\mathbb{R}^6$ are dominoes.

Lemma 4.10. *Given $D \in \mathcal{D}_{n,k,2}$, label the edges of the southeast border of D by $1, \dots, n$ from northeast to southwest, and let $s_1 < \cdots < s_k$ be the labels of the vertical steps. Also let $V \in \mathrm{Gr}_{k,n}$. Then $V \in S_D$ if and only if V has a basis $\{v^{(1)}, \dots, v^{(k)}\}$ such that for all $i \in [k]$,*

- *$v^{(i)}_j = 0$ for all $j \neq s_i, s_i + 1, n$; and*
- *$v^{(i)}_{s_i}$, $v^{(i)}_{s_i+1}$, and $(-1)^{k-i} v^{(i)}_n$ are all positive.*

Note that such a vector $v^{(i)}$ is a sum of two dominoes.

Proof. Let A be the $k \times n$ matrix from Theorem 2.11, which represents a general element of S_D. For $i = 1, \dots, k-1$, we perform the following row operations on A to obtain a new $k \times n$ matrix A', which represents the same element of $\mathrm{Gr}_{k,n}^{\geq 0}$ that A does:

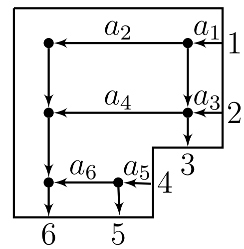
- if $s_{i+1} > s_i + 1$, we leave row s_i unchanged;

- if $s_{i+1} = s_i + 1$, we add the appropriate scalar multiple of row s_{i+1} to row s_i in order to make the (s_i, j)-entry zero, where $j > s_i$ is minimum such that $j \neq s_1, \dots, s_k$.

For example, for the Ⅎ-diagram

$$D = \begin{array}{|c|c|c|} \hline + & 0 & + \\ \hline + & 0 & + \\ \hline + & + & \\ \cline{1-2} \end{array}$$

with associated network



,

the parameterization matrices A and A' are

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & -a_1 & 0 & 0 & a_1(a_2+a_4) \\ 0 & 1 & a_3 & 0 & 0 & -a_3 a_4 \\ 0 & 0 & 0 & 1 & a_5 & a_5 a_6 \end{bmatrix} \end{array} \rightsquigarrow A' = \begin{array}{c} \\ 1 \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & a_1/a_3 & 0 & 0 & 0 & a_1 a_2 \\ 0 & 1 & a_3 & 0 & 0 & -a_3 a_4 \\ 0 & 0 & 0 & 1 & a_5 & a_5 a_6 \end{bmatrix} \end{array}.$$

Given $V \in S_D$, we can specialize the edge variables to positive real numbers so that A' represents V. Then we may take $v^{(1)}, \dots, v^{(k)}$ to be the rows of A'. Conversely, given any $v^{(1)}, \dots, v^{(k)}$ as in the statement of the lemma, we can find positive values of the edge variables so that the rows of A' are $v^{(1)}, \dots, v^{(k)}$, after we scale row s_i by $v^{(i)}_{s_i}$. ■

4.4. **Disjointness for** $m = 2$**.** We show that the images of the cells $\mathcal{C}_{n,k,2}$ inside the $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}(Z)$ are disjoint. This is a warmup to Section 11, where we prove disjointness for $m = 4$ when $k = 2$.

Lemma 4.11. *If $v \in \mathbb{R}^n$ is a sum of $k \geq 1$ dominoes, then* $\operatorname{var}(v) \leq k - 1$.

We leave the proof as an exercise to the reader. We will provide a more general version of Lemma 4.11 in Lemma 11.3.

Proposition 4.12. *Let $Z \in \operatorname{Mat}^{>0}_{k+2,n}$. Then $\tilde{Z}$ maps the cells $\mathcal{C}_{n,k,2}$ of $\operatorname{Gr}^{\geq 0}_{2,n}$ injectively into the amplituhedron $\mathcal{A}_{n,k,2}(Z)$, and their images are pairwise disjoint.*

Proof. We must show that given $D, D' \in \mathcal{D}_{n,k,2}$ and $V \in S_D, V' \in S_{D'}$ such that $\tilde{Z}(V) = \tilde{Z}(V')$, we have $V = V'$. Let $\{v^{(1)}, \dots, v^{(k)}\}$ be the distinguished basis of V from Lemma 4.10.

Claim. For any $i \in [k]$ and $v' \in V'$, we have $\operatorname{var}(v^{(i)} - v') \leq k + 1$.

Proof of Claim. Let $\{w^{(1)}, \dots, w^{(k)}\}$ be the distinguished basis of V' from Lemma 4.10, and write $v' = \sum_{i=1}^k c_i w^{(i)}$ for some $c_1, \dots, c_k \in \mathbb{R}$. Then $v^{(i)} - v' = v^{(i)} - \sum_{i=1}^k c_i w^{(i)}$. The right-hand side can be written as a sum of $k+2$ or fewer dominoes, so $\operatorname{var}(v^{(i)} - v') \leq k+1$ by Lemma 4.11. □

For $i \in [k]$, let $v'^{(i)} \in V'$ be the matching vector for $v^{(i)}$, as in Lemma 3.4. Then by the claim and Lemma 3.4, we have $v^{(i)} = v'^{(i)}$, i.e. $v^{(i)} \in V'$. Hence $V \subseteq V'$, and so $V = V'$. ■

5. Binary trees and BCFW plabic graphs

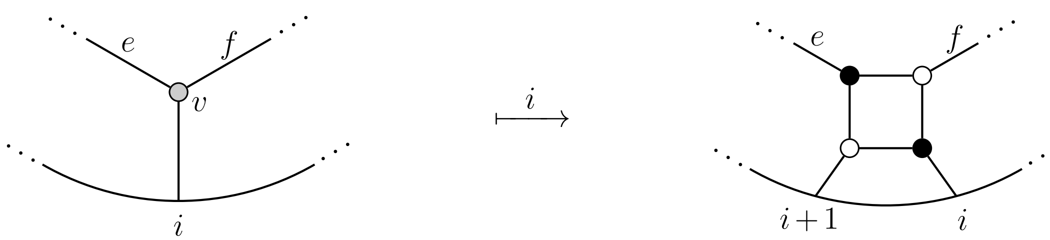
In the case $m = 4$ (which is the case of interest in physics), there is a distinguished collection of $4k$-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ called *BCFW cells*. These cells are named after Britto, Cachazo, Feng, and Witten, who gave recursion relations for computing scattering amplitudes [BCF05, BCFW05]. These relations were translated in [AHBC+16] into a recursion on plabic graphs, which we define in Section 5.1. The resulting graphs are easily seen to be in bijection with complete binary trees (well-known Catalan objects), as we explain in Section 5.2. However, these plabic graphs do not literally label the BCFW cells in the sense of Definition 2.9; rather, we must perform a 'shift by 2' to the decorated permutation corresponding to the graph, which makes it difficult to explicitly describe the BCFW cells from the BCFW recursion.[6] In Section 6, we will use a different family of Catalan objects, namely pairs of noncrossing lattice paths inside a rectangle, to explicitly describe the BCFW cells in terms of $\oplus$-diagrams.

5.1. **The recursive description of BCFW cells from plabic graphs.**

The *BCFW recursion* [BCF05, BCFW05] is a recursive operation that produces, for each pair (k, n) with $0 \leq k \leq n-4$, a set $\mathcal{C}_{n,k,4}$ of $4k$-dimensional positroid cells of $\mathrm{Gr}_{k,n}^{\geq 0}$, called the (k, n)-*BCFW cells*.[7] This operation is described in [AHBC+16, Section 17.2/16.2[8]].

Definition 5.1. We define a local operation on plabic graphs which we call *blowing up*. Let G be a plabic graph of type (k, n) (see Definition 2.9), and $i \in [n]$ a boundary vertex incident to a unique internal vertex v, which has degree 3. We *blow up G at i* by locally replacing v by a square, as follows, while leaving the rest of the graph unchanged:

(5.2)



(The color of v may be black or white.) This produces a plabic graph G' of type $(k', n+1)$, whose boundary vertices are labeled by $[n+1]$ so that i and $i+1$ are labeled as above (i.e. labels $1, \dots, i-1$ from G remain unchanged, and labels $i+1, \dots, n$ from G get increased by 1). In the blowup, the orientation of the square is important: its vertices alternate in color, with i incident to a black vertex and $i+1$ to a white vertex. Note that by (2.10), $k' = k$ if v is black, while $k' = k+1$ if v is white.

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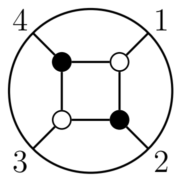
[6] Bai and He [BH15, (3.3)] found a recursion on the plabic graphs which do literally index the BCFW cells. However, their recursion is more complicated and does not obviously correspond to a family of Catalan objects, so we prefer to work with the original version of the recursion.

[7] In fact, there are many different ways to carry out the BCFW recursion, which each lead to a possibly different set of positroid cells of $\mathrm{Gr}_{k,n}^{\geq 0}$. For example one can cyclically permute all boundary labels of the plabic graphs one obtains in the recursion. In this paper we fix a canonical way to perform the recursion.

[8] These two section numbers refer, respectively, to the published book and the arXiv preprint.

Definition 5.3 (BCFW recursion)**.** For positive integers k and n such that $2 \leq k \leq n-2$, we recursively define a set $\tilde{\mathcal{G}}_{n,k,4}$ of plabic graphs of type (k,n) as follows.[9]

1. If $n = 4$ and $k = 2$, then $\tilde{\mathcal{G}}_{n,k,4}$ contains a unique element, the plabic graph



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2. For $n \geq 5$, $\tilde{\mathcal{G}}_{n,k,4}$ is the set of all plabic graphs obtained either by blowing up a plabic graph in $\tilde{\mathcal{G}}_{n-1,k,4}$ at some $i \neq 1, n-1$ incident to a black vertex, or by blowing up a plabic graph in $\tilde{\mathcal{G}}_{n-1,k-1,4}$ at some $i \neq 1, n-1$ incident to a white vertex. (We may obtain such a plabic graph multiple times in this way, but we only count it once in $\tilde{\mathcal{G}}_{n,k,4}$.) We emphasize that we never blow up at the first or last boundary positions.

See Figure 7 for a depiction of the graphs $\tilde{\mathcal{G}}_{n,k,4}$ for $n \leq 6$. Note that we can read the k statistic from each graph as the number of black vertices incident to the boundary.

Definition 5.4. Let $k, n \geq 0$ satisfy $k \leq n-4$, and $c_n := (n \;\; n{-}1 \;\; \cdots \;\; 2 \;\; 1)$ be the long cycle in the symmetric group on $[n]$. We define the *BCFW permutations* of type (k,n) (for $m = 4$) as

$$\Pi_{n,k,4} := \{c_n^2 \pi_G : G \in \tilde{\mathcal{G}}_{n,k+2,4}\},$$

where above we color any fixed points of $c_n^2\pi_G$ black. (Note that for any plabic graph G coming from the BCFW recursion, we have $\pi_G(i) \neq i, i+1 \pmod{n}$ for all $i \in [n]$, so indeed multiplying π_G by c_n^2 on the left decreases the number of anti-excedances by 2.) The collection $\mathcal{C}_{n,k,4}$ of cells $S_\pi \subseteq \mathrm{Gr}_{k,n}^{\geq 0}$ corresponding to BCFW permutations $\pi \in \Pi_{n,k,4}$ are called the (k,n)-*BCFW cells* (for $m = 4$).[10]

Conjecture 5.5 ([AHT14, Section 5])**.** *Let $Z \in \mathrm{Mat}_{k+4,n}^{>0}$, where $k, n \geq 0$ satisfy $k \leq n-4$. Then the images under $\tilde{Z}$ of the BCFW cells $\mathcal{C}_{n,k,4}$ "triangulate" the $m = 4$ amplituhedron, i.e. they are pairwise disjoint, and together they cover a dense subset of $\mathcal{A}_{n,k,4}(Z)$.*

Interestingly, the number of BCFW cells in $\mathrm{Gr}_{k,n}^{\geq 0}$ is a *Narayana number*. The Narayana numbers $N_{a,b} := \frac{1}{a}\binom{a}{b}\binom{a}{b-1}$ refine the Catalan numbers, i.e. $\sum_{b=1}^{a} N_{a,b}$ is the Catalan number $\frac{1}{a+1}\binom{2a}{a}$ [Sta15, A46].

Lemma 5.6 ([AHBC⁺16, (17.7)/(16.8)])**.** *For $m = 4$, the number of (k,n)-BCFW cells is the Narayana number $N_{n-3,k+1} = \frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$.*

[9]In our notation $\tilde{\mathcal{G}}_{n,k,4}$, the tilde is to remind us that these graphs do **not** directly label the $m = 4$ BCFW cells. Rather, we must first 'shift by 2'; see Definition 5.4.

[10]Some authors use the term 'BCFW cells' to refer to the images $\tilde{Z}(S_\pi)$ in the amplituhedron $\mathcal{A}_{n,k,4}(Z)$ (though in general it is not known that these images are indeed topological cells).

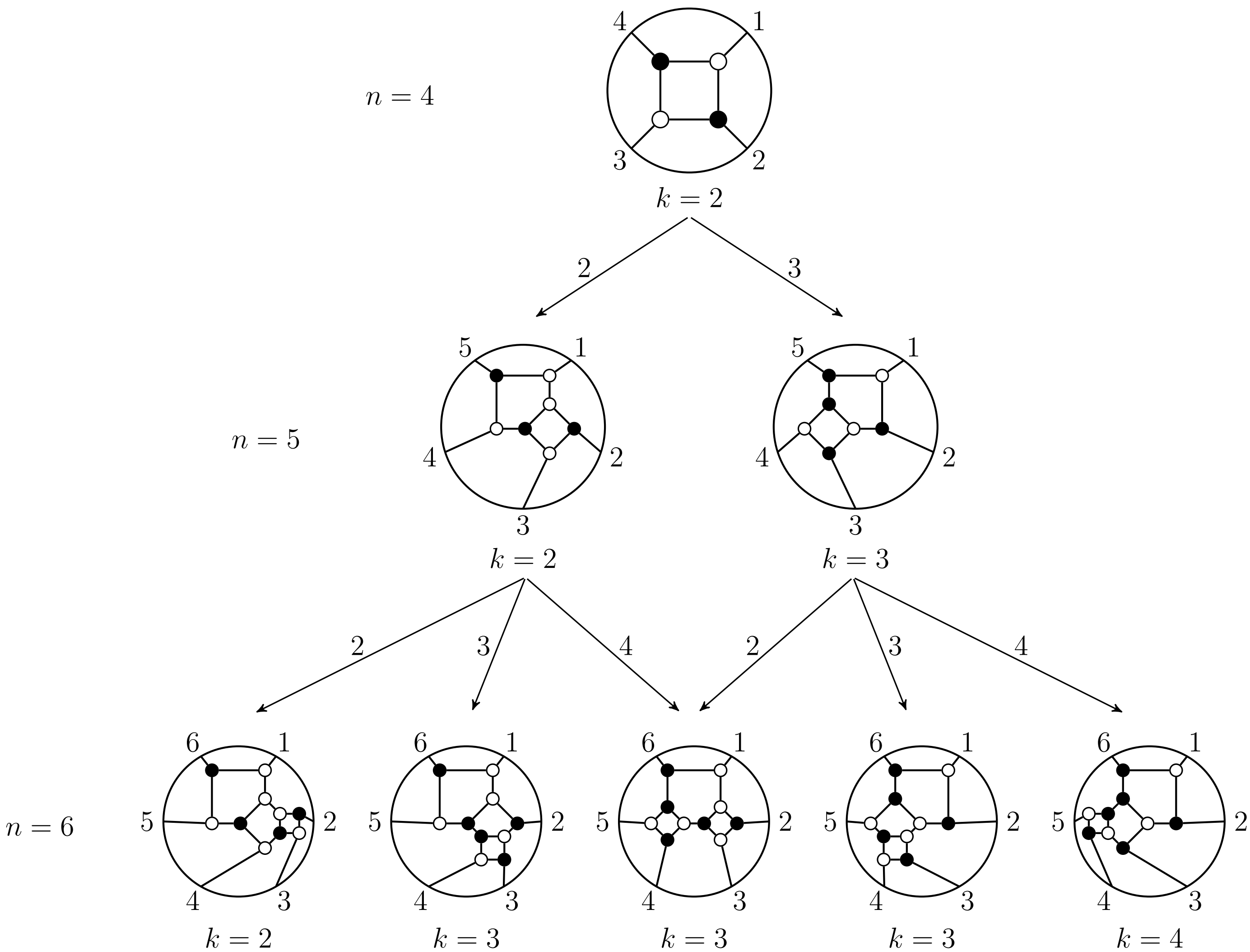


FIGURE 7. The first two steps of the BCFW recursion. The Catalan sequence $1, 2, 5, 14, \dots$ gives the number of graphs in each row. Note that the k statistic has not yet been 'shifted by 2'. Observe that the middle graph in the bottom row can be obtained via two different sequences of blowups.

5.2. **Complete binary trees and a bijection to BCFW graphs.** In this section we explain how to index the plabic graphs $\tilde{\mathcal{G}}_{n,k,4}$ coming from the BCFW recursion by *complete binary trees*. This construction also appears in a similar form in lecture notes written by Morales based on a course taught by Postnikov [Mor, Figure 110].

Definition 5.7. A *complete* (or *plane*) *binary tree* T is a rooted tree such that every vertex either has 2 ordered child vertices, one joined by a horizontal (left) edge and the other by a vertical (up) edge, or 0 child vertices. We require that the root vertex has 2 children. (See Figure 8 for an example.) We call childless vertices *leaves*, and other vertices *internal vertices*. An edge of T is called *external* if it is incident to a leaf; otherwise it is called *internal.* If T has $n-2$ leaves ($n \geq 4$), we label them by $2, 3, \ldots, n-1$ clockwise, such that the leaf 2 is joined to the root vertex by a path of horizontal edges (and similarly, $n-1$ is joined to the root by a path of vertical edges). We let $\mathcal{T}_{n,k,4}$ denote the set of complete binary trees T with $n-2$ leaves, exactly $k+1$ of which are incident to a horizontal edge.

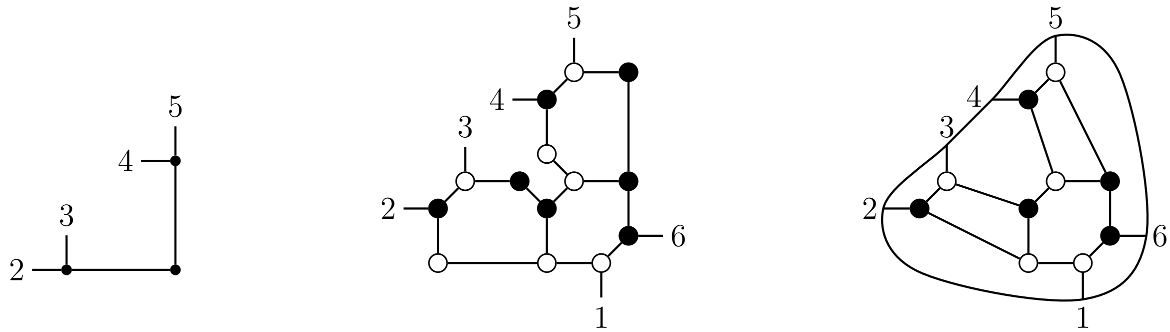
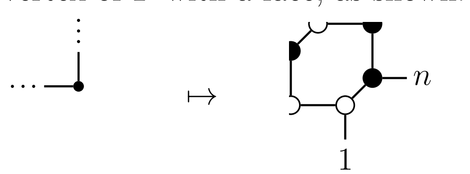


FIGURE 8. A complete binary tree $T \in \mathcal{T}_{6,1,4}$ and its plabic graph $G(T) \in \tilde{\mathcal{G}}_{6,3,4}$ (before and after contracting bivalent vertices and enclosing the graph in a disk). Note that $G(T)$ is the middle graph in the bottom row of Figure 7.

Definition 5.8. Given a complete binary tree $T \in \mathcal{T}_{n,k,4}$, we define a plabic graph $G(T)$ with n boundary vertices, as follows.

- We replace the root vertex of T with a face, as shown:



  (As we will explain shortly, the 'half-vertices' on the right are intentional.)
- We replace each internal vertex of T with a face, as shown:

- We replace each leaf of T, as shown:

  In both of the resulting local pictures of a plabic graph above, the vertex not incident to i is incident to exactly one other vertex (which appears just outside this local picture), of the same color; we contract these two vertices. (This is to avoid having any bivalent vertices in the resulting plabic graph.)

- We draw a curve through $1, \dots, n$, enclosing the resulting graph in a disk.

This gives a plabic graph $G(T)$. See Figure 8 for an example. (Above, we have depicted only 'half vertices' of $G(T)$, since each internal vertex of $G(T)$, aside from those incident to 1 and n, comes from two vertices of T.)

Lemma 5.9. *The map $T \mapsto G(T)$ from Definition 5.8 gives a bijection $\mathcal{T}_{n,k,4} \to \tilde{\mathcal{G}}_{n,k+2,4}$.*

Proof. It follows from Definition 5.8 that adding child vertices to a leaf i of T corresponds to blowing up $G(T)$ at i. That is, the BCFW recursion acts on complete binary trees by adding children. ■

Note that it is straightforward to recover T from $G(T)$: the graph formed by the internal edges of T is dual to the graph formed by the internal faces of $G(T)$.

6. Pairs of noncrossing lattice paths and BCFW Ⅎ-diagrams

In this section, we index the $m = 4$ BCFW cells by pairs of noncrossing lattice paths inside a rectangle. We explain how to obtain a $\oplus$-diagram of a BCFW cell from such a pair.

Definition 6.1. Fix $a, b \in \mathbb{N}$. A *lattice path* W inside an $a \times b$ rectangle is a path that moves from the northeast corner to the southwest corner, taking unit steps west and south. We represent W by a word of length $a+b$ on the alphabet $\{H, V\}$, with exactly a letters V (corresponding to the vertical steps) and b letters H (corresponding to the horizontal steps). For example, the upper lattice path W_U in Figure 9 is given by $W_U = HHVHHVVH$.

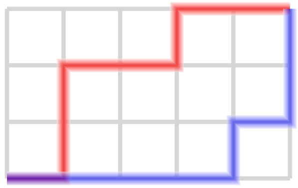
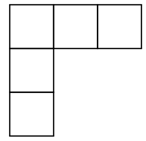
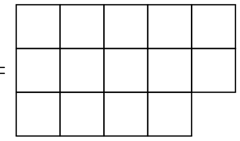
Let $\mathcal{L}_{n,k,4}$ denote the set of all pairs (W_U, W_L) of *noncrossing* lattice paths inside a $k \times (n-k-4)$ rectangle, where W_U denotes the upper path and W_L denotes the lower path. That is, W_L is weakly below W_U (but the two paths are allowed to overlap); see Figure 9. In terms of words, this means that for any $i \in [n-4]$, there are at least as many V's among the first i letters of W_L as among the first i letters of W_U.

We give an injective map from elements of $\mathcal{L}_{n,k,4}$ to $4k$-dimensional positroid cells of $\mathrm{Gr}_{k,n}^{\geq 0}$. We will prove that the collection of cells in its image are precisely the (k,n)-BCFW cells.

Definition 6.2. Given $(W_U, W_L) \in \mathcal{L}_{n,k,4}$, let Y_U be the Young diagram inside a $k \times (n-k-4)$ rectangle whose southeast border is Y_U, and similarly define Y_L. We associate a $\oplus$-diagram D of type (k,n) (recall Definition 2.3) to (W_U, W_L) as follows. (See Figure 9 for an example.)

- Step 1. The Young diagram of D is obtained from Y_L by adding $m = 4$ extra columns at the left of height k. Place a $+$ at the far left end and far right end of each row of D, leaving the other boxes empty.
- Step 2. Consider each column of Y_U in turn, reading the columns from left to right. For each column of Y_U of height i, place a top-justified column of i 0's in D as far right as possible.
- Step 3. In each row of D, place two $+$'s as far to the right as possible.
- Step 4. Fill any remaining empty boxes of D with a 0.

We denote D by $\Omega_{\mathcal{LD}}(W_U, W_L)$. This defines an injection $\Omega_{\mathcal{LD}}(W_U, W_L)$ from $\mathcal{L}_{n,k,4}$ to the set of $\oplus$-diagrams of type (k,n). Let $\mathcal{D}_{n,k,4}$ be the image of $\Omega_{\mathcal{LD}}(W_U, W_L)$. We will show

$(W_U, W_L) =$  $Y_U =$  $Y_L =$ 

Step 1.

+								+
+								+
+							+	

Step 2.

+						0		+
+						0		+
+						0	+	

Step 2.

+						0	0	+
+						0		+
+						0	+	

Step 2.

+					0	0	0	+
+						0		+
+						0	+	

Step 3.

+			+	+	0	0	0	+
+					+	0	+	+
+				+	+	0	+	

Step 4.

+	0	0	+	+	0	0	0	+
+	0	0	0	0	+	0	+	+
+	0	0	0	+	+	0	+	

FIGURE 9. The map $\Omega_{\mathcal{LD}}(W_U, W_L)$ from Definition 6.2 takes a pair of lattice paths $(W_U, W_L) \in \mathcal{L}_{n,k,4}$ to a reduced $\oplus$-diagram in $\mathcal{D}_{n,k,4}$, corresponding to a $4k$-dimensional cell of $\mathrm{Gr}_{k,n}^{\geq 0}$. Here $k = 3$ and $n = 12$.

in Lemma 6.4 that each $D \in \mathcal{D}_{n,k,4}$ is a reduced $\oplus$-diagram with exactly $4k$ +'s. Hence by Lemma 2.6, D corresponds to a $4k$-dimensional cell of $\mathrm{Gr}_{k,n}^{\geq 0}$, and we can use Ⅎ-moves to find the Ⅎ-diagram of D. We will also show in Lemma 6.4 that these $4k$-dimensional cells are all distinct.

Theorem 6.3. *The $\oplus$-diagrams $\mathcal{D}_{n,k,4}$ index the (k,n)-BCFW cells $\mathcal{C}_{n,k,4}$.*

Theorem 6.3 follows from Lemma 5.9 and Theorem 7.4, the latter of which we will prove in Section 7. First, we show that the $\oplus$-diagrams in $\mathcal{D}_{n,k,4}$ are reduced and represent distinct positroid cells.

Lemma 6.4. *The $\oplus$-diagrams in $\mathcal{D}_{n,k,4}$ are reduced, and each correspond to a distinct $4k$-dimensional cell of $\mathrm{Gr}_{k,n}^{\geq 0}$.*

Proof. We show that each $D \in \mathcal{D}_{n,k,4}$ is reduced. Let D' be the $\oplus$-diagram of type $(k, n-1)$ obtained from D by deleting the leftmost column. Then D' is a Ⅎ-diagram: each 0 added in Step 2 of Definition 6.2 has no + above it in the same column, and each 0 added in Step 4 has no + to its left in the same row (except for the + at the left end of the row, which has been deleted to form D' from D). Hence D' is reduced, so its pipe dream $P(D')$ has no double crossings. We form $P(D)$ from $P(D')$ by adding a column of elbows at the left, which introduces no new crossings. Hence D is reduced. Since D has $4k$ +'s, it corresponds to a $4k$-dimensional cell of $\mathrm{Gr}_{k,n}^{\geq 0}$.

Now we show that the diagrams in $\mathcal{D}_{n,k,4}$ index distinct cells of $\mathrm{Gr}_{k,n}^{\geq 0}$. Suppose that $D_1, D_2 \in \mathcal{D}_{n,k,4}$ index the same cell of $\mathrm{Gr}_{k,n}^{\geq 0}$, and as above let D_1' and D_2' be the Ⅎ-diagrams of type $(k, n-1)$ formed by deleting the leftmost column of D_1 and D_2, respectively. In the construction of the pipe dreams $P(D_1)$ and $P(D_2)$ from Definition 2.3, the edges of the west border of the Young diagram in each pipe dream are labeled with the anti-excedances of the corresponding decorated permutation. Since $\pi_{D_1} = \pi_{D_2}$, the pipe dreams $P(D_1)$ and $P(D_2)$ have the same shape and their edges are labeled in the same way. Hence $\pi_{D_1'} = \pi_{D_2'}$, whence $D_1' = D_2'$ by Lemma 2.5. This implies $D_1 = D_2$. ■

7. From binary trees to pairs of lattice paths

In this section we prove Theorem 6.3. Our strategy is to construct a map $\Omega_{\mathcal{TL}} : \mathcal{T}_{n,k,4} \to \mathcal{L}_{n,k,4}$ which takes a complete binary tree T to a pair of noncrossing lattice paths inside a $k \times (n-k-4)$ rectangle, such that the decorated permutation of the $\oplus$-diagram $\Omega_{\mathcal{LD}}(\Omega_{\mathcal{TL}}(T))$ equals $c_n^2 \pi_{G(T)}$ (recall Definition 5.4).

Definition 7.1. Given a complete binary tree T, we let r_H and r_V denote the horizontal and vertical child vertices of the root r of T. We let T_H denote the subtree of T rooted at r obtained from T by deleting all children of r_V. Also, if r_H is an internal vertex of T_H, we let T_H' be the subtree of T rooted at r_H formed from T_H by deleting r and its two incident edges. We similarly define the subtrees T_V and T_V' of T, by switching the roles of r_H and r_V.

We associate two lattice paths $W_U(T), W_L(T)$ to T by the following recursive definition. (Recall from Definition 6.1 that we identify a lattice path of length l with its corresponding word in $\{H, V\}^l$.) Below, $\cdot$ denotes concatenation of words.

- If r_H and r_V are both leaves, then $W_U(T)$ and $W_L(T)$ are the empty words.
- If r_H is not a leaf and r_V is a leaf, then
$$W_U(T) := H \cdot W_U(T'_H), \qquad W_L(T) := W_L(T'_H) \cdot H.$$
- If r_H is a leaf and r_V is not a leaf, then
$$W_U(T) := V \cdot W_U(T'_V), \qquad W_L(T) := V \cdot W_L(T'_V).$$
- Otherwise,
$$W_U(T) = W_U(T_H) \cdot W_U(T_V), \qquad W_L(T) = W_L(T_H) \cdot W_L(T_V).$$

Note that in fact $W_U(T) = W_U(T_H) \cdot W_U(T_V)$ and $W_L(T) = W_L(T_H) \cdot W_L(T_V)$ for all T.

We let $\Omega_{\mathcal{TL}} : \mathcal{T}_{n,k,4} \to \mathcal{L}_{n,k,4}$ be the map which sends T to $(W_U(T), W_L(T))$. It follows from the definitions of $W_U(T)$ and $W_L(T)$ that they are both words in $\{H,V\}^{n-4}$ with precisely k V's, and moreover that for any $i \in [n-4]$, there are at least as many V's among the first i letters of $W_L(T)$ as among the first i letters of $W_U(T)$. Hence the pair of lattice paths $(W_U(T), W_L(T))$ is indeed noncrossing, and represents an element of $\mathcal{L}_{n,k,4}$.

For example, for the complete binary tree T in Figure 10, we have $W_U(T) = HVHVH$ and $W_L(T) = HVHHV$.

Remark 7.2. Given $T \in \mathcal{T}_{n,k,4}$, we can alternatively find $W_U(T), W_L(T) \in \{H,V\}^{n-4}$ as follows. We obtain $W_U(T)$ by reading the internal edges of T in a depth-first search starting at the root, preferentially reading horizontal edges over vertical edges; we record an H for each horizontal edge and a V for each vertical edge. We obtain $W_L(T)$ by reading the leaves $3, 4, \dots, n-2$ of T in order, recording an H for each leaf incident to a vertical edge, and a V for each leaf incident to a horizontal edge.

Proposition 7.3. *The map $\Omega_{\mathcal{TL}} : \mathcal{T}_{n,k,4} \to \mathcal{L}_{n,k,4}$ is a bijection.*

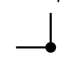
Proof. It follows from Lemma 5.6 and Proposition 8.6(1) that $\mathcal{T}_{n,k,4}$ and $\mathcal{L}_{n,k,4}$ have the same cardinality, namely the Narayana number $N_{n-3,k+1}$.[11] Hence it suffices to show that $\Omega_{\mathcal{TL}}$ is injective. We can prove this by induction on n, using the recursive definitions of $W_U(T)$ and $W_L(T)$. The key observation is that when we regard the pair of lattice paths $(W_U(T), W_L(T))$ as the concatenation of the pairs $(W_U(T_H), W_L(T_H))$ and $(W_U(T_V), W_L(T_V))$, we recover where $(W_U(T_V), W_L(T_V))$ begins as the first occurrence of overlapping vertical steps in $(W_U(T), W_L(T))$. (If there are no overlapping vertical steps, then r_V is a leaf.) To see this, note that $W_U(T_V)$ and $W_L(T_V)$ both begin with a vertical step (if they have any steps at all); conversely, the paths $W_U(T_H)$ and $W_L(T_H)$ do not have any overlapping vertical steps, since otherwise the paths $W_U(T'_H)$ and $W_L(T'_H)$ would cross each other. ∎

Now we state and prove the main result of this section.

Theorem 7.4. *For $T \in \mathcal{T}_{n,k,4}$, we have*
$$\pi_D = c_n^2 \pi_{G(T)},$$
where $D = \Omega_{\mathcal{LD}}(\Omega_{\mathcal{TL}}(T)) \in \mathcal{D}_{n,k,4}$ and $c_n := (n \;\; n{-}1 \;\; \cdots \;\; 2 \;\; 1)$ is the long cycle in the symmetric group on $[n]$. (As in Definition 5.4, we color the fixed points of $c_n^2 \pi_{G(T)}$ black.)

[11] For further references on the enumeration of $\mathcal{L}_{n,k,4}$, see [Sta15, pp. 66-67].

As we have already noted, this implies Theorem 6.3.

Proof. We proceed by induction on n. Given $T \in \mathcal{T}_{n,k,4}$, let $D := \Omega_{\mathcal{LD}}(\Omega_{\mathcal{TL}}(T)) \in \mathcal{D}_{n,k,4}$ be its associated $\oplus$-diagram. Note that every row of D contains at least one $+$, so π_D has no white fixed points. Hence it suffices to show that the equality $\pi_D = c_n^2 \pi_{G(T)}$ holds for (undecorated) permutations. If $n = 4$, then T is necessarily the tree , and D is the empty $\oplus$-diagram inside a 0×4 rectangle. Hence $\pi_{G(T)} = 3412$ and $\pi_D = 1234 = c_4^2 \pi_{G(T)}$. This proves the base case.

Now suppose that $n \geq 5$ and Theorem 7.4 holds for smaller values of n. We consider two different cases, depending on whether the last letter of $W_U(T)$ is V or H.

Case 1: $W_U(T)$ ends in V. Recall from Remark 7.2 that we obtain $W_U(T)$ by reading the internal edges of T in a depth-first search. Let e be the internal edge of T corresponding to the last letter of $W_U(T)$, and let w be the child vertex incident to e (i.e. w is a child of the other vertex incident to e). Then the children of w are both leaves, and are labeled by p and $p+1$ for some $2 \leq p \leq n-2$. Let $T' \in \mathcal{T}_{n-1,k-1,4}$ be obtained from T by replacing e and its children by a vertical boundary edge incident to a leaf (which will be labeled by p). Since $W_U(T)$ ends in V, we obtain $(W_U(T'), W_L(T'))$ from $(W_U(T), W_L(T))$ by deleting the last vertical step of each path. Let $D' \in \mathcal{D}_{n-1,k-1,4}$ be the associated $\oplus$-diagram. (See Figure 10.) By the induction hypothesis, we have $\pi_{D'} = c_{n-1}^2 \pi_{G(T')}$. (We will regard permutations of $[n-1]$ as permutations of $[n]$ which fix n.)

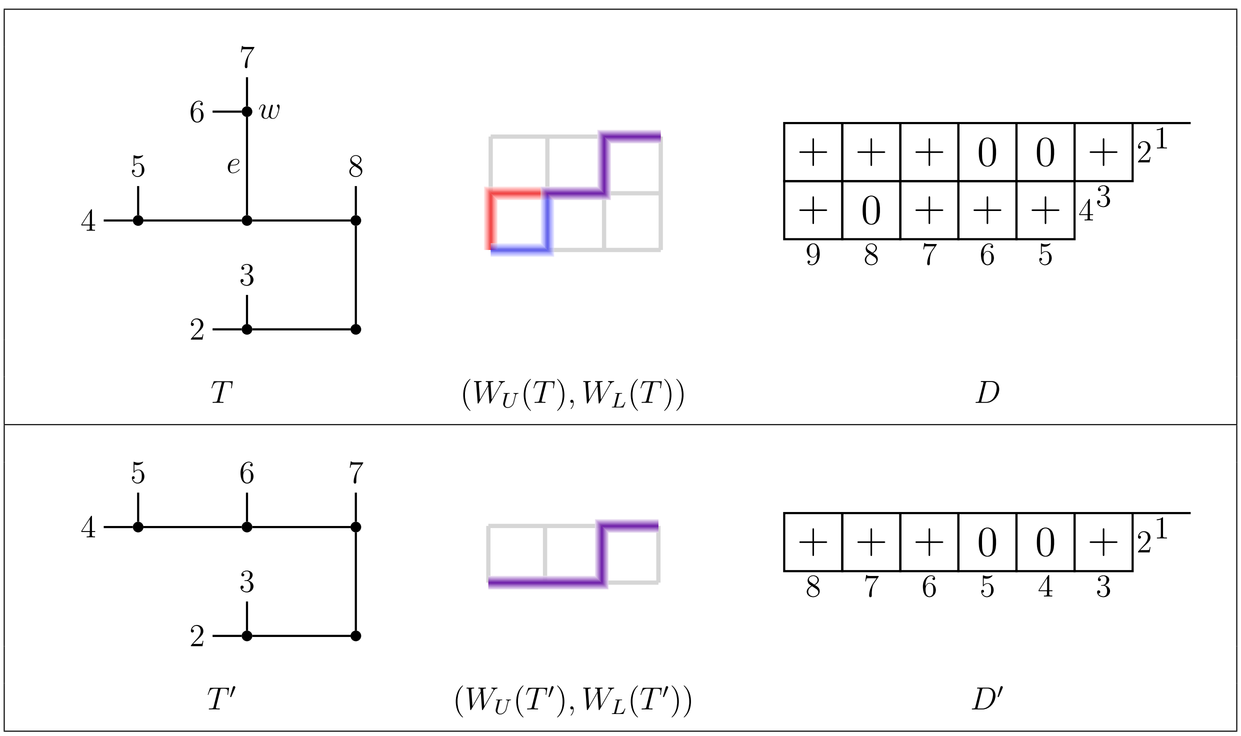
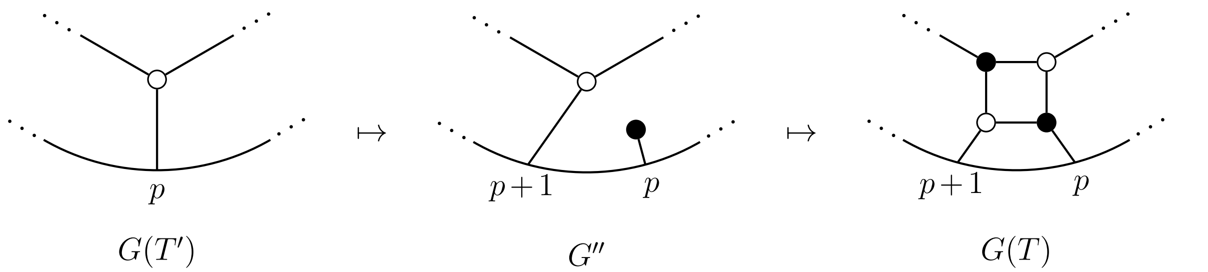


FIGURE 10. An example of Case 1 in the proof of Theorem 7.4. Here $k = 2$, $n = 9$, $p = 6$.

In terms of plabic graphs, $G(T)$ is obtained from $G(T')$ by blowing up at p. Let us introduce the intermediate graph G'', obtained from $G(T')$ by inserting a lollipop in between boundary vertices $p-1$ and p, and increasing the labels of the boundary vertices $p, p+1, \dots, n-1$ of $G(T')$ by 1. In terms of permutations, we have

$$\pi_{G''} = (p\ \ p{+}1\ \ \cdots\ \ n)\pi_{G(T')}(n\ \ n{-}1\ \ \cdots\ \ p).$$

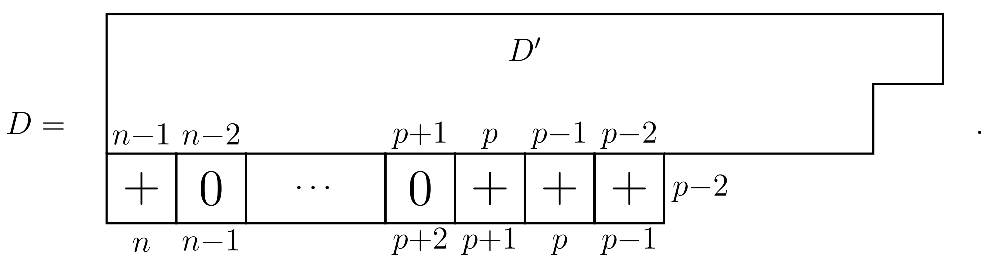
Then $G(T)$ is obtained from $G(T')$ via G'' as follows (cf. (5.2)):



Since e corresponds to the last letter of $W_U(T)$, and this letter is V, the boundary vertices $p+1, p+2, \dots, n-1$ of $G(T)$ are each incident to a white vertex. Hence $\pi_{G(T)}(n) = p$, and we see that $\pi_{G(T)} = \pi_{G''}(p\ \ p{+}1\ \ n)$. Thus

$$\begin{aligned} \pi_{G(T)} &= (p\ \ p{+}1\ \ \cdots\ \ n)\pi_{G(T')}(n\ \ n{-}1\ \ \cdots\ \ p)(p\ \ p{+}1\ \ n) \\ &= (p\ \ p{+}1\ \ \cdots\ \ n)\pi_{G(T')}(n{-}1\ \ n{-}2\ \ \cdots\ \ p{+}1). \end{aligned} \tag{7.5}$$

On the other hand, by Definition 6.2, D is obtained from D' by appending a new row, as follows:



Above we have given the labels of the southeast borders of D and D' starting at $p-2$. We can then verify from Definition 2.3 that

$$\pi_D = (p{-}2\ \ p{-}1\ \ \cdots\ \ n)\pi_{D'}(n{-}1\ \ n{-}2\ \ \cdots\ \ p{+}1).$$

Putting this together with the induction hypothesis $\pi_{D'} = c_{n-1}^2 \pi_{G(T')}$ and (7.5), we obtain

$$\begin{aligned} \pi_D &= (p{-}2\ \ p{-}1\ \ \cdots\ \ n)c_{n-1}^2\pi_{G(T')}(n{-}1\ \ n{-}2\ \ \cdots\ \ p{+}1) \\ &= c_n^2(p\ \ p{+}1\ \ \cdots\ \ n)\pi_{G(T')}(n{-}1\ \ n{-}2\ \ \cdots\ \ p{+}1) = c_n^2\pi_{G(T)}. \end{aligned}$$

Case 2: $W_U(T)$ ends in H. Suppose that $W_U(T)$ ends in precisely s H's, where $s \geq 1$. Let $e_{n-s-3}, e_{n-s-2}, \dots, e_{n-4}$ be the internal edges of T corresponding to the last s letters of $W_U(T)$, when we read the internal edges in a depth-first search as in Remark 7.2. These edges appear in a horizontal path in T as in Figure 11, where the rightmost vertex is either the root vertex (in which case this picture is the entirety of T), or is joined to its parent vertex by a vertical edge. The children of the vertices on this path are all leaves, which

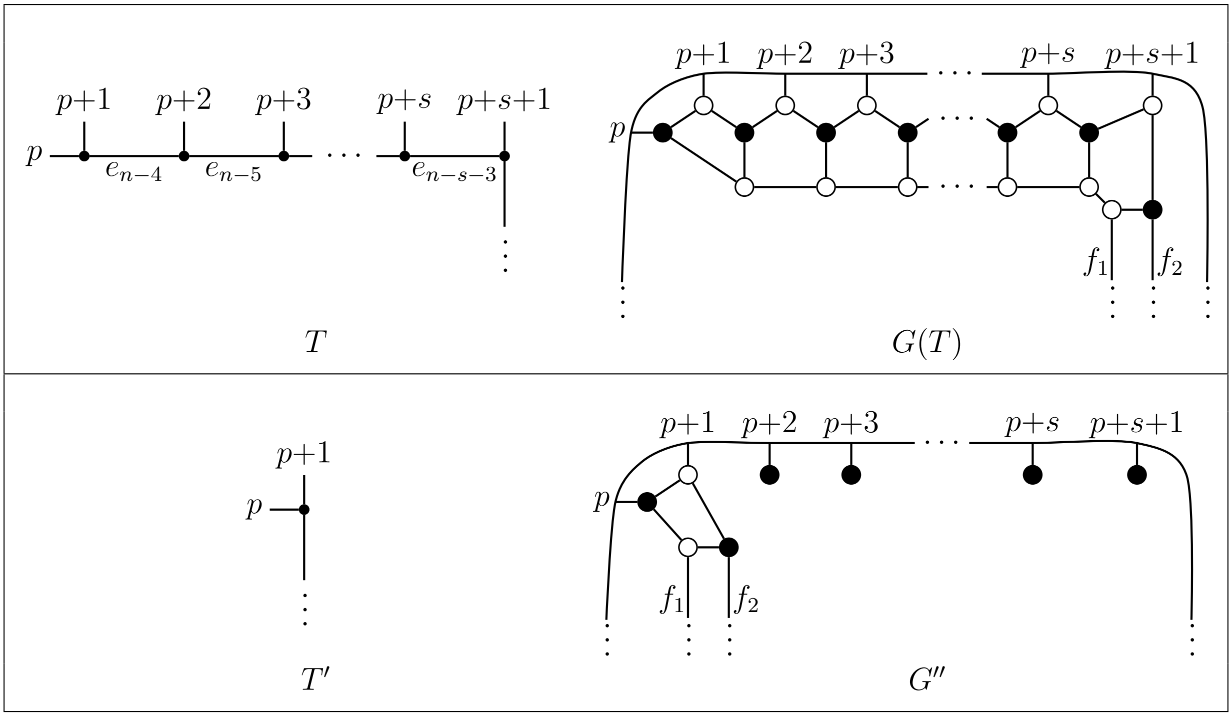


FIGURE 11. The complete binary trees T and T', and the plabic graphs $G(T)$ and G'', in Case 2 of the proof of Theorem 7.4. If $k = 0$, then the rightmost vertex shown in T and T' is the root vertex, and the edges f_1 and f_2 are incident to boundary vertices 1 and n.

are labeled by $p, p+1, \dots, p+s+1$ for some p. Proceeding in a similar manner to Case 1, we let $T' \in \mathcal{T}_{n-s,k,4}$ be obtained from T by replacing the entirety of this horizontal path and its children by a vertex incident to two leaves labeled p and $p+1$ (see Figure 11). Note that $(W_U(T'), W_L(T'))$ is obtained from $(W_U(T), W_L(T))$ by deleting the last s steps of each path (which are all horizontal). Let $D' \in \mathcal{D}_{n-s,k,4}$ be the associated $\oplus$-diagram. (See Figure 12.) By the induction hypothesis, we have $\pi_{D'} = c_{n-s}^2 \pi_{G(T')}$. (As in Case 1, we will regard permutations of $[n-s]$ as permutations of $[n]$ which fix $n-s+1, \dots, n$.)

In order to relate $\pi_{G(T)}$ and $\pi_{G(T')}$, we again introduce a plabic graph G'', obtained from $G(T')$ by inserting s lollipops in between boundary vertices $p+1$ and $p+2$, and increasing the labels of the boundary vertices $p+2, p+3, \dots, n-s$ of $G(T')$ by s. We have

$$\pi_{G''} = (p{+}2 \;\; p{+}3 \;\; \cdots \;\; n)^s \pi_{G(T')} (n \;\; n{-}1 \;\; \cdots \;\; p{+}2)^s,$$

and we see from Figure 11 that $\pi_{G(T)} = \pi_{G''}(p \;\; p{+}1 \;\; \cdots \;\; p{+}s{+}1)^2$. This gives

$$\begin{aligned} \pi_{G(T)} &= (p{+}2 \;\; p{+}3 \;\; \cdots \;\; n)^s \pi_{G(T')} (n \;\; n{-}1 \;\; \cdots \;\; p{+}2)^s (p \;\; p{+}1 \;\; \cdots \;\; p{+}s{+}1)^2 \\ &= (p{+}2 \;\; p{+}3 \;\; \cdots \;\; n)^s \pi_{G(T')} (n \;\; n{-}1 \;\; \cdots \;\; p)^s. \end{aligned} \tag{7.6}$$

On the other hand, by Definition 6.2, D is obtained from D' by adding s columns of all 0's; the bottom edges of these columns are labeled by $p, p+1, \dots, p+s-1$ when we label

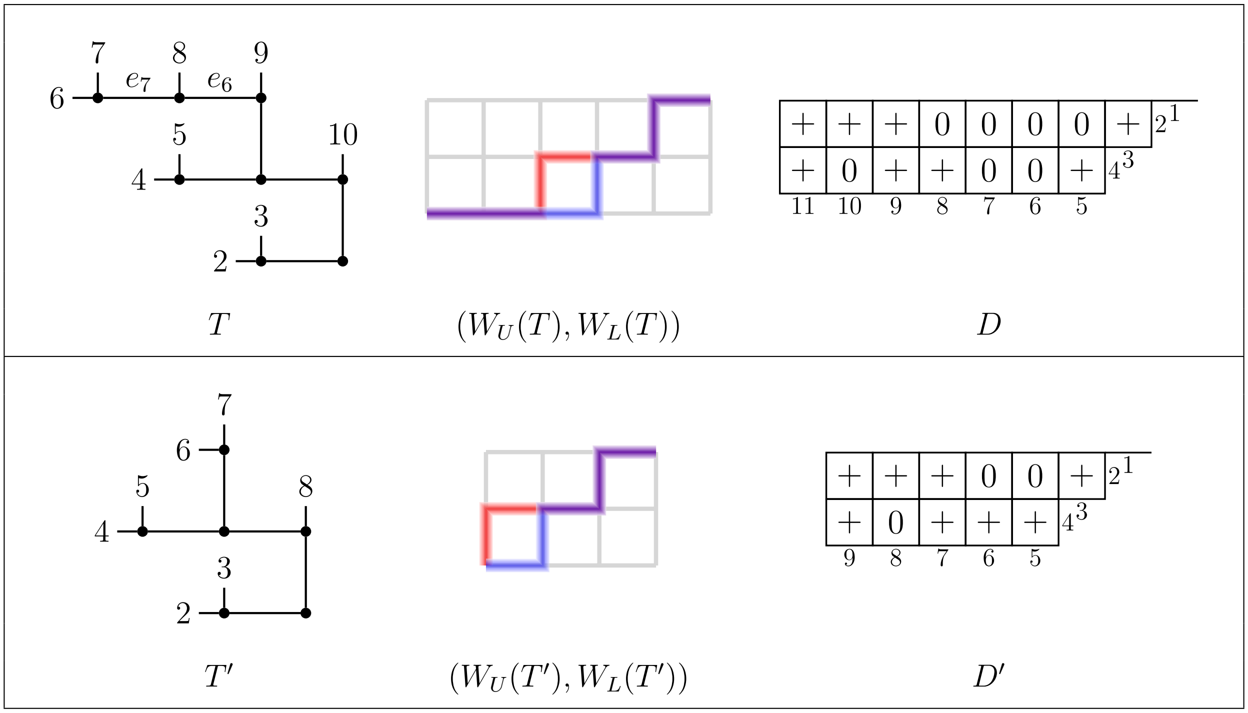


FIGURE 12. An example of Case 2 in the proof of Theorem 7.4. Here $k = 2$, $n = 11$, $p = 6$, $s = 2$.

the southeast border of D by $1, \dots, n$. Hence

$$\pi_D = (p \;\; p{+}1 \;\; \cdots \;\; n)^s \pi_{D'} (n \;\; n{-}1 \;\; \cdots \;\; p)^s.$$

Putting this together with the induction hypothesis $\pi_{D'} = c^2_{n-s}\pi_{G(T')}$ and (7.6), we obtain

$$\begin{aligned} \pi_D &= (p \;\; p{+}1 \;\; \cdots \;\; n)^s c^2_{n-s} \pi_{G(T')} (n \;\; n{-}1 \;\; \cdots \;\; p)^s \\ &= c^2_n (p{+}2 \;\; p{+}3 \;\; \cdots \;\; n)^s \pi_{G(T')} (n \;\; n{-}1 \;\; \cdots \;\; p)^s = c^2_n \pi_{G(T)}. \end{aligned}$$ ■

8. NUMBER OF CELLS IN A DECOMPOSITION OF $\mathcal{A}_{n,k,m}$ FOR ARBITRARY EVEN m

Recall from Section 4 that the $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}$ has a decomposition with $\binom{n-2}{k}$ top-dimensional cells, and from Lemma 5.6 that the (conjectural) BCFW decomposition of the $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}$ contains $\frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$ top-dimensional cells. On the other hand, when $k = 1$ the amplituhedron $\mathcal{A}_{n,1,m}$ is a cyclic polytope $C(n, m)$. Bayer [Bay93, Corollary 10] (see also [Ram97, Corollary 1.2(ii)]) showed that if m is even, any triangulation of $C(n, m)$ has exactly $\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$ top-dimensional simplices. In this section we conjecture a generalization of the above statements. We also give several families of combinatorial objects which are in bijection with the top-dimensional cells in our conjectural decomposition of $\mathcal{A}_{n,k,m}$.

For $a, b, c \in \mathbb{N}$, define

$$M(a,b,c) := \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}.$$

Note that $M(a,b,c)$ is symmetric in a, b, c.

Conjecture 8.1. *For even m, there is a cell decomposition of the amplituhedron $\mathcal{A}_{n,k,m}$, whose top-dimensional cells are the images of precisely $M(k, n-k-m, \frac{m}{2})$ cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ of dimension km.*

We give a summary of all special cases in which Conjecture 8.1 is known or conjecturally known:

special case	$M(k, n-k-m, \frac{m}{2})$	**explanation**
$m = 0$	1	$\mathcal{A}$ is a point
$m = 2$	$\binom{n-2}{k}$	[AHTT, Section 7][12]
$m = 4$	$\frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$	conjectured in [AHT14]
$k = 0$	1	$\mathcal{A}$ is a point
$k = n-m$	1	$\mathcal{A} \cong \mathrm{Gr}_{k,n}^{\geq 0}$
$k = 1$	$\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$	$\mathcal{A} \cong$ cyclic polytope $C(n,m)$

Remark 8.2. Conjecture 8.1 only deals with the case of even m. For odd m, it is possible that a decomposition of $\mathcal{A}_{n+1,k,m+1}$ with $M(k, n-k-m, \frac{m+1}{2})$ top-dimensional cells could be used to give a decomposition of $\mathcal{A}_{n,k,m}$ with the same number of top-dimensional cells. This is the case when $m = 1$, as two of us showed in [KW17]. It is also the case when $k = 1$, since for odd m there is a triangulation of the cyclic polytope $C(n,m)$ with $\binom{n-\frac{m+1}{2}}{\frac{m+1}{2}}$ top-dimensional simplices [Ram97, Corollary 1.2(ii)]. When $m = 3$, we came up with a natural construction of $M(k, n-k-3, 2)$ cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ whose images we had hoped would give a decomposition of $\mathcal{A}_{n,k,3}$. However, these images inside $\mathcal{A}_{n,k,3}$ are not disjoint (see Section 12). We mention that even in the case of cyclic polytopes, triangulations are not as well behaved in odd dimension: for even m every triangulation of $C(n,m)$ has $\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$ top-dimensional simplices, while for odd m the number of top-dimensional simplices in a triangulation can lie anywhere between $\binom{n-1-\frac{m+1}{2}}{\frac{m-1}{2}}$ and $\binom{n-\frac{m+1}{2}}{\frac{m+1}{2}}$ [Ram97, Corollary 1.2(ii)].

The symmetry of $M(a,b,c)$ raises the following question.

[12]Arkani-Hamed, Thomas, and Trnka show that the $\binom{n-2}{k}$ top-dimensional cells in $\mathcal{A}_{n,k,2}$ are disjoint and cover a dense subset of the amplituhedron. It is not known whether this induces a cell decomposition.

Question 8.3. *Conjecture 8.1 suggests that there is a symmetry for amplituhedra $\mathcal{A}_{n,k,m}$ among the parameters k, $n-k-m$, and $\frac{m}{2}$. Is there an explanation for this symmetry?*

In the case $m = 4$, the symmetry between k and $n-k-m$ comes from the well-known *parity* of the scattering amplitude (see [AHTT, Section 11]). Below, we give a combinatorial explanation of this symmetry in terms of complete binary trees and decorated permutations. The possible symmetries between $\frac{m}{2}$ and the other two parameters are completely mysterious.

Proposition 8.4. *Let $c_n := (n\ \ n{-}1\ \ \cdots\ \ 2\ \ 1)$ be the long cycle in the symmetric group on $[n]$, and w_n the permutation given by $w_n(i) := n+1-i$ for $i \in [n]$.*
(i) Given $T \in \mathcal{T}_{n,k,4}$, let $T' \in \mathcal{T}_{n,n-k-4,4}$ be obtained by reflecting T through a line of slope -1 (i.e. by switching the roles of horizontal and vertical children). Then $\pi_{G(T')} = w_n \pi_{G(T)} w_n$.
(ii) The map $\pi \mapsto c_n^4 w_n \pi w_n$ is an involution from $\Pi_{n,k,4}$ to $\Pi_{n,n-k-4,4}$.

Proof. By Definition 5.8, $G(T')$ is obtained from $G(T)$ by reflecting the plabic graph, interchanging black and white vertices, and reversing the order of the ground set. Reflecting and interchanging colors individually invert the decorated permutation (by Definition 2.9), and reversing the order of the ground set corresponds to conjugating by w_n. This proves part (i). Then using the fact that $w_n c_n w_n = c_n^{-1}$, we get $c_n^2 \pi_{G(T')} = c_n^4 w_n (c_n^2 \pi_{G(T)}) w_n$. Hence by Lemma 5.9, the involution $T \mapsto T'$ from $\mathcal{T}_{n,k,4}$ to $\mathcal{T}_{n,n-k-4,4}$, which sends each tree to its reflection, induces the involution $\pi \mapsto c_n^4 w_n \pi w_n$ from $\Pi_{n,k,4}$ to $\Pi_{n,n-k-4,4}$. ■

Recall from Section 4 that $\Pi_{n,k,2}$ is a set of decorated permutations which give a decomposition of $\mathcal{A}_{n,k,2}$. We can verify that the analogue of Proposition 8.4(ii) holds for $m = 2$, i.e. $\pi \mapsto c_n^2 w_n \pi w_n$ is an involution from $\Pi_{n,k,2}$ to $\Pi_{n,n-k-2,2}$. This motivates the following question.

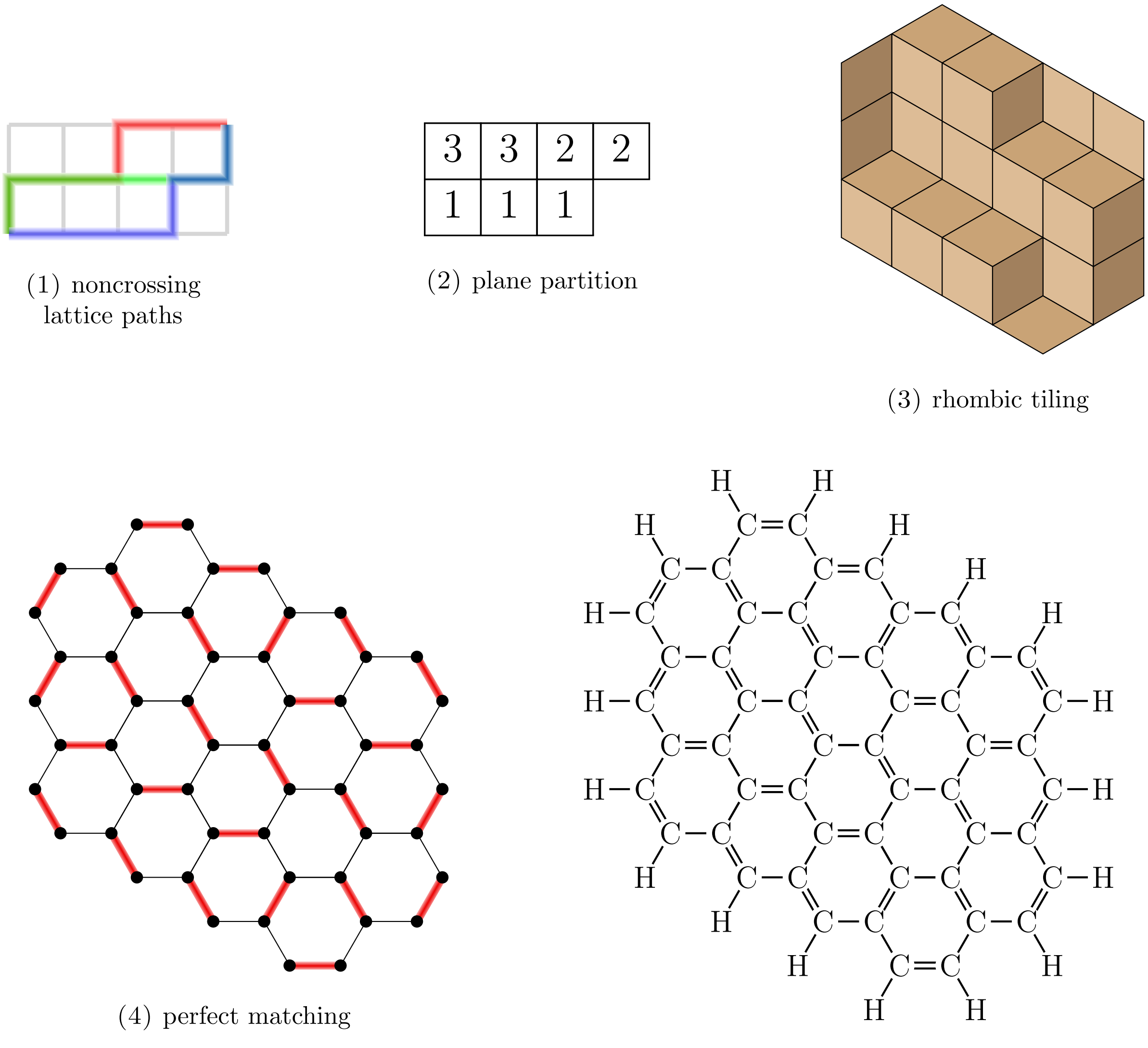
Question 8.5. *Fix n and m with m even, and assume for each k we can find a collection $\Pi_{n,k,m}$ of decorated permutations corresponding to km-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$, whose images induce a cell decomposition of $\mathcal{A}_{n,k,m}(Z)$. Can we choose $\Pi_{n,k,m}$ so that the map $\pi \mapsto c_n^m w_n \pi w_n$ is an involution from $\Pi_{n,k,m}$ to $\Pi_{n,n-k-m,m}$?*

We do not expect Question 8.5 has a positive answer if m is odd. Indeed, if we let $\Pi_{n,k,1}$ be the set of decorated permutations corresponding to the $m = 1$ BCFW cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ defined in [KW17], then $\pi \mapsto c_n w_n \pi w_n$ does not in general take $\Pi_{n,k,1}$ to $\Pi_{n,n-k-1,1}$. (For example, $\Pi_{3,1,1} = \{(1\ \ 2), (2\ \ 3)\}$, and $c_3 w_3 (1\ \ 2) w_3 = 321 \notin \Pi_{3,1,1}$.)

8.1. **Combinatorial and geometric interpretations of $M(a,b,c)$.** The number $M(a,b,c)$ has many interpretations which appear in the literature. We present some of them below. We refer to the article of Propp [Pro99] for further background on this subject.

Proposition 8.6 ([Mac16, Sections IX and X]). *Fix $a, b, c \in \mathbb{N}$. Then $M(a,b,c)$ equals the number of the following objects (see Figure 13):*

(1) collections of precisely c noncrossing lattice paths inside an $a \times b$ rectangle;
(2) plane partitions *which fit inside an $a \times b \times c$ box;*



(1) noncrossing lattice paths

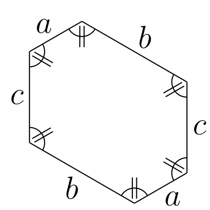
(2) plane partition

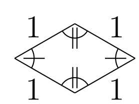
(3) rhombic tiling

(4) perfect matching

(5) Kekulé structure

FIGURE 13. The objects appearing in Proposition 8.6. Here $(a, b, c) = (2, 4, 3)$.

(3) *tilings of the hexagon* $H(a, b, c) :=$ 

by rhombi of the form 

;

(4) *perfect matchings of the honeycomb lattice* $\mathcal{O}(a, b, c)$ *defined below;*

(5) Kekulé structures *of a hexagon-shaped benzenoid with parameters* a, b, c.

We briefly define the objects in Proposition 8.6 and describe bijections between them, with reference to Figure 13. Noncrossing lattice paths (1) were defined in Definition 6.1. A *plane* *partition* (2) is a filling of the boxes of a Young diagram Y with positive integers, such the numbers along each row (from left to right) and along each column (from top to bottom) are weakly decreasing. We get from a collection of noncrossing lattice paths (1) to a plane partition (2) by taking the Young diagram whose southeast border is the bottom lattice path, and writing in each box the number of lattice paths passing below it. We can depict a plane partition as a stacking of unit cubes in the nonnegative orthant of $\mathbb{R}^3$ (3), by stacking d unit cubes on top of each box of Y filled with a d. We are considering stackings of cubes contained in an $a \times b \times c$ box, i.e. such that Y is contained inside an $a \times b$ rectangle and the entries in its boxes are bounded above by c.[13] From a plane partition regarded as a stacking of cubes inside an $a \times b \times c$ box, we get a rhombic tiling of $H(a, b, c)$ (3) by orthogonally projecting the exterior surface of the stacking onto a symmetric affine plane.

We define the *honeycomb lattice* $\mathcal{O}(a, b, c)$ as the graph dual to the tiling of $H(a, b, c)$ by unit equilateral triangles; that is, the vertices of $\mathcal{O}(a, b, c)$ correspond to the triangles tiling $H(a, b, c)$, and two vertices of $\mathcal{O}(a, b, c)$ are adjacent precisely when the corresponding triangles share an edge. Alternatively, we obtain $\mathcal{O}(a, b, c)$ by gluing together regular hexagons into a hexagonal arrangement, as shown in (4). A *perfect matching* of a graph is a subset of its edges which meets every vertex exactly once. From a rhombic tiling of $H(a, b, c)$ (3), we obtain a perfect matching of $\mathcal{O}(a, b, c)$ (4) by including an edge in the matching if and only if the corresponding equilateral triangles in $H(a, b, c)$ are covered by the same rhombus.

Finally, from $\mathcal{O}(a, b, c)$, we obtain a *hexagon-shaped benzenoid* (with parameters a, b, c) by replacing each vertex by a carbon atom, and each edge by a bond between carbon atoms. Moreover, every carbon atom should be bonded to exactly three other atoms, so for each carbon atom bonded to only two others, we add a hydrogen atom bonded to it (by a single bond). A *Kekulé structure* of such a benzenoid specifies whether each bond between carbon atoms is a single bond or a double bond, subject to the condition that each carbon atom is tetravalent, i.e. it participates in exactly one double bond. (The tetravalency condition was first proposed by Kekulé [Kek65, Kek66].) We have a bijection from perfect matchings (4) to Kekulé structures (5), which sends matched edges to double bonds and unmatched edges to single bonds.[14]

Remark 8.7. We note that there is a natural structure of *distributive lattice* on the various combinatorial objects enumerated by $M(a, b, c)$ [Pro, Theorem 2 and Example 2.1]. In terms of stackings of cubes inside an $a \times b \times c$ box, a cover relation in the lattice structure corresponds to adding a single unit cube. It would be interesting to explore what this distributive lattice

[13] MacMahon was the first to enumerate any of the objects in Proposition 8.6, by showing that the number of plane partitions which fit inside an $a \times b \times c$ box equals $M(a, b, c)$ [Mac16, Sections IX and X].

[14] In general, a *Kekulé structure* corresponds to a perfect matching of any graph formed by gluing together regular hexagons, not necessarily of $\mathcal{O}(a, b, c)$. Independently of work on plane partitions, the chemists Gordon and Davison [GD52] gave bijections between Kekulé structures (5), perfect matchings (4), and collections of noncrossing lattice paths (1). (We thank Greg Kuperberg for bringing this to our attention.) They also state a formula suggested to them by Everett for the number of perfect matchings of $\mathcal{O}(a, b, b)$, and say that it is "a special case of a more general equation established by Mr. M. Woodger" in a forthcoming paper. Unfortunately, Woodger's work was never published. In later work on Kekulé structures, Cyvin [Cyv86, (8)] rediscovered MacMahon's formula for the number of perfect matchings of $\mathcal{O}(a, b, c)$, and it was reproven in [BGCT88]. We refer to [CG88, KBT02] for more on Kekulé structures.

structure tells us about the relative position of the corresponding cells in a decomposition of the amplituhedron. Perhaps cover relations in the distributive lattice are related to whether the corresponding cells are adjacent in the amplituhedron.

Remark 8.8. In addition to the combinatorial interpretations given in Proposition 8.6, $M(a,b,c)$ also equals the dimension of the degree c component of the homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}_{a,a+b}]$ [Hod43]. So, the number of top-dimensional cells in the (conjectured) BCFW decomposition of $\mathcal{A}_{n,k,4}$ is equal to the dimension of the degree 2 part of $\mathbb{C}[\mathrm{Gr}_{k,n-4}]$. It would be interesting to give a geometric explanation of this statement.

9. Disjointness for BCFW cells when $k = 1$

For completeness, and as a further warmup to proving that the images of the $k = 2$ BCFW cells are disjoint in the amplituhedron (Section 11), we prove disjointness in the case $k = 1$. This follows from the work of Rambau on triangulations of cyclic polytopes [Ram97]: any amplituhedron $\mathcal{A}_{n,1,4}(Z)$ is a 4-dimensional cyclic polytope with n vertices [Stu88], and the triangulation given by the BCFW recursion is among of the ones identified by Rambau. We give his argument below, rephrased in the language of sign variation and dominoes.

When $k = 1$, the BCFW cells have the following explicit description, which we can verify from Theorem 6.3 (see also [AHT14, (5.2)]).

Lemma 9.1. *For $k = 1$, the positroids (see Definition 2.1) of the (k,n)-BCFW cells are precisely $M = \{\{i\},\{i+1\},\{j\},\{j+1\},\{n\}\}$ for all $i,j \in [n-2]$ with $i+1 < j$.*

Then Conjecture 5.5 when $k = 1$ says that every 4-dimensional cyclic polytope with n vertices is triangulated by the simplices whose vertex sets are of the form $\{i, i+1, j, j+1, n\}$. This is a special case of [Ram97, Theorem 4.2][15].

Proposition 9.2 ([Ram97])**.** *Let $Z \in \mathrm{Mat}_{5,n}^{>0}$. Then $\tilde{Z}$ maps the $(1,n)$-BCFW cells $\mathcal{C}_{n,1,4}$ injectively into the amplituhedron $\mathcal{A}_{n,1,4}(Z)$, and their images are pairwise disjoint.*

Proof (cf. [Ram97, Remark 3.8])**.** Let $V, V' \in \mathrm{Gr}_{1,n}^{\geq 0}$ be subspaces each contained in a $(1,n)$-BCFW cell, such that $\tilde{Z}(V) = \tilde{Z}(V')$. We must show that $V = V'$. Let $v \in \mathbb{R}^n$ be a basis vector of V, and $v' \in V'$ its matching vector as in Lemma 3.4. By Lemma 9.1, we can write $v - v'$ as a sum of 5 or fewer dominoes (recall Definition 4.9). Hence $\mathrm{var}(v - v') \leq 4$ by Lemma 4.11, whence $v = v'$ by Lemma 3.4. ■

Remark 9.3. This argument generalizes to all m, to show that the images in the amplituhedron $\mathcal{A}_{n,1,m}(Z)$ of certain m-dimensional cells of $\mathrm{Gr}_{1,n}^{\geq 0}$ are mutually disjoint. When m is even, these cells are indexed by the collection of positroids of the form $\{\{i_1\},\{i_1+1\},\dots,\{i_{m/2}\},\{i_{m/2}+1\},\{n\}\}$. When m is odd, we can take the collection of positroids of the form $\{\{i_1\},\{i_1+1\},\dots,\{i_{(m+1)/2}\},\{i_{(m+1)/2}+1\}\}$, or alternatively the collection of positroids of the form $\{\{1\},\{i_1\},\{i_1+1\},\dots,\{i_{(m-1)/2}\},\{i_{(m-1)/2}+1\},\{n\}\}$. These in fact give triangulations of $\mathcal{A}_{n,1,m}(Z)$ by [Ram97, Theorem 4.2].

[15] In more detail, the BCFW triangulation is an iterated extension (in the sense of [Ram97, Definition 4.1]) of the triangulation of a 1-dimensional cyclic polytope into intervals between consecutive vertices.

10. Domino bases for BCFW cells when $k = 2$

In this section we give a basis classification of the BCFW cells $\mathcal{C}_{n,2,4}$ (the case $k = 2$) in terms of domino bases. We propose a generalization to all k in the appendix (Conjecture A.7). We recall the definition of a domino (Definition 4.9), and introduce some new terminology.

Definition 10.1. We say that $d \in \mathbb{R}^n \setminus \{0\}$ is a *domino* if there exists $i \in [n]$ such that $d_j = 0$ for all $j \neq i, i+1$, and d_i and d_{i+1} are nonzero and have the same sign. (If $i = n$, then we require that $d_j = 0$ for all $j < n$, and that d_n is nonzero.) In this case, we call d an *i-domino*, we call i the *index* of d, and we call the common sign of d_i and d_{i+1} the *sign* of d. We also regard the zero vector as a domino, with sign zero.

For example, $(0, -1, -2, 0, 0) \in \mathbb{R}^5$ is a negative 2-domino.

Definition 10.2. Given $v \in \mathbb{R}^n$ with $n \geq 1$, let $\overline{v} \in \mathbb{R}^n$ denote the vector obtained from v by setting coordinate n to 0. For $v \in \mathbb{R}^n$, we say that v is *orthodox* if $\overline{v}$ is a sum of two nonzero dominoes of the same sign with disjoint support, and *deviant* if $\overline{v}$ is a difference of two such nonzero dominoes.

Theorem 10.3. *Each BCFW cell S of $\mathrm{Gr}_{2,n}^{\geq 0}$ falls into one of 9 classes, shown in Figure 14. Any element $V \in S$ can be written as the row span of a $2 \times n$ matrix with rows d and e, whose sign patterns are specified precisely in Figure 14. In the figure, a vertical line represents a (possibly empty) block of 0's. We call d and e the* standard basis vectors *of V. Note that d is either orthodox or deviant; we call V either* orthodox *or* deviant*, accordingly.*

Moreover, we can write the row vectors d and e in terms of linearly independent positive dominoes $d^{(1)}, d^{(2)}, d^{(3)}, d^{(4)} \in \mathbb{R}^n$, such that the following holds (where i_j is the index of $d^{(j)}$):

- *if V is orthodox, then $i_1 + 1 < i_2 \leq i_3 < i_4 - 1$, $\overline{d} = d^{(1)} + d^{(2)}$ with $d_n < 0$, and $\overline{e} = d^{(3)} + d^{(4)}$ with $e_n > 0$;*
- *if V is deviant, then $i_1 + 1 < i_2 + 1 < i_3 \leq i_4$, $\overline{d} = d^{(1)} - d^{(4)}$ with $d_n < 0$, and $e = d^{(1)} + d^{(2)} + d^{(3)}$ (with $e_n = 0$).*

We call $d^{(1)}, d^{(2)}, d^{(3)}, d^{(4)}$ the fundamental dominoes *of V.*

For example, the matrix $\begin{bmatrix} 0 & 0 & 3 & 7 & 0 & 1 & 2 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 & 2 & 0 & 2 \end{bmatrix}$ represents an orthodox element of $\mathrm{Gr}_{2,11}^{\geq 0}$ in a Class 3 BCFW cell of Figure 14.

Proof. By Theorem 6.3, the BCFW cells $\mathcal{C}_{n,2,4}$ of $\mathrm{Gr}_{2,n}^{\geq 0}$ correspond to the $\oplus$-diagrams $\mathcal{D}_{n,2,4}$, and a straightforward case analysis using Definition 6.2 implies that these are precisely the $\oplus$-diagrams in Figure 14. It remains to show that for each such $\oplus$-diagram, an arbitrary element of the corresponding positroid cell can be represented by the $2 \times n$ matrix to its right in Figure 14, and that we can find dominoes $d^{(1)}, d^{(2)}, d^{(3)}, d^{(4)}$ satisfying the desired properties. We can ignore the black vertical bars in the $\oplus$-diagrams, which correspond exactly to the vertical bars in the matrices. Since the proofs for each of the 9 classes are similar, we work out the details in two representative cases (Classes 3 and 8), and leave the others as an exercise. The idea is to use Ⅎ-moves (Lemma 2.6) to turn each $\oplus$-diagram into a Ⅎ-diagram, from which we obtain a hook diagram (Definition 2.8) and matrix parameterization of the corresponding cell (Theorem 2.11). From here we can verify the required properties.

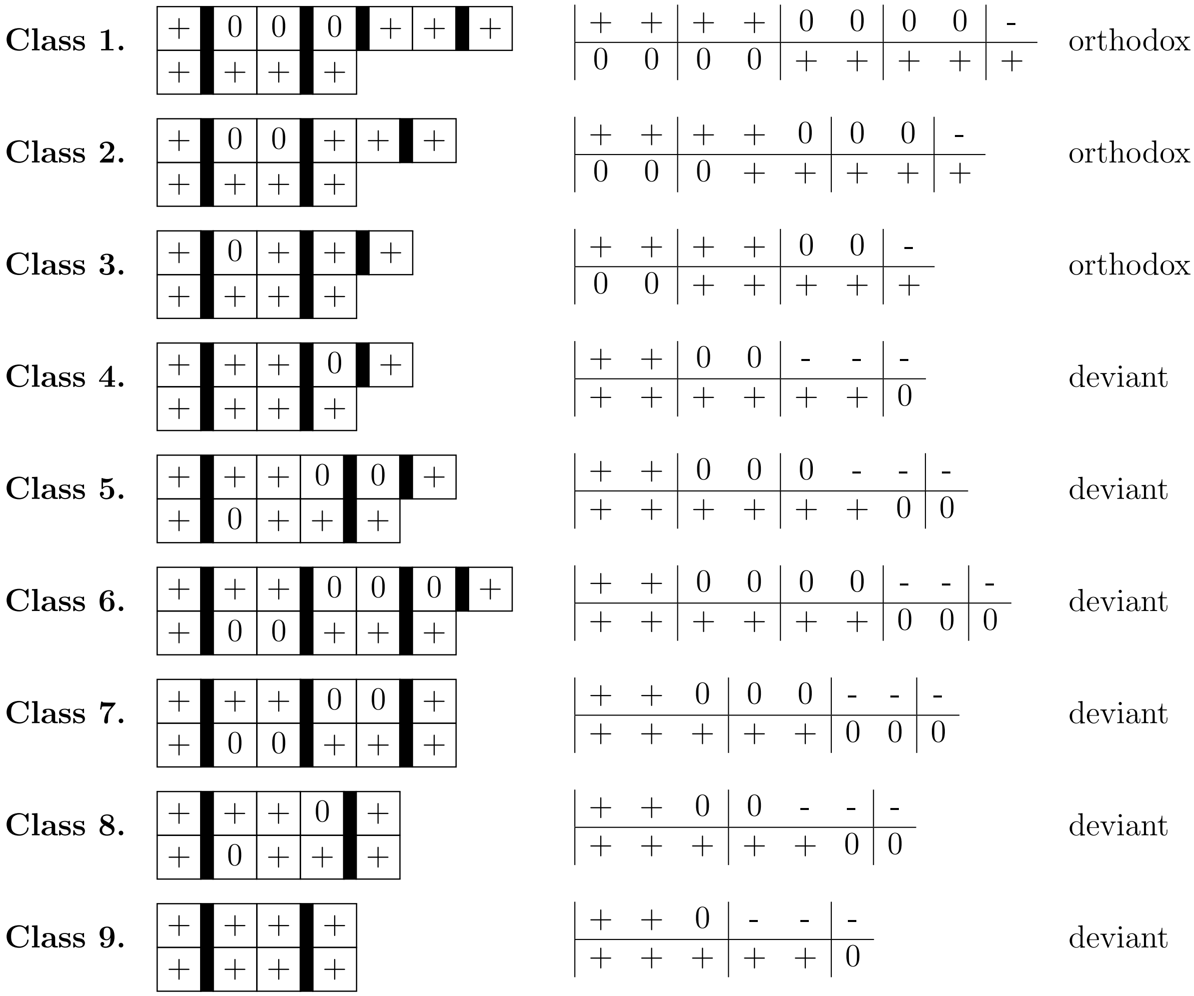
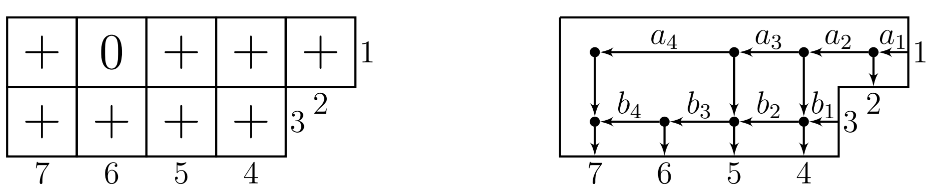


FIGURE 14. The $\oplus$-diagrams and standard basis vectors for the 9 classes of BCFW cells for $k = 2$. The black vertical bars in the $\oplus$-diagrams represent a (possibly empty) block of 0's of the appropriate height. Similarly, the vertical bars in the matrices represent a (possibly empty) block of 0's.

Class 3. In this case the $\oplus$-diagram is a Ⅎ-diagram, with the following hook diagram:



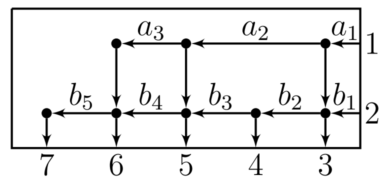
By Theorem 2.11, an arbitrary element of the corresponding cell of $\mathrm{Gr}_{2,7}^{\geq 0}$ is represented by

$$\begin{bmatrix} 1 & a_1 & 0 & -a_1a_2 & -a_1a_2(a_3+b_2) & -a_1a_2b_3(a_3+b_2) & -a_1a_2(a_3a_4+b_3b_4(a_3+b_2)) \\ 0 & 0 & 1 & b_1 & b_1b_2 & b_1b_2b_3 & b_1b_2b_3b_4 \end{bmatrix},$$

where the a_i's and b_i's are positive real numbers. Let $v, w \in \mathbb{R}^7$ denote the first and second rows of this matrix. If we replace v by $v' := v + \frac{a_1 a_2 (a_3+b_2)}{b_1 b_2} w$, then we obtain a matrix whose sign pattern matches the matrix in Figure 14. Therefore any element V in this cell has basis vectors $d := v'$ and $e := w$, and we can write $\overline{d} = d^{(1)} + d^{(2)}$ and $\overline{e} = d^{(3)} + d^{(4)}$ for some positive dominoes $d^{(1)}, d^{(2)}, d^{(3)}, d^{(4)}$. We can check that their indices satisfy the stated inequalities, and they are linearly independent (the fact that $\Delta_{\{3,4\}} \neq 0$ implies that $d^{(2)}$ and $d^{(3)}$ are linearly independent).

Class 8. After applying a ⅃-move to the $\oplus$-diagram, we obtain the following ⅃-diagram and hook diagram:

0	+	+	0	+	1
+	+	+	+	+	2
7	6	5	4	3	



.

Therefore an arbitrary element of the corresponding cell of $\mathrm{Gr}_{2,7}^{\geq 0}$ is represented by

$$\begin{bmatrix} 1 & 0 & -a_1 & -a_1 b_2 & -a_1(a_2 + b_2 b_3) & -a_1(a_2(a_3+b_4) + b_2 b_3 b_4) & -a_1 b_5 (a_2(a_3+b_4) + b_2 b_3 b_4) \\ 0 & 1 & b_1 & b_1 b_2 & b_1 b_2 b_3 & b_1 b_2 b_3 b_4 & b_1 b_2 b_3 b_4 b_5 \end{bmatrix},$$

where the a_i's and b_i's are positive real numbers. Let $v, w \in \mathbb{R}^7$ denote the first and second rows of this matrix. If we replace v by $v' := v + \frac{a_1}{b_1} w$ and then w by $w' := \frac{a_1 a_2 (a_3+b_4)}{b_1 b_2 b_3 b_4} w + v'$, we obtain a matrix whose sign pattern matches the matrix in Figure 14. Therefore any element V in this cell has basis vectors $d := v'$ and $e := w'$, and we can write $\overline{d} = d^{(1)} - d^{(4)}$ and $\overline{e} = d^{(1)} + d^{(2)} + d^{(3)}$ for some positive dominoes $d^{(1)}, d^{(2)}, d^{(3)}, d^{(4)}$. We can again check that their indices satisfy the stated inequalities and that they are linearly independent. ■

Definition 10.4. Let $V \in \mathrm{Gr}_{2,n}^{\geq 0}$ come from a BCFW cell, with fundamental dominoes $d^{(1)}, d^{(2)}, d^{(3)}, d^{(4)}$ from Theorem 10.3. Note that for any $v \in V$, we can write $\overline{v} = \sum_{j=1}^{4} \alpha_j d^{(j)}$ for some unique $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$. We let $\mathrm{Dom}_V(v)$ denote the sequence of dominoes $(\alpha_1 d^{(1)}, \alpha_2 d^{(2)}, \alpha_3 d^{(3)}, \alpha_4 d^{(4)})$, and $\mathrm{dom}_V(v) \in \{0, +, -\}^4$ the sign vector of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

We state two corollaries of Theorem 10.3 that we will use in Section 11.

Corollary 10.5. *Suppose that $V \in \mathrm{Gr}_{2,n}^{\geq 0}$ comes from a BCFW cell, and $v \in V$ such that $\mathrm{dom}_V(v)$ has no zero components. Then we have the following:*

- *if V is orthodox, $\mathrm{dom}_V(v)$ equals $\pm(+,+,+,+)$ or $\pm(+,+,-,-)$;*
- *if V is deviant, $\mathrm{dom}_V(v)$ equals $\pm(+,+,+,-)$ or $\pm(-,+,+,+)$ or $\pm(+,+,+,+)$.*

Corollary 10.6. *Suppose that $V \in \mathrm{Gr}_{2,n}^{\geq 0}$ comes from a BCFW cell with standard basis vectors $d, e \in \mathbb{R}^n$ from Theorem 10.3, and $v \in V$ such that $\overline{v}$ has support size at most 4 (in particular, this includes orthodox and deviant v). Then we have the following:*

- *if V is orthodox, then v is a scalar multiple of d or e;*
- *if V is deviant, then v is a scalar multiple of d or $d - e$.*

11. Disjointness for BCFW cells when $k = 2$

This section is devoted to the proof of the following result.

Theorem 11.1. *For $m = 4$, $\tilde{Z}$ maps the BCFW cells $\mathcal{C}_{n,2,4}$ of $\mathrm{Gr}_{2,n}^{\geq 0}$ injectively into the amplituhedron $\mathcal{A}_{n,2,4}(Z)$, and their images are pairwise disjoint.*

11.1. Lemmas on dominoes.

We begin by proving some useful results on dominoes. We already have Lemma 4.11, but we will need more powerful tools.

Definition 11.2. Let $D \subseteq \mathbb{R}^n$ be a finite multiset of dominoes, and $v \in \mathbb{R}^n$ the sum of the dominoes in D. Given $I = \{i_1 < \cdots < i_k\} \subseteq [n]$, an *$I$-alternating domino sequence for v* (with respect to D) is a sequence $(d^{(1)}, \dots, d^{(k)})$ of distinct nonzero dominoes in D such that

- $d^{(j)}_{i_j}$ has the same sign as v_{i_j} for all $j \in [k]$;
- the *sign sequence* $(\mathrm{sign}(d^{(1)}), \dots, \mathrm{sign}(d^{(l)})) \in \{+, -\}^l$ alternates in sign.

We call k the *length* of $(d^{(1)}, \dots, d^{(k)})$.

Lemma 11.3. *Suppose that $D \subseteq \mathbb{R}^n$ is a finite multiset of dominoes, $v \in \mathbb{R}^n$ is the sum of the dominoes in D, and $I \subseteq [n]$. Then v alternates in sign restricted to I if and only if v has an I-alternating domino sequence. The indices of dominoes in any I-alternating domino sequence for v are weakly increasing, with each index appearing at most twice.*

We think of an I-alternating domino sequence $(d^{(1)}, \dots, d^{(k)})$ for v as a witness for the corresponding alternating subsequence of v.

Example 11.4. Let D be the set of dominoes $d^{(1)} := (1, 1, 0, 0, 0)$, $d^{(2)} := (0, -3, -1, 0, 0)$, $d^{(3)} := (0, 1, 2, 0, 0)$, $d^{(4)} := (0, 0, 0, -2, -4)$, and let $v = (1, -1, 1, -2, -4)$ be their sum. Note that v alternates in sign restricted to $I := \{1, 2, 3, 5\}$. The unique corresponding I-alternating domino sequence is $(d^{(1)}, d^{(2)}, d^{(3)}, d^{(4)})$, with corresponding indices $1, 2, 2, 4$. ◊

Proof. Write $I = \{i_1 < \cdots < i_k\}$.

($\Rightarrow$): Suppose that v alternates in sign on I. Then for each $j \in [k]$, we can find $d^{(j)} \in D$ such that $d^{(j)}_{i_j}$ is nonzero and has the same sign as v_{i_j}. Letting a_j be the index of $d^{(j)}$ for $j \in [k]$, we have $a_j \in \{i_j, i_j - 1\}$, so $a_j < a_{j'}$ for $j' \geq j + 2$. Moreover, $d^{(j)}$ and $d^{(j+1)}$ have opposite signs for $j \in [k-1]$, so $d^{(1)}, \dots, d^{(k)}$ are all distinct. Hence $(d^{(1)}, \dots, d^{(k)})$ is an I-alternating domino sequence for v.

($\Leftarrow$): Suppose that $(d^{(1)}, \dots, d^{(k)})$ is an I-alternating domino sequence for v, with corresponding indices $a_1, \dots, a_k$. Then $\mathrm{sign}(v|_I)$ equals the sign sequence of $(d^{(1)}, \dots, d^{(k)})$, which alternates in sign. Moreover, since $a_j \in \{i_j, i_j - 1\}$ for all $j \in [k]$, we have $a_1 \leq \cdots \leq a_k$ and $a_j < a'_j$ for $j' \geq j + 2$. Since $d^{(j)}$ and $d^{(j+1)}$ have opposite signs for $j \in [k-1]$, this also shows that $d^{(1)}, \dots, d^{(k)}$ are all distinct. ∎

Lemma 11.5. *Suppose that $(d^{(1)}, \dots, d^{(k)})$ is an I-alternating domino sequence for $v \in \mathbb{R}^n$, where $I = \{i_1 < \cdots < i_k\}$. Choose $j' \in [k]$ and any subset $J \subseteq [k] \setminus \{j'\}$. Then the vector $v - \sum_{j \in J} d^{(j)}$ has the same sign in coordinate $i_{j'}$ as the domino $d^{(j')}$.*

Proof. We know from Definition 11.2 that $v_{i_{j'}}$ has the same sign as $d^{(j')}$. It now suffices to observe that for all $j \in J$, $d^{(j)}_{i_{j'}}$ is either zero or has the opposite sign as $d^{(j')}$. ■

We will also use the following facts about dominoes and sign variation, which are straightforward to verify.

Lemma 11.6. *Suppose that $v \in \mathbb{R}^n$, and $d \in \mathbb{R}^n$ is an i-domino.*
(i) We have $\operatorname{var}(v+d) \le \operatorname{var}(v) + 2$.
(ii) If $v_j = 0$ for all $j \le i$ or $v_j = 0$ for all $j > i$, then $\operatorname{var}(v+d) \le \operatorname{var}(v) + 1$.

Definition 11.7. A *shuffle* of two sequences $(s_1, \dots, s_k)$ and $(t_1, \dots, t_l)$ is a sequence of length $k+l$ formed by permuting $s_1, \dots, s_k, t_1, \dots, t_l$, such that $s_1, \dots, s_k$ appear in the same relative order, and $t_1, \dots, t_l$ appear in the same relative order.

For example, the shuffles of (a, b) and (c, d) are precisely (a, b, c, d), (a, c, b, d), (a, c, d, b), (c, a, b, d), (c, a, d, b), and (c, d, a, b).

Lemma 11.8. *Suppose that $V, V' \in \mathrm{Gr}_{2,n}^{\ge 0}$ come from BCFW cells, and $v \in V$, $v' \in V$. Note that $\overline{v+v'}$ is the sum of the dominoes in the multiset D of nonzero dominoes appearing in* $\mathrm{Dom}_V(v)$ *and* $\mathrm{Dom}_{V'}(v')$. *Then any alternating domino sequence for $\overline{v+v'}$ with respect to D in which all the dominoes in D appear is obtained by shuffling* $\mathrm{Dom}_V(v)$ *and* $\mathrm{Dom}_V(v')$ *and deleting all zero dominoes.*

Proof. Suppose that $(f^{(1)}, \dots, f^{(l)})$ is an I-alternating domino sequence for $\overline{v+v'}$ with respect to D which uses all the dominoes in D, where $I = \{i_1 < \cdots < i_l\}$. Then the indices of $(f^{(1)}, \dots, f^{(l)})$ are weakly increasing, so we must show that if two dominoes in $\mathrm{Dom}_V(v)$ or two dominoes in $\mathrm{Dom}_{V'}(v')$ have the same index and both appear in $(f^{(1)}, \dots, f^{(l)})$, then they appear in the same relative order. It suffices to prove this for $\mathrm{Dom}_V(v)$. We proceed by contradiction and suppose that there exist dominoes $f^{(j)}, f^{(j+1)} \in D$ with the same index (they are necessarily adjacent in the domino sequence, by Lemma 11.3), but appear in $\mathrm{Dom}_V(v)$ in the opposite order, i.e. $f^{(j+1)}$ before $f^{(j)}$. We will deduce that $\operatorname{var}(v) \ge 2$, which contradicts Theorem 3.3(i).

Perhaps after multiplying all of $v, v', f^{(1)}, \dots, f^{(l)}$ by -1, we may assume that $f^{(j)}$ is negative and $f^{(j+1)}$ is positive. Note that $\overline{v}$ equals $\overline{v+v'}$ minus the sum of the dominoes appearing in $\mathrm{Dom}_{V'}(v')$. Since $(f^{(1)}, \dots, f^{(l)})$ are precisely all dominoes in D, by Lemma 11.5, $v_{i_j} < 0$ and $v_{i_{j+1}} > 0$. We now let $d^{(1)}, d^{(2)}, d^{(3)}, d^{(4)}$ be the fundamental dominoes of V, and refer to Theorem 10.3. First we suppose that V is orthodox. Since $f^{(j)}$ and $f^{(j+1)}$ have the same index, V must be of Class 3. Then $f^{(j+1)}$ is a positive scalar multiple of $d^{(2)}$, and $f^{(j)}$ is a negative scalar multiple of $d^{(3)}$. Hence $\overline{v} = \alpha(d^{(1)} + d^{(2)}) - \beta(d^{(3)} + d^{(4)})$ for some $\alpha, \beta > 0$. We see that if i is the index of $d^{(1)}$, then $i < i_j$ and $v_i > 0$, which gives $\operatorname{var}(v) \ge 2$, as desired. Instead suppose that V is deviant. Then it must be of Class 4 or 9. Then $f^{(j+1)}$ is a positive scalar multiple of $d^{(3)}$, and $f^{(j)}$ is a negative scalar multiple of $d^{(4)}$. Hence $\overline{v} = \gamma(d^{(1)} - d^{(4)}) + \delta(d^{(1)} + d^{(2)} + d^{(3)})$ for some $\gamma, \delta > 0$. Again v is positive at the index of $d^{(1)}$, giving $\operatorname{var}(v) \ge 2$. ■

Corollary 11.9. *Suppose that $V, V' \in \mathrm{Gr}_{2,n}^{\geq 0}$ come from BCFW cells, and $v \in V$, $v' \in V$ are distinct such that $Z(v) = Z(v')$ and the sum of the support sizes of $\mathrm{dom}_V(v)$ and $\mathrm{dom}_{V'}(-v')$ is at most 6, i.e. at most 6 dominoes appear with nonzero coefficient in $\mathrm{Dom}_V(v)$ and $\mathrm{Dom}_{V'}(-v')$. Then the sum of the support sizes is exactly 6, and some shuffle of $\mathrm{dom}_V(v)$ and $\mathrm{dom}_{V'}(-v')$ alternates in sign (ignoring the zero components).*

This will be useful for us, since if d, e are the standard basis vectors of V from Theorem 10.3, then $\mathrm{dom}_V(d)$ has support size 2, and also $\mathrm{dom}_V(e)$ has support size 2 if V is orthodox.

Proof. By Theorem 3.3(ii), we have $\mathrm{var}(v-v') \geq 6$, and so $\mathrm{var}(\overline{v - v'}) \geq 5$. By Lemma 11.3, $\overline{v - v'}$ has an alternating domino sequence of length 6, which necessarily uses all the (nonzero) dominoes in $\mathrm{Dom}_V(v)$ and $\mathrm{Dom}_{V'}(-v')$, since by assumption at most 6 dominoes appear. By Lemma 11.8, the sequence is obtained by shuffling $\mathrm{Dom}_V(v)$ and $\mathrm{Dom}_{V'}(-v')$ and deleting all zero dominoes. The corresponding sign sequence (which alternates in sign) is therefore obtained by shuffling $\mathrm{dom}_V(v)$ and $\mathrm{dom}_{V'}(-v')$ and deleting all zero components. ∎

11.2. **Orthodox vs. orthodox cells.**

Lemma 11.10. *Suppose that $V, V' \in \mathrm{Gr}_{2,n}^{\geq 0}$ are distinct and orthodox. Then $\tilde{Z}(V) \neq \tilde{Z}(V')$.*

Proof. Suppose otherwise that $\tilde{Z}(V) = \tilde{Z}(V')$. Let $d, e \in \mathbb{R}^n$ be the standard basis vectors of V from Theorem 10.3, which are orthodox, and take matching vectors $v, w \in V'$ for d, e, i.e. $Z(d) = Z(v)$ and $Z(e) = Z(w)$. We claim that $d = v$. Otherwise, by Corollary 11.9, some shuffle of $\mathrm{dom}_V(d)$ and $\mathrm{dom}_{V'}(-v)$ alternates in sign after deleting its two zero components, and in particular contains exactly three $+$'s and three $-$'s. But $\mathrm{dom}_V(d)$ contains two $+$'s, and by Corollary 10.5 $\mathrm{dom}_{V'}(v)$ contains an even number of $+$'s. This contradiction shows $d = v$. Similarly, $e = w$. Since d and e span V, we get $V \subseteq V'$, a contradiction. ∎

11.3. **Deviant vs. deviant cells.**

Lemma 11.11. *Suppose that $V, V' \in \mathrm{Gr}_{2,n}^{\geq 0}$ are distinct and deviant. Then $\tilde{Z}(V) \neq \tilde{Z}(V')$.*

Proof. Suppose otherwise that $\tilde{Z}(V) = \tilde{Z}(V')$. Denote the standard basis vectors and fundamental dominoes of V and V' by d, e, $d^{(1)}$, $d^{(2)}$, $d^{(3)}$, $d^{(4)}$, and d', e', ${d^{(1)}}'$, ${d^{(2)}}'$, ${d^{(3)}}'$, ${d^{(4)}}'$, respectively. Take matching vectors $v, w \in V'$ for d, e. We claim that $d = v$. Otherwise, by Corollary 11.9, some shuffle of $\mathrm{dom}_V(d)$ and $\mathrm{dom}_{V'}(-v)$ alternates in sign after deleting its two zero components, and in particular contains exactly three $+$'s and three $-$'s. But $\mathrm{dom}_V(d)$ contains one $+$, and by Corollary 10.5 $\mathrm{dom}_{V'}(v)$ contains either at least three $+$'s or at most one $+$. This contradiction shows $d = v$.

Since e has support size 3, we cannot apply Corollary 11.9 to e and w, but we can deduce that $e \neq w$, since otherwise $V = V'$. Hence $\mathrm{var}(e - w) \geq 6$ by Lemma 3.4. Since $d = v$, we may rescale our fundamental dominoes and standard basis vectors so that $d = d'$, $d^{(1)} = {d^{(1)}}'$, and $d^{(4)} = {d^{(4)}}'$. We can write $d = d' = d^{(1)} - d^{(4)} - f$ for some positive n-domino f, and $w = \alpha d' + \beta e'$ for some $\alpha, \beta \in \mathbb{R}$. Then

$$e - w = (1 - \alpha - \beta)d^{(1)} + d^{(2)} + d^{(3)} + \alpha d^{(4)} - \beta {d^{(2)}}' - \beta {d^{(3)}}' + \alpha f.$$

Since $\mathrm{var}(e-w) \ge 6$, by Lemma 11.3 we can arrange the 7 dominoes summed above into an alternating domino sequence $(f^{(1)}, \dots, f^{(7)})$. The indices of the dominoes weakly increase, so $f^{(1)} = (1-\alpha-\beta)d^{(1)}$ and $f^{(7)} = \alpha f$. Since $f^{(1)}$ and $f^{(7)}$ have the same sign, we get $\mathrm{sign}(1-\alpha-\beta) = \mathrm{sign}(\alpha)$. Hence the sign sequence of $(f^{(1)}, \dots, f^{(7)})$ is a permutation of

$$\mathrm{sign}(\alpha), +, +, \mathrm{sign}(\alpha), -\mathrm{sign}(\beta), -\mathrm{sign}(\beta), \mathrm{sign}(\alpha).$$

This sequence must contain either 3 or 4 +'s, and therefore α and β are negative. But this contradicts $\mathrm{sign}(1-\alpha-\beta) = \mathrm{sign}(\alpha)$. ∎

11.4. **Orthodox vs. deviant cells.** This is the hardest case. Until now, our strategy has been to pick a standard basis vector, look at its matching vector, and obtain a contradiction. It turns out this is not sufficient here, and we have to work harder to prove disjointness.

We start by examining the supports $\mathrm{supp}(v) := \{i \in [n] : v_i \neq 0\}$ of vectors $v \in \mathbb{R}^n$.

Lemma 11.12. *Suppose that $V, V' \in \mathrm{Gr}_{2,n}^{\ge 0}$ come from BCFW cells, and $\tilde{Z}(V) = \tilde{Z}(V')$. Then $\min_{v\in V\setminus\{0\}} \mathrm{supp}(v) = \min_{v\in V'\setminus\{0\}} \mathrm{supp}(v)$, $\max_{v\in V\setminus\{0\}} \mathrm{supp}(\overline{v}) = \max_{v\in V'\setminus\{0\}} \mathrm{supp}(\overline{v})$.*

Proof. Suppose otherwise, so that without loss of generality, we may assume that either $\min_{v\in V\setminus\{0\}} \mathrm{supp}(v) < \min_{v\in V'\setminus\{0\}} \mathrm{supp}(v)$ or $\max_{v\in V\setminus\{0\}} \mathrm{supp}(\overline{v}) > \max_{v\in V'\setminus\{0\}} \mathrm{supp}(\overline{v})$. Let d, e, $d^{(1)}$, $d^{(2)}$, $d^{(3)}$, $d^{(4)}$ be the standard basis vectors and fundamental dominoes of V. First we consider the case $\min_{v\in V\setminus\{0\}} \mathrm{supp}(v) < \min_{v\in V'\setminus\{0\}} \mathrm{supp}(v)$. Write $\overline{d} = d^{(1)} + f$, where $f := d^{(2)}$ if V is orthodox and $f := -d^{(4)}$ if V is deviant, and let $v' \in V'$ be the matching vector for d. Note that the index of $d^{(1)}$ is strictly less than $\min(\mathrm{supp}(v'))$. Hence by applying Lemma 11.6(i) to f, Lemma 11.6(ii) to $d^{(1)}$, and Theorem 3.3(i) to v', we obtain

$$\mathrm{var}(\overline{d-v'}) = \mathrm{var}(\overline{-v'} + f + d^{(1)}) \le \mathrm{var}(\overline{v'}) + 2 + 1 \le 1 + 2 + 1 = 4.$$

But we also have $d \neq v'$, so $\mathrm{var}(\overline{d-v'}) \ge \mathrm{var}(d-v') - 1 \ge 5$ by Lemma 3.4, a contradiction. We can treat the case $\max_{v\in V\setminus\{0\}} \mathrm{supp}(\overline{v}) > \max_{v\in V'\setminus\{0\}} \mathrm{supp}(\overline{v})$ by a similar argument, where if V is orthodox we replace d with e. ∎

Lemma 11.13. *Suppose that $V \in \mathrm{Gr}_{k,n}^{\ge 0}$ is orthodox and $V' \in \mathrm{Gr}_{k,n}^{\ge 0}$ is deviant with $\tilde{Z}(V) = \tilde{Z}(V')$. Then $V \cap V'$ contains no orthodox vectors.*

Proof. Suppose otherwise that there exists an orthodox $v \in V \cap V'$. Denote the standard basis vectors and fundamental dominoes of V and V' by d, e, $d^{(1)}$, $d^{(2)}$, $d^{(3)}$, $d^{(4)}$, and d', e', ${d^{(1)}}'$, ${d^{(2)}}'$, ${d^{(3)}}'$, ${d^{(4)}}'$, respectively. By Corollary 10.6, v is a scalar multiple of both $e' - d'$ and either d or e. After rescaling the vectors appropriately, we may assume that $\overline{v}$ equals both $e' - d'$ and either d or e.

Let $w \in V$ be the matching vector for d', and write $-w = \alpha d + \beta e$ for some $\alpha, \beta \in \mathbb{R}$. By Corollary 10.6 we have $w \neq d'$, so $\mathrm{var}(\overline{d'-w}) \ge 5$ by Lemma 3.4, i.e. $\overline{d'-w}$ alternates in sign on some $I = \{i_1 < \cdots < i_6\} \subseteq [n-1]$. By Lemma 11.3 and Lemma 11.8, $\overline{d'-w}$ has an I-alternating domino sequence obtained by shuffling $({d^{(1)}}', -{d^{(4)}}')$ and $(\alpha d^{(1)}, \alpha d^{(2)}, \beta d^{(3)}, \beta d^{(4)})$, which we see must equal

$$(\alpha d^{(1)}, {d^{(1)}}', \alpha d^{(2)}, \beta d^{(3)}, -{d^{(4)}}', \beta d^{(4)}),$$

with $\beta > 0$. In particular, the index of $d^{(2)}$ is strictly less than that of $d^{(4)'}$, which implies $e' - d' \neq d$. Hence $e' - d' = e$. Now let $x := (1-\beta)d' + \beta e' \in V'$, so that $\operatorname{var}(x) \leq 1$ by Theorem 3.3(i). We can write $\overline{x} = \overline{d' - w} - \alpha d^{(1)} - \alpha d^{(2)}$, so by Lemma 11.5,

$$\operatorname{sign}(x|_{\{i_2,i_4,i_5,i_6\}}) = \operatorname{sign}(d^{(1)'}, \beta d^{(3)}, -d^{(4)'}, \beta d^{(4)}) = (+,+,-,+).$$

This implies $\operatorname{var}(x) \geq 2$. ■

Lemma 11.14. *Let $V \in \operatorname{Gr}_{2,n}^{\geq 0}$ be orthodox and $V' \in \operatorname{Gr}_{2,n}^{\geq 0}$ be deviant. Then $\tilde{Z}(V) \neq \tilde{Z}(V')$.*

Proof. Suppose otherwise that $\tilde{Z}(V) = \tilde{Z}(V')$. Denote the standard basis vectors and fundamental dominoes of V and V' by d, e, $d^{(1)}$, $d^{(2)}$, $d^{(3)}$, $d^{(4)}$, and d', e', $d^{(1)'}$, $d^{(2)'}$, $d^{(3)'}$, $d^{(4)'}$, respectively. Let $v, w \in V'$ be the matching vectors for d, e, whence $v \neq d$ and $w \neq e$ by Lemma 11.13. Hence $\operatorname{var}(\overline{d-v}), \operatorname{var}(\overline{e-w}) \geq 5$ by Lemma 3.4, so we can take $I, J \in \binom{[n]}{6}$ such that $\overline{d-v}$ alternates on I and $\overline{e-w}$ alternates on J.

Let us write $-v = \alpha d' + \beta e'$ and $-w = \gamma d' + \delta e'$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. By Lemma 11.3 and Lemma 11.8, $\overline{d-v}$ has an I-alternating domino sequence obtained by shuffling $(d^{(1)}, d^{(2)})$ and $((\alpha+\beta)d^{(1)'}, \beta d^{(2)'}, \beta d^{(3)'}, -\alpha d^{(4)'})$. By Lemma 11.12, $d^{(1)}$ and $d^{(1)'}$ have the same support, say $\{i, i+1\}$. Hence this shuffle equals

$$((\alpha+\beta)d^{(1)'}, d^{(1)}, \beta d^{(2)'}, d^{(2)}, \beta d^{(3)'}, -\alpha d^{(4)'})$$

with $\alpha < 0$ and $\beta < 0$. Since $(d-v)_i$ has the same sign as $(\alpha+\beta)d^{(1)'}$, we have $(d-v)_i < 0$. Similarly, $\overline{e-w}$ has a J-alternating domino sequence obtained by shuffling $(d^{(3)}, d^{(4)})$ and $((\gamma+\delta)d^{(1)'}, \delta d^{(2)'}, \delta d^{(3)'}, -\gamma d^{(4)'})$. By Lemma 11.12, $d^{(4)}$ and $d^{(4)'}$ have the same support, so this shuffle equals

$$((\gamma+\delta)d^{(1)'}, \delta d^{(2)'}, d^{(3)}, \delta d^{(3)'}, d^{(4)}, -\gamma d^{(4)'}),$$

with $\gamma > 0$, $\delta < 0$, and $\gamma + \delta > 0$. Since $(\gamma+\delta)d^{(1)'}$ is the only domino above whose support contains i, we have $(e-w)_i > 0$.

Now let

$$x := \delta(d-v) - \beta(e-w) = \delta d - \beta e + (\alpha\delta - \beta\gamma)d', \tag{11.15}$$

so that $x \in \ker(Z)$. Note that $x \neq 0$, since otherwise $d' \in V$, contradicting Corollary 10.6. Hence $\operatorname{var}(\overline{x}) \geq 5$ by Theorem 3.3(ii), i.e. x alternates in sign on K for some $K \in \binom{[n]}{6}$. By Lemma 11.3 and Lemma 11.8, $\overline{x}$ has a K-alternating domino sequence obtained by shuffling $(\delta d^{(1)}, \delta d^{(2)}, -\beta d^{(3)}, -\beta d^{(4)})$ and $((\alpha\delta - \beta\gamma)d^{(1)'}, -(\alpha\delta - \beta\gamma)d^{(4)'})$, which necessarily equals

$$(\delta d^{(1)}, (\alpha\delta - \beta\gamma)d^{(1)'}, \delta d^{(2)}, -\beta d^{(3)}, -(\alpha\delta - \beta\gamma)d^{(4)'}, -\beta d^{(4)}).$$

Hence x_i has the same sign as $\delta d^{(1)}$, so $x_i < 0$. But $\beta < 0$, $\delta < 0$, $(d-v)_i < 0$, and $(e-w)_i > 0$, so we see from (11.15) that $x_i > 0$. ■

We can now deduce Theorem 11.1 from Theorem 10.3, Lemma 11.10, Lemma 11.11, and Lemma 11.14.

12. A non-triangulation for $m = 3$

In this section we show that the $m = 3$ amplituhedron is *not* triangulated by a seemingly natural collection of cells coming from the BCFW cells for $m = 4$. As background, we recall from Section 4 (in particular, see Remark 4.8) that we have two sets of ⅃-diagrams $\mathcal{D}_{n,k,2}$ and $\mathcal{D}_{n,k,1}$, which give triangulations of $\mathcal{A}_{n,k,2}(Z)$ and $\mathcal{A}_{n,k,1}(Z)$, respectively. Moreover, we have a bijection $\mathcal{D}_{n+1,k,2} \to \mathcal{D}_{n,k,1}$, which takes a ⅃-diagram D and deletes its leftmost column.

Analogously, let $\mathcal{D}_{n,k,3}$ be the set of ⅃-diagrams formed from $\oplus$-diagrams in $\mathcal{D}_{n+1,k,4}$ by deleting the leftmost column, so that we have a bijection $\mathcal{D}_{n+1,k,4} \to \mathcal{D}_{n,k,3}$ given by deleting the leftmost column. (The fact that this is a well-defined bijection follows from the proof of Lemma 6.4.) To our surprise, we found that the images under $\tilde{Z}$ of the cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ corresponding to $\mathcal{D}_{n,k,3}$ do not triangulate the $m = 3$ amplituhedron $\mathcal{A}_{n,k,3}(Z)$, since their images are not mutually disjoint.

For example, consider the $\oplus$-diagrams

$$D_1 := \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline + & + & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + \\ \hline + & 0 & 0 & + & + & 0 & 0 & + & \multicolumn{3}{c}{} \\ \cline{1-8} \end{array}\,, \qquad D_2 := \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline + & + & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + \\ \hline + & 0 & 0 & 0 & 0 & + & + & 0 & 0 & + & \multicolumn{1}{c}{} \\ \cline{1-10} \end{array}$$

in $\mathcal{D}_{13,2,4}$, which are both Class 6 $\oplus$-diagrams from Figure 14. Let D'_1 and D'_2 be the ⅃-diagrams of type $(2, 12)$ formed from D_1 and D_2 by deleting the leftmost column. Given $Z \in \mathrm{Mat}_{5,12}^{\geq 0}$, by the claim in [Kar17, Lemma 4.1], the sign patterns of the nonzero vectors in $\ker(Z)$ are precisely those with at least 5 sign changes. In particular $\ker(Z)$ contains a vector $v \in \mathbb{R}^{12}$ with $\mathrm{sign}(v) = (+, +, -, -, +, +, -, -, +, +, -, -)$.

Now let $V_1, V_2 \in \mathrm{Gr}_{2,12}^{\geq 0}$ be represented by the matrices

$$\begin{bmatrix} v_1 & v_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_{11} & v_{12} \\ v_1 & v_2 & 0 & 0 & v_5 & v_6 & 0 & 0 & v_9 & v_{10} & v_{11} & v_{12} \end{bmatrix}, \quad \begin{bmatrix} v_1 & v_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_{11} & v_{12} \\ 0 & 0 & -v_3 & -v_4 & 0 & 0 & -v_7 & -v_8 & 0 & 0 & 0 & 0 \end{bmatrix},$$

respectively. We can check using the network parameterizations coming from the ⅃-diagrams (Theorem 2.11) that $V_1 \in S_{D'_1}$, $V_2 \in S_{D'_2}$. The difference of the two matrices above is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} \end{bmatrix},$$

whose rows are both in $\ker(Z)$. Hence $\tilde{Z}(V_1) = \tilde{Z}(V_2)$, showing that the images of $S_{D'_1}$ and $S_{D'_2}$ in $\mathcal{A}_{12,2,3}(Z)$ intersect.

Problem 12.1. *Can we find $3k$-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$, naturally in bijection with the BCFW cells $\mathcal{C}_{n+1,k,4}$, whose images under $\tilde{Z}$ 'triangulate' the $m = 3$ amplituhedron $\mathcal{A}_{n,k,3}(Z)$?*

Appendix. Dyck paths and BCFW domino bases (with Hugh Thomas[16])

Definition A.1. A *Dyck path* P is a path in the plane from $(0, 0)$ to $(2n, 0)$ for some $n \geq 0$, formed by n up steps $(1, 1)$ and n down steps $(1, -1)$, which never passes below the x-axis.

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A local maximum of P is called a *peak*. Let $\mathcal{P}_{n,k,4}$ denote the set of Dyck paths with $2(n-3)$ steps and precisely $n-3-k$ peaks. For example, the Dyck path P shown in Figure 15 is in $\mathcal{P}_{12,3,4}$.

The cardinality of $\mathcal{P}_{n,k,4}$ equals the Narayana number $N_{n-3,k+1}$ [Sta15, A46], the number of (k,n)-BCFW cells. We will give a bijection $\mathcal{L}_{n,k,4} \longleftrightarrow \mathcal{P}_{n,k,4}$, which thereby allows us to label the (k,n)-BCFW cells by the Dyck paths $\mathcal{P}_{n,k,4}$. We then provide a way to conjecturally obtain k basis vectors for any element of a BCFW cell from its Dyck path.

A.1. **From pairs of lattice paths to Dyck paths.**

Definition A.2. To any pair of noncrossing lattice paths $(W_U, W_L) \in \mathcal{L}_{n,k,4}$, we associate a Dyck path $P(W_U, W_L) \in \mathcal{P}_{n,k,4}$ by the following recursive definition. (We use $+$ and $-$ to denote up and down steps of a Dyck path, and $\cdot$ to denote concatenation of paths.)

- If W_U and W_L are the trivial paths of length zero, then $P(W_U, W_L) := +-$.
- If W_U and W_L both begin with a vertical step, then we can write $W_U = V \cdot W'_U$, $W_L = V \cdot W'_L$. We set
$$P(W_U, W_L) := + \cdot P(W'_U, W'_L) \cdot -.$$
- Otherwise, let (W''_U, W''_L) be the final portion of (W_U, W_L) starting at the first overlapping vertical step of W_U and W_L. (If W_U and W_L have no overlapping vertical steps, then we let W''_U and W''_L be the trivial paths.) Then we can write $W_U = H \cdot W'_U \cdot W''_U$, $W_L = W'_L \cdot H \cdot W''_L$. We set
$$P(W_U, W_L) := P(W'_U, W'_L) \cdot P(W''_U, W''_L).$$

For example, see Figure 15. This gives us a map $\Omega_{\mathcal{LP}} : \mathcal{L}_{n,k,4} \to \mathcal{P}_{n,k,4}$, which sends (W_U, W_L) to $P(W_U, W_L)$.

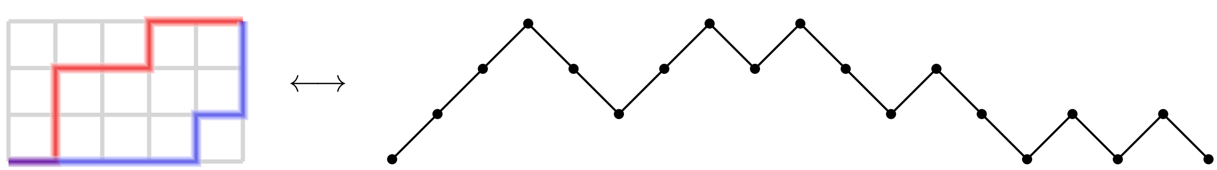


FIGURE 15. The pair of noncrossing lattice paths in $\mathcal{L}_{12,3,4}$ from Figure 9, and its corresponding Dyck path in $\mathcal{P}_{12,3,4}$.

In order to show that $\Omega_{\mathcal{LP}}$ is a bijection, we will define its inverse, which has an elegant description. As far as we know, these inverse bijections have not appeared in the literature.

Definition A.3. Let P be a Dyck path. For any point p on P, we let $\mathrm{shadow}_p(P)$ be the waterline when we turn P upside-down and fill it with as much water as possible without submerging p; this is a piecewise linear curve between the endpoints of P. Let $\mathrm{touch}_p(P)$ be the number of down steps of P whose right endpoint lies on $\mathrm{shadow}_p(P)$ to the right of p, but otherwise does not intersect $\mathrm{shadow}_p(P)$. Then we associate a pair of noncrossing lattice paths $(W_U(P), W_L(P))$ to P as follows.

- $W_L(P)$ is obtained by reading the up steps of P from left to right, recording V for every up step followed by an up step and H for every up step followed by a down step (except for the final up step).
- Let $p_1, \dots, p_k$ be the left endpoints of the up steps of P which are followed by an up step, ordered from left to right. We let $W_U(P)$ be such that the distance between the vertical edges of $W_U(P)$ and $W_L(P)$ in row i equals $\mathrm{touch}_{p_i}(P) - 1$, for $1 \le i \le k$.

Note that this gives a map $\Omega_{\mathcal{PL}} : \mathcal{P}_{n,k,4} \to \mathcal{L}_{n,k,4}$, which sends P to $(W_U(P), W_L(P))$.

Example A.4. Let $P \in \mathcal{P}_{12,3,4}$ be the Dyck path in Figure 15. Reading the up steps of P from left to right (and ignoring the final up step), we get $W_L(P) = VVHVHHHH$. To find $W_U(P)$, we let p_1, p_2, p_3 denote the left endpoints of the three up steps of P which are followed by an up step. (See Figure 16.) Then $\mathrm{shadow}_{p_1}(P)$ is the portion of the x-axis between the endpoints of P, and $\mathrm{touch}_{p_1}(P) = 3$. We have $\mathrm{shadow}_{p_2}(P) = \mathrm{shadow}_{p_3}(P)$, and we can calculate that $\mathrm{touch}_{p_2}(P) = 5$, $\mathrm{touch}_{p_3}(P) = 4$. (Note that $\mathrm{touch}_{p_2}(P) = 1 + \mathrm{touch}_{p_3}(P)$, since p_3 contributes to $\mathrm{touch}_{p_2}(P)$ but not to $\mathrm{touch}_{p_3}(P)$. Also, we point out that the first down step of P which ends on the x-axis does not contribute to $\mathrm{touch}_{p_2}(P)$ or $\mathrm{touch}_{p_3}(P)$, since the entire step lies on the shadow.) Therefore $W_U(P)$ is such that the distances between $W_U(P)$ and $W_L(P)$ in rows 1, 2, and 3 are, respectively, $3-1$, $5-1$, and $4-1$. This agrees with Figure 15. $\diamondsuit$

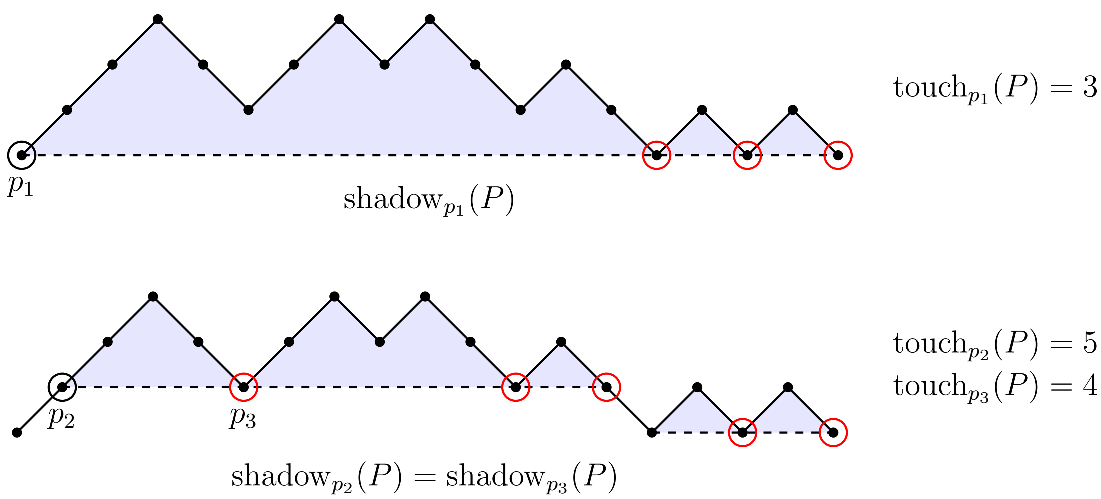


FIGURE 16. Calculating $\mathrm{shadow}_{p_i}(P)$ and $\mathrm{touch}_{p_i}(P)$ for the Dyck path P from Figure 15.

Proposition A.5. *The map $\Omega_{\mathcal{LP}} : \mathcal{L}_{n,k,4} \to \mathcal{P}_{n,k,4}$ is a bijection with inverse $\Omega_{\mathcal{PL}}$.*

Proof. We can show that $\Omega_{\mathcal{PL}} \circ \Omega_{\mathcal{LP}}$ is the identity on $\mathcal{L}_{n,k,4}$ by induction, using the recursive nature of Definition A.2. The result then follows from the fact that $|\mathcal{L}_{n,k,4}| = N_{n-3,k+1} = |\mathcal{P}_{n,k,4}|$. (Alternatively, we can verify that $\Omega_{\mathcal{LP}} \circ \Omega_{\mathcal{PL}}$ is the identity on $\mathcal{P}_{n,k,4}$.) ■

By Theorem 6.3, the (k, n)-BCFW cells are labeled by the $\oplus$-diagrams $\mathcal{D}_{n,k,4}$. Therefore the bijection $\Omega_{\mathcal{LP}} \circ \Omega_{\mathcal{LD}}^{-1} : \mathcal{D}_{n,k,4} \to \mathcal{P}_{n,k,4}$ allows us to label the (k, n)-BCFW cells by $\mathcal{P}_{n,k,4}$.

A.2. **From Dyck paths to domino bases.** Recall the definition of an i-domino from Definition 10.1.

Definition A.6. Given $P \in \mathcal{P}_{n,k,4}$, label the up steps of P by $1, \ldots, n-3$ from left to right, and label each down step so that it has the same label as the next up step. (We label the final down step by $n-2$.) We match each up step of P to the next down step at the same height. Let $\mathsf{up}_1 < \cdots < \mathsf{up}_k$ be the labels of the up steps of P which are followed by an up step, and for $i \in [k]$ let down_i be the label of the matching down step of the up step up_i.

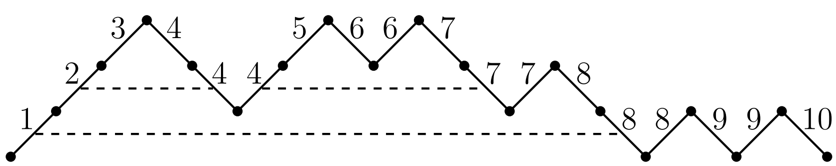
Now for $i = 1, \ldots, k$, we define the following dominoes in $\mathbb{R}^n$. Let $d^{(i)}$ be a positive up_i-domino, and $e^{(i)}$ a down_i-domino which has sign $(-1)^{|\{j \in [k] : \mathsf{up}_i < \mathsf{up}_j < \mathsf{down}_i\}|}$. If the up step up_i of P begins on the x-axis, then we let $f^{(i)}$ be the n-domino $(0, \ldots, 0, (-1)^{k-i})$. Otherwise, take $i' < i$ so that $\mathsf{up}_{i'}$ labels the last up step of P before up_i which finishes at the same height that up_i begins, and let $f^{(i)} := (-1)^{i-i'-1} d^{(i')}$. We set

$$v^{(i)} := d^{(i)} + e^{(i)} + f^{(i)} \in \mathbb{R}^n.$$

We call any k-tuple of linearly independent vectors $(v^{(1)}, \ldots, v^{(k)})$ which we can obtain in this way a *P-domino basis*.

Conjecture A.7. *Let $S \subseteq \mathrm{Gr}_{k,n}^{\geq 0}$ be a (k, n)-BCFW cell labeled by the Dyck path $P \in \mathcal{P}_{n,k,4}$. Then any $V \in S$ has a P-domino basis.*

Example A.8. Let $P \in \mathcal{P}_{12,3,4}$ be the Dyck path from Figure 15. Then the edges of P are labeled as follows:



We see that

$$\mathsf{up}_1 = 1,\ \mathsf{up}_2 = 2,\ \mathsf{up}_3 = 4, \quad \mathsf{down}_1 = 8,\ \mathsf{down}_2 = 4,\ \mathsf{down}_3 = 7.$$

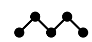
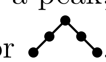
Then according to Definition A.6, the P-domino bases $(v^{(1)}, v^{(2)}, v^{(3)})$ are precisely those given by the rows of the matrix

$$\begin{bmatrix} v^{(1)} \\ v^{(2)} \\ v^{(3)} \end{bmatrix} = \begin{array}{c} \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \begin{bmatrix} \alpha & \beta & 0 & 0 & 0 & 0 & 0 & \gamma & \delta & 0 & 0 & 1 \\ \alpha & \beta+\varepsilon & \zeta & \eta & \theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha & -\beta & 0 & \iota & \kappa & 0 & \lambda & \mu & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \tag{A.9}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \iota, \kappa, \lambda, \mu > 0$. Let us explain in detail how the third row, $v^{(3)}$, is obtained. First, $d^{(3)}$ is a positive 4-domino, which we take to be $(\ldots, 0, \iota, \kappa, 0, \ldots)$. Now, $e^{(3)}$ is a 7-domino, and is positive since none of the elements of $\{1, 2, 4\}$ are strictly between 4 and 7. We take $e^{(3)}$ to be $(\ldots, 0, \lambda, \mu, 0, \ldots)$. Finally, the last up step of P before up_3 which finishes at the same height that up_3 begins is up_1. Therefore $f^{(3)} = (-1)^{3-1-1}d^{(1)}$, and we have already taken $d^{(1)} = (\alpha, \beta, 0, \dots)$.

We can check that every element of the BCFW cell in $\mathrm{Gr}_{3,12}^{\geq 0}$ labeled by P can be represented by a unique matrix of the form (A.9), by taking the $\oplus$-diagram in Figure 9, performing ⅃-moves to obtain a ⅃-diagram, constructing the network parameterization matrix from Theorem 2.11, and carrying out appropriate row operations. This verifies Conjecture A.7 in this particular case. It seems likely that the same method can be used to prove Conjecture A.7 in general. To do so, one would need to figure out a systematic way to perform the required ⅃-moves and row operations. We also observe that constraining all parameters to be positive is not sufficient for the element of $\mathrm{Gr}_{3,12}$ represented by (A.9) to lie in the corresponding BCFW cell; we also have the nontrivial inequality $\eta\kappa > \theta\iota$. $\Diamond$

We note in the cases $k = 1$ and $k = 2$, Conjecture A.7 follows from the same arguments used to establish Lemma 9.1 and Theorem 10.3. When $k = 2$, the P-domino basis vectors $v^{(1)}$, $v^{(2)}$ of an element V of a BCFW cell are the rows of the corresponding matrix in Figure 14, up to rescaling the rows by positive constants.

Remark A.10. If we take $P \in \mathcal{P}_{n,k,4}$ and delete the $2(n-3-k)$ edges incident to a peak, we obtain a Dyck path P' with $2k$ steps. When $k = 2$, P' equals either  or , depending on whether the BCFW cell of P is orthodox or deviant, respectively. In general, it may make sense to divide the (k,n)-BCFW cells into $C_k = \frac{1}{k+1}\binom{2k}{k}$ classes, based on P'.

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