SHELLING TOTALLY NONNEGATIVE FLAG VARIETIES

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ABSTRACT. In this paper we study the face partially ordered set (poset) $\mathcal{Q}^J$ of cells in Rietsch's [22] cell decomposition of the totally nonnegative part of an arbitrary flag variety $(G/P_J)_{\geq 0}$. Our goal is to use combinatorial techniques to understand what $(G/P_J)_{\geq 0}$ and its cell decomposition "look like." The order complex $\|\mathcal{Q}^J\|$ is a simplicial complex which can be thought of as a combinatorial approximation of $(G/P_J)_{\geq 0}$. Using tools such as Bjorner's EL-labellings [1] and Dyer's reflection orders [7], we prove that $\mathcal{Q}^J$ has the most favorable combinatorial properties: namely, it is graded, thin, and EL-shellable. It follows that $\mathcal{Q}^J$ is Eulerian. Additionally, our results imply that $\|\mathcal{Q}^J\|$ is homeomorphic to a ball, and moreover, that $\mathcal{Q}^J$ is the face poset of a regular CW complex homeomorphic to a ball. In particular, this paper resolves Postnikov's conjecture that the face poset of the totally nonnegative part of the Grassmannian is shellable and Eulerian.

1. INTRODUCTION

The classical theory of total positivity concerns matrices in which all minors are nonnegative. In the past decade Lusztig has extended this subject [14, 15, 16] by introducing the totally nonnegative variety $G_{\geq 0}$ in an arbitrary reductive group G as well as the totally nonnegative part $(G/P)_{\geq 0}$ of a real flag variety G/P, which he called a "remarkable polyhedral subspace" [16]. Rietsch has constructed a cell decomposition [21] for the totally nonnegative part of an arbitrary flag variety $(G/P_J)_{\geq 0}$, and has described the order relation for closures of cells. The partially ordered set (poset) of cells, which we denote by $\mathcal{Q}^J$, is intimately connected to the Bruhat order of the corresponding Weyl group W: for example, when $G/P_J = G/B$ is the complete flag variety, $\mathcal{Q}^J$ is the interval poset of the Bruhat order, and when G/P_J is the type A Grassmannian, $\mathcal{Q}^J$ is Postnikov's *cyclic Bruhat order*.

Lusztig [14] has proved that the totally nonnegative part of the (full) flag variety is contractible, which implies the same result for any partial flag variety [23]. However, it is unknown whether $(G/P_J)_{\geq 0}$ is homeomorphic to a ball. In addition, very little is known about the topology of the closures of the individual cells, or how the cells are glued together.

The goal of this paper is to apply combinatorial methods to $\mathcal{Q}^J$ in order to understand what $(G/P_J)_{\geq 0}$ looks like. Indeed, the past thirty years have seen a

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wealth of literature designed to facilitate the interplay between combinatorics and geometry (see [6], [1], [4], [5], [2]). In particular, in a 1984 paper [2], Bjorner recognized that regular CW complexes are combinatorial objects in the following sense: if Q is the poset of closed cells in a regular CW decomposition of a space P, then the order complex (or nerve) $\|Q\|$ is homeomorphic to P. Furthermore, he gave criteria [2] for recognizing when a poset is the face poset of a regular CW complex: for example, if a poset is *thin* and *shellable* then it is the face poset of some regular CW complex homeomorphic to a sphere.

This theory has been applied to intervals in the Bruhat order of a Coxeter group. Bjorner and Wachs [4] proved that such intervals are thin and *CL-shellable*; Dyer later used *reflection orders* to prove Bjorner's conjecture that such intervals are actually *EL-shellable*. Bjorner [2] pointed out that such intervals are therefore face posets of regular CW complexes homeomorphic to spheres and asked for a natural geometric construction of such a CW complex. Sixteen years later, Fomin and Shapiro [8] had the insight that such a construction might come from total positivity, and conjectured that links in the Bruhat decomposition of the totally nonnegative part of the unipotent radical of a semisimple group G have the desired properties. Although they could not prove regularity of these stratified spaces, they proved contractibility of both the closed and open strata in type A.

In this paper we begin a line of research that in many ways parallels the story of the Bruhat order. Our main results are the following.

Theorem 1.1. *$\mathcal{Q}^J$ is thin.*

Theorem 1.2. *$\mathcal{Q}^J$ is EL-shellable.*

As a corollary, we deduce that $\mathcal{Q}^J$ is *Eulerian*. Additionally, it follows from our main results that $\|\mathcal{Q}^J\|$ is homeomorphic to a ball, and that moreover, $\mathcal{Q}^J$ is the face poset of a regular CW complex homeomorphic to a ball. While these properties do not completely determine the topology of $(G/P_J)_{\geq 0}$, they are the strongest results that one can hope to obtain in this direction by a purely combinatorial study of the poset $\mathcal{Q}^J$.

We note that part of the technical difficulty of proving our main results lies in the need to understand the Bruhat order of a Coxeter group W when thinking of W as the product $W^J \times W_J$. Here, W_J is a parabolic subgroup of W and W^J is the set of minimal length coset representatives of W/W_J. See for example Corollary 3.9, Corollary 3.11, and Proposition 3.13.

The structure of this paper is as follows. We begin in Section 2 by reviewing tools from poset topology. In Section 3 we review the Bruhat order, and prove some technical lemmas necessary for our main theorems. In Section 4 we define totally nonnegative flag varieties and Rietsch's cell decomposition of them [22]. In Section 5 we introduce the poset $\mathcal{Q}^J$ of cells in this cell decomposition. In Sections 6 and 7 we prove our main results: that the poset $\mathcal{Q}^J$ is thin and EL-shellable. We then use Bjorner's theorem [2] to conclude that $\mathcal{Q}^J$ is the face poset of a regular CW complex homeomorphic to a ball.

Additionally, we include an appendix which gives a detailed explanation of some of the combinatorics of $\mathcal{Q}^J$ in the type A Grassmannian case, relating our present work to Postnikov's [18].

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2. Background on poset topology

Poset topology is the study of combinatorial properties of a partially ordered set, or *poset*, which reflect the topology of an associated simplicial complex. In this section we will review some of the basic definitions and results of poset topology.

Let P be a poset with order relation $<$. We will use the symbol $\lessdot$ to denote a covering relation in the poset: $x \lessdot y$ means that $x < y$ and there is no z such that $x < z < y$. Additionally, if $x < y$ then $[x, y]$ denotes the *interval* from x to y; that is, the set $\{z \in P \mid x \leq z \leq y\}$.

The natural geometric object that one associates to a poset P is the realization of its *order complex* (or *nerve*). The order complex $\Delta(P)$ is defined to be the simplicial complex whose vertices are the elements of P and whose simplices are the chains $x_0 < x_1 < \cdots < x_k$ in P. We will denote the realization of $\Delta(P)$ by $\|\Delta(P)\|$, or just $\|P\|$, and if we say that P possesses some topological property, we will mean that $\|P\|$ possesses that property. As we will see later, it is particularly useful to study $\|P\|$ when P is the face poset of a cell complex.

Recall that a poset is said to be *bounded* if it has a least element and a greatest element. These elements are denoted $\hat{0}$ and $\hat{1}$, respectively. A finite poset is said to be *pure* if all maximal chains have the same length, and *graded*, if in addition, it is finite and bounded. Any element x of a graded poset P has a well-defined *rank* $\rho(x)$ equal to the common length of all unrefinable chains from $\hat{0}$ to x in P.

We now introduce the property of being *thin*.

Definition 2.1. *A poset P is called thin if every interval of length 2 is a diamond, i.e. if for any $p < q$ such that $\operatorname{rank}(q) - \operatorname{rank}(p) = 2$, there are exactly two elements in the open interval (p, q).*

Another useful property is that of *shellability*. This property is desirable to have because it implies, for example, that Δ is Cohen-Macaulay.

Definition 2.2. *A pure finite simplicial complex Δ is said to be shellable if its maximal faces can be ordered $F_1, F_2, \ldots, F_n$ in such a way that $F_k \cap (\cup_{i=1}^{k-1} F_i)$ is a nonempty union of maximal proper faces of F_k for $k = 2, 3, \ldots, n$.*

One technique that can be used to prove that an order complex $\Delta(P)$ is shellable is the notion of *lexicographic shellability*, or EL-shellability, which was first introduced by Bjorner [1]. Let P be a graded poset, and let $\mathcal{E}(P)$ be the set of edges of the Hasse diagram of P, i.e. $\mathcal{E}(P) = \{(x,y) \in P \times P \mid x \gtrdot y\}$. An *edge labeling* of P is a map $\lambda : \mathcal{E}(P) \to \Lambda$ where Λ is some poset (usually the integers). Given an edge labeling λ, each maximal chain $c = (x_0 \gtrdot x_1 \gtrdot \cdots \gtrdot x_k)$ of length k can be associated with a k-tuple $\sigma(c) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{k-1}, x_k))$. We say that c is an *increasing chain* if the k-tuple $\sigma(c)$ is increasing; that is, if $\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{k-1}, x_k)$. The edge labeling allows us to order the maximal chains of any interval of P by ordering the corresponding k-tuples lexicographically. If $\sigma(c_1)$ lexicographically precedes $\sigma(c_2)$ then we say that c_1 lexicographically precedes c_2 and we denote this by $c_1 <_L c_2$.

Definition 2.3. *An edge labeling is called an EL-labeling (edge lexicographical labeling) if for every interval $[x, y]$ in P,*

(1) *there is a unique increasing maximal chain c in $[x, y]$, and*
(2) *$c <_L c'$ for all other maximal chains c' in $[x, y]$.*

If a graded poset P admits an EL-labeling then its order complex is shellable [1]. Therefore a graded poset that admits an EL-labeling is said to be *EL-shellable.*

We now turn our attention to posets that come from CW complexes. First we review the definition of a *regular* CW complex.

Definition 2.4. *A CW complex is regular if the closure $\overline{C}$ of each cell C is homeomorphic to a closed ball and if additionally $\overline{C} \setminus C$ is homeomorphic to a sphere.*

Given a cell complex $\mathcal{K}$, we define its *face poset* $\mathcal{F}(\mathcal{K})$ to be the set of closed cells ordered by containment and augmented by a least element $\hat{0}$. In general, the order complex $\|\mathcal{F}(\mathcal{K}) - \{\hat{0}\}\|$ does not reveal the topology of $\mathcal{K}$. However, the following result shows that regular CW complexes are combinatorial objects in the sense that the incidence relation of cells determines their topology.

Proposition 2.5. [3, Proposition 4.7.8] *Let $\mathcal{K}$ be a regular CW complex. Then $\mathcal{K}$ is homeomorphic to $\|\mathcal{F}(\mathcal{K}) - \{\hat{0}\}\|$.*

It is natural to ask when a poset is the face poset of a regular CW complex. Bjorner [2] gave a complete answer to this question, which we now explain.

Definition/Theorem 2.6. [2] *A poset P is said to be a CW poset if*

(1) *P has a least element $\hat{0}$,*
(2) *P is nontrivial, i.e. has more than one element, and*
(3) *for all $x \in P - \{\hat{0}\}$, the open interval $(\hat{0}, x)$ is homeomorphic to a sphere.*

Furthermore, P is a CW poset if and only if it is isomorphic to the face poset of a regular CW complex.

Bjorner gave some additional criteria for determining when a poset P is a CW poset. We list below the criterion which will be most useful to us later.

Proposition 2.7. [2] *Let P be a nontrivial, finite, pure poset of length $d+1$ with least element $\hat{0}$. Let $\hat{P}$ denote the poset $P \cup \{\hat{1}\}$, where $\hat{1}$ is a new greatest element. If $\hat{P}$ is shellable and thin then P is isomorphic to $\mathcal{F}(\mathcal{K})$, where $\mathcal{K}$ is a regular d-dimensional CW complex homeomorphic to the d-sphere.*

A finite graded poset with $\hat{0}$ and $\hat{1}$ is called *Eulerian* if every interval of length at least one has the same number of elements of odd rank as of even rank. It is an elementary topological result [24] that if a poset is a CW poset, then it is Eulerian.

Lemma 2.8. [24] *A CW poset is Eulerian.*

3. Background on the Bruhat order

In this section we will review properties of the Bruhat order $\leq$ of a Coxeter group, define reflection orders, and prove some lemmas that we will need later. We will assume knowledge of the basic definitions of Coxeter systems and Bruhat order; we refer the reader to [11] for details.

Fix a Coxeter group W generated by a set of simple reflections $\{s_i \mid i \in I\}$ and let T be the set of all reflections. Let $\ell : W \to \mathbb{N}$ denote the length function: $\ell(w)$ is the length of a minimal reduced expression for w. Recall that length is the rank function for the Bruhat order of a Coxeter group (which is a graded poset). One of the key facts about the nature of reduced expressions in W is the *Strong Exchange Property*, which is due to Verma [26].

Theorem 3.1. *(Strong Exchange Property) Let $w = s_1 \dots s_r$ (where $i \in I$ for $1 \leq i \leq r$), not necessarily a reduced expression. Suppose a reflection $t \in T$ satisfies $\ell(wt) < \ell(w)$. Then there is an index i for which $wt = s_1 \dots \hat{s_i} \dots s_r$ (omitting s_i). If the expression for w is reduced, then i is unique.*

Without loss of generality, assume that W is realized geometrically as a group of isometries of a real vector space V. Let Π denote the set of simple roots, Φ^+ the set of positive roots, and for non-isotropic $\alpha \in V$, let $r_\alpha : V \to V$ denote the reflection in α. Note that the map $\alpha \mapsto r_\alpha$ is a bijection between Φ^+ and T.

Dyer [7] proved that intervals in the Bruhat order of a Coxeter group are EL-shellable, strengthening the earlier CL-shellability result of Bjorner and Wachs [4]. Dyer's primary tool was his notion of "reflection orders," certain total orderings of T. To describe reflection orders, we use the fact that positive roots are naturally in bijection with T. The orders on the positive roots which correspond to reflection orders on T may be characterized as follows: "the restriction of the order to the positive roots lying on the plane spanned by any two positive roots is one of the two possible orders in which a ray from the origin, undergoing a full rotation in the plane beginning at a negative root, would sweep through the positive roots on that plane" [7]. The algebraic definition of a reflection order is the following.

Definition 3.2. [7] *A total order $\preceq$ on T is called a reflection order if for any dihedral reflection subgroup W' of W either $r \prec rsr \prec rsrsr \prec \cdots \prec srsrs \prec srs \prec s$ or $s \prec srs \prec srsrs \prec \cdots \prec rsrsr \prec rsr \prec r$.*

Dyer proved the existence of many different reflection orders.

Proposition 3.3. [7] *Let J, K be disjoint subsets of I and let $W_J = \langle s_j \mid j \in J \rangle$, $W_K = \langle s_k \mid k \in K \rangle$ be the corresponding parabolic subgroups of W. Then there is a reflection order $\preceq$ on T such that*

(1) *$t \prec t'$ if $t \in W_J \cap T$ and $t' \in T \setminus W_J$, and*
(2) *$t \prec t'$ if $t \in T \setminus W_K$ and $t' \in W_K \cap T$.*

In other words, there is a reflection order such that reflections contained in W_J come first, and reflections contained in W_K come last.

Dyer used reflection orders to prove the following lemma.

Lemma 3.4. [7] *Let $u, w \in W$ with $u \leq w$ and $\ell(w) - \ell(u) = 2$. Let $\preceq$ be a reflection order. Then there exist unique $x, y \in W$ such that $u \lessdot x \lessdot w$, $u \lessdot y \lessdot w$, $x^{-1}u \prec w^{-1}x$ and $y^{-1}u \succ w^{-1}y$. Moreover, $w^{-1}y \prec w^{-1}x$ and $x^{-1}u \prec y^{-1}u$.*

In particular, this lemma implies the following result (originally proved in [4]):

Corollary 3.5. [4, 7] *The Bruhat order of W is thin.*

Dyer then extended this result to EL-shellability, as follows.

Proposition 3.6. [7] *Fix a reflection order $\preceq$ and regard the set T of reflections as a poset under $\preceq$. Label each edge $x \gtrdot y$ of the Bruhat order by the reflection $x^{-1}y$. Then this edge labeling is an EL-labeling; therefore the Bruhat order is EL-shellable.*

We will now prove some properties of Bruhat order which will be useful to us later. Let us fix (W, I) and a subset $J \subset I$. We will denote by W^J the set of minimal-length coset representatives of W/W_J. Recall that each element $w \in W$ has a unique factorization of the form $w'u$, where $w' \in W^J$ and $u \in W_J$. Also recall that the projection from W to W^J is order-preserving. Use Proposition 3.3 to choose a reflection order $\preceq$ such that *reflections contained in W_J come last.* As above, we will label the edge between two elements $v \gtrdot u$ with the reflection $v^{-1}u$.

Lemma 3.7. *Suppose that $w, w' \in W^J$, and $w'v \lessdot w$ for some $v \in W_J$. Let λ be the label of the edge between w and $w'v$. Then if m is any chain in W from w to w' which passes through $w'v$, the labels of m following λ are greater than λ.*

Proof. Clearly any chain in W from w to w' through $w'v$ will have the form $w \gtrdot w'v \gtrdot w'v_1 \gtrdot w'v_2 \gtrdot \cdots \gtrdot w'v_{m-1} \gtrdot w'v_m = w'$ where $v_i \in W_J$ and $v_m = e$. This is because the projection from W to W^J is order-preserving.

By the definition of our labeling, $\lambda = w^{-1}w'v$, which is clearly not in W_J. Now note that the edge between any $w'v_i$ and $w'v_{i+1}$ is labeled by the element $(w'v_i)^{-1}w'v_{i+1} = v_i^{-1}v_{i+1}$, which is in W_J. Because we chose $\preceq$ to be a reflection order in which elements of W_J come last, the lemma follows. $\square$

This lemma implies the following results.

Corollary 3.8. *Suppose that $w, w' \in W^J$, and $w \gtrdot w'v$ for some $v \in W_J$. Let λ be the label of the edge between w and $w'v$. Then the unique chain from w to w' in W with an increasing label begins with λ.*

Proof. By Lemma 3.7, we know that for any chain from w to w' which goes through $w'v$, the edge labels after the initial λ are greater than λ. Furthermore, by Proposition 3.6, there exists a unique increasing chain from $w'v$ to w'. Therefore this gives rise to an increasing chain from w to w' whose initial edge label is λ. This increasing chain must necessarily be the unique one. □

The following corollary is essential to our later arguments.

Corollary 3.9. *Fix $w, w' \in W^J$ with $w > w'$. Then there can be at most one element in W covered by w which has the form $w'v$ (for any $v \in W_J$).*

Proof. Suppose that there were another element covered by w of the form $w'u$ for some $u \in W_J$. Let the label of the edge from w to $w'v$ be $\lambda := w^{-1}w'v$ and let the label of the edge from w to $w'u$ be $\mu := w^{-1}w'u$. Then Corollary 3.8 implies that the unique increasing chain in W from w to w' begins with λ and also begins with μ, which implies that $\lambda = \mu$. But this implies that $w'v = w'u$. □

Lemma 3.10. *Fix $w', w \in W^J$ with $w' < w$. Suppose that $w'u \lessdot wv$ where $u, v \in W_J$. Then we can write u in the form rv where $r \in W_J$ and $\ell(u) = \ell(r) + \ell(v)$.*

Proof. Let us choose a reduced expression for wv, say $s_1 s_2 \dots s_m s_{m+1} \dots s_n$ where $w = s_1 s_2 \dots s_m$ and $v = s_{m+1} \dots s_n$. Here, the s_i are simple reflections. Then by the Strong Exchange property, $w'u = s_1 s_2 \dots \hat{s}_a \dots s_m s_{m+1} \dots s_n$ for a uniquely determined $a \leq m$. Clearly $s_1 s_2 \dots \hat{s}_a \dots s_m$ is reduced, and hence $s_1 s_2 \dots \hat{s}_a \dots s_m \lessdot w$. We can now write $s_1 s_2 \dots \hat{s}_a \dots s_m$ uniquely in the form tr where $t \in W^J$ and $r \in W_J$ and $\ell(t) + \ell(r) = m - 1$. But now $w'u = trv$ which implies that $t = w'$ and $u = rv$. Since $\ell(t) + \ell(r) = m - 1$, $\ell(v) = n - m$, and $\ell(w'u) = n - 1$, we must have that $\ell(rv) = \ell(r) + \ell(v)$. □

Corollary 3.11. *Fix $w', w \in W^J$ with $w' < w$ and fix $v \in W_J$. Then wv covers at most one element of the form $w'u$ (for any $u \in W_J$).*

Proof. This follows from Corollary 3.9 and Lemma 3.10. □

Corollary 3.12. *Fix $w', w \in W^J$ with $w' < w$ and fix $u \in W_J$. Then $w'u$ is covered by at most one element of the form wv (for any $v \in W_J$).*

Proof. Assume that $w'u$ is covered by two elements wv and $w\tilde{v}$. Then by Lemma 3.10, we can write $u = rv$ and also $u = \tilde{r}\tilde{v}$, where $\ell(u) = \ell(r) + \ell(v)$ and $\ell(u) = \ell(\tilde{r}) + \ell(\tilde{v})$. Since wv and $w\tilde{v}$ cover $w'rv = w'\tilde{r}\tilde{v}$, it follows that w covers $w'r$ and $w'\tilde{r}$. But now by Corollary 3.9, we must have that $r = \tilde{r}$. It follows that $v = \tilde{v}$. □

Proposition 3.13. *Suppose that $w'u' < wu$ where $w, w' \in W^J$, $u, u' \in W_J$, and $w' < w$. Then there exists an element xv such that $w'u' \leq xv \lessdot wu$ where $x < w$, $x \in W^J$, and $v \in W_J$.*

Proof. Choose a reflection order $\preceq$ such that reflections in W_J come last, and label edges $v \gtrdot u$ of the interval $[w'u', wu]$ with the reflection $v^{-1}u$. Since this labeling is an EL-labeling, there is an increasing chain from wu to $v'u'$. Write the element of this chain which is covered by wu in the form xv, where $x \in W^J$ and $v \in W_J$.

Therefore $w'u' \leq xv \lessdot wu$. Clearly $x \leq w$. Since $w' < w$, the labeling of any chain from wu to $w'u'$ must contain at least one edge label which is *not* in W_J. By our choice of labeling, it follows that the label of $wu \gtrdot xv$ is not in W_J, and hence $x < w$. □

4. The cell decomposition of $(G/P_J)_{\geq 0}$

In this section we introduce the totally nonnegative part of a flag variety $(G/P_J)_{\geq 0}$ and its cell decomposition proved by Rietsch. Note that the reader who is interested only in the combinatorics of this object may skip to Section 5.

Remark 4.1. *Before giving the general definition of $(G/P_J)_{\geq 0}$, let us first point out that in Type A, the totally nonnegative part of a flag variety G/P_J can be defined to be the subset of G/P_J in its Plucker embedding such that all Plucker coordinates are nonnegative.*

Let G be a semisimple linear algebraic group over $\mathbb{C}$ split over $\mathbb{R}$, with split torus T. We identify G (and related spaces) with their real points and consider them with their real topology. Let $X(T) = \mathrm{Hom}(T, \mathbb{R}^*)$ and $\Phi \subset X(T)$ the set of roots. Choose a system of positive roots Φ^+. We denote by B^+ the Borel subgroup corresponding to Φ^+ and by U^+ its unipotent radical. We also have the opposite Borel subgroup B^- such that $B^+ \cap B^- = T$, and its unipotent radical U^-.

Denote the set of simple roots by $\Pi = \{\alpha_i \mid i \in I\} \subset R^+$. For each $\alpha_i \in \Pi$ there is an associated homomorphism $\phi_i : \mathrm{SL}_2 \to G$. Consider the 1-parameter subgroups in G (landing in U^+, U^-, and T, respectively) defined by

$$x_i(m) = \phi_i \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix},\ y_i(m) = \phi_i \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix},\ \alpha_i^\vee(t) = \phi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix},$$

where $m \in \mathbb{R}, t \in \mathbb{R}^*, i \in I$. The datum $(T, B^+, B^-, x_i, y_i; i \in I)$ for G is called a *pinning*. The standard pinning for SL_d consists of the diagonal, upper-triangular, and lower-triangular matrices, along with the simple root subgroups $x_i(m) = I_d + mE_{i,i+1}$ and $y_i(m) = I_d + mE_{i+1,i}$ where I_d is the identity matrix and $E_{i,j}$ has a 1 in position (i,j) and zeroes elsewhere.

The Weyl group $W = N_G(T)/T$ acts on $X(T)$ permuting the roots Φ. The simple reflections $s_i \in W$ are given explicitly by $s_i := \dot{s}_i T$ where $\dot{s}_i := \phi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and any $w \in W$ can be expressed as a product $w = s_{i_1} s_{i_2} \dots s_{i_m}$ with $\ell(w)$ factors. We set $\dot{w} = \dot{s}_{i_1} \dot{s}_{i_2} \dots \dot{s}_{i_m}$.

We can identify the flag variety G/B with the variety $\mathcal{B}$ of Borel subgroups, via

$$gB^+ \Longleftrightarrow g \cdot B^+ := gB^+ g^{-1}.$$

We have the Bruhat decompositions

$$\mathcal{B} = \sqcup_{w \in W} B^+ \dot{w} \cdot B^+ = \sqcup_{w \in W} B^- \dot{w} \cdot B^+$$

of $\mathcal{B}$ into B^+-orbits called *Bruhat cells*, and B^--orbits called *opposite Bruhat cells.*

Definition 4.2. *For $v, w \in W$ define*

$$R_{v,w} := B^+\dot{w} \cdot B^+ \cap B^-\dot{v} \cdot B^+.$$

The intersection $R_{v,w}$ is non-empty precisely if $v \leq w$, and in that case is irreducible of dimension $\ell(w) - \ell(v)$, see [12].

Let $J \subset I$. The parabolic subgroup $W_J \subset W$ corresponds to a parabolic subgroup P_J in G containing B^+. Namely, $P_J = \sqcup_{w \in W_J} B^+\dot{w}B^+$. Consider the variety $\mathcal{P}^J$ of all parabolic subgroups of G conjugate to P_J. This variety can be identified with the partial flag variety G/P_J via

$$gP_J \Longleftrightarrow gP_Jg^{-1}.$$

We have the usual projection from the full flag variety to a partial flag variety which takes the form $\pi = \pi^J : \mathcal{B} \to \mathcal{P}^J$, where $\pi(B)$ is the unique parabolic subgroup of type J containing B.

We now give the relevant definitions from total positivity.

Definition 4.3. [16] *The totally nonnegative part $U^-_{\geq 0}$ of U^- is defined to be the semigroup in U^- generated by the $y_i(t)$ for $t \in \mathbb{R}_{\geq 0}$.*

The totally nonnegative part of $\mathcal{B}$ (which we denote by $\mathcal{B}_{\geq 0}$ or by $(G/B)_{\geq 0}$) is defined by

$$\mathcal{B}_{\geq 0} := \overline{\{u \cdot B^+ \mid u \in U^-_{\geq 0}\}},$$

where the closure is taken inside $\mathcal{B}$ in its real topology.

The totally nonnegative part of a partial flag variety $\mathcal{P}^J$ (which we denote by $\mathcal{P}^J_{\geq 0}$ or by $(G/P_J)_{\geq 0}$) is defined to be $\pi^J(\mathcal{B}_{\geq 0})$.

There are natural decompositions of $\mathcal{B}_{\geq 0}$ and $\mathcal{P}^J_{\geq 0}$ which were introduced by Lusztig [16, 15].

Definition 4.4. [16] *For $v, w \in W$ with $v \leq w$, let*

$$R_{v,w;>0} := R_{v,w} \cap \mathcal{B}_{\geq 0}.$$

We write W^J (respectively W^J_{max}) for the set of minimal (respectively maximal) length coset representatives of W/W_J.

Definition 4.5. [15] *Let $\mathcal{I}^J \subset W^J_{max} \times W_J \times W^J$ be the set of triples (x, u, w) with the property that $x \leq wu$. Given $(x, u, w) \in \mathcal{I}^J$, we define $P^J_{x,u,w;>0} := \pi^J(R_{x,wu;>0}) = \pi^J(R_{xu^{-1},w;>0})$.*

Rietsch [21] proved that $R_{v,w;>0}$ and $P^J_{x,u,w;>0}$ are semi-algebraic cells of dimension $\ell(w) - \ell(v)$ and $\ell(wu) - \ell(x)$, respectively. Furthermore, Rietsch [22] has given a concrete description of the order relation on cells.

Theorem 4.6. [22] *We have that $P^J_{x,u,w;>0} \subset \overline{P^J_{x',u',w';>0}}$ precisely if there exist $u_1, u_2 \in W_J$ with $u_1u_2 = u$ and $\ell(u) = \ell(u_1) + \ell(u_2)$, such that the following holds:*

$$x'u'^{-1} \leq xu_2^{-1} \leq wu_1 \leq w'.$$

Furthermore, the closure of a cell is a union of cells.

5. The poset of cells $\mathcal{Q}^J$

Motivated by Theorem 4.6, we now define the poset which is the main focus of this paper. We will give two equivalent descriptions of the poset: Definition 5.1 is simpler to write down, but Definition 5.2 is actually easier to work with. It is this latter definition that we will use in proving our results.

Fix a Coxeter system (W, I) and a subset $J \subset I$ such that W_J is finite. Write W^J (respectively W^J_{max}) for the set of minimal (respectively maximal) length coset representatives of W/W_J.

Definition 5.1. *Let $\mathcal{Q}^J$ denote the poset with elements $Q_{x,w}$ indexed by pairs $(x, w) \in W^J_{max} \times W$ such that $x \leq w$ in the Bruhat order. We say that $Q_{x,w} < Q_{x',w'}$ if and only if there exists $z \in W_J$ such that $x' \leq xz \leq wz \leq w'$.*

This definition first appeared in the paper [9] of Goodearl and Yakimov in a Poisson geometry setting. When W is a Weyl group of a semisimple linear algebraic group, this poset describes the order relation for closures of strata in Lusztig's stratification [15] of the partial flag variety G/P_J. (However, we allow W to be an arbitrary Coxeter group because our combinatoial results hold in this greater level of generality; we thank Anders Bjorner for this observation.) Goodearl and Yakimov were interested in this stratification because it describes the torus-orbits of symplectic leaves for a natural Poisson structure on G/P_J.

Independently and simultaneously, another description of the order relation for this stratification was given by Rietsch in [22]. Additionally, Rietsch showed that the cells in her cell decomposition of $(G/P_J)_{\geq 0}$ [21] obey the same order relation [22]. See Section 4 for details. Her description of the order relation is as follows.

Definition 5.2. *Let $\mathcal{I}^J \subset W^J_{max} \times W_J \times W^J$ be the set of triples (x, u, w) with the property that $xu^{-1} \leq w$. Let $\mathcal{Q}^J$ denote the poset with elements $Q_{x,u,w}$ indexed by $\mathcal{I}^J$, where we say that $Q_{x,u,w} \leq Q_{x',u',w'}$ if and only if there exist $u_1, u_2 \in W_J$ with $u_1 u_2 = u$ and $\ell(u) = \ell(u_1) + \ell(u_2)$, such that $x'u'^{-1} \leq xu_2^{-1} \leq wu_1 \leq w'$.*

Remark 5.3. *Rietsch has pointed out that a symmetric version of her argument in* [21] *shows that the order relation can also be described by replacing the inequality above by $x' \leq xu_2^{-1} \leq wu_1 \leq w'u'$.*

Remark 5.4. *For technical reasons (see Section 2), we will add a least element $\hat{0}$ to the poset $\mathcal{Q}^J$, and stipulate that $\hat{0} < Q_{x,u,w}$ for all $Q_{x,u,w} \in \mathcal{Q}^J$.*

Remark 5.5. *Because Definitions 5.1 and 5.2 describe the order relation on the same stratification of G/P_J, they must be the same. It is also possible to prove the equivalence of the definitions combinatorially, see* [10].

Besides having a unique least element $\hat{0}$, $\mathcal{Q}^J$ also has a unique greatest element: this element is $\hat{1} := Q_{u_0,u_0,w_0^J}$, where u_0 is the longest element in W_J and w_0^J is the longest element in W^J. In Section 6 we will prove that $\mathcal{Q}^J$ is a graded poset, where the rank of $Q_{x,u,w}$ is simply $\ell(wu) - \ell(x) = \ell(w) - \ell(xu^{-1})$.

See Example A.7 and Figure 5 for the case of the Grassmannian $Gr_{2,4}(\mathbb{R})$.

Remark 5.6. *When G/P_J is the (type A) Grassmannian, $\mathcal{Q}^J$ is the poset of cells of the totally nonnegative Grassmannian, studied by Postnikov* [18] *and the author* [27]. *This poset is also called the cyclic Bruhat order, and Postnikov* [18] *has shown that it can be described in terms of many different combinatorial objects, such as decorated permutations and* Ⅎ*-diagrams (certain $0-1$ tableaux). See Appendix A.*

Remark 5.7. *When G/P_J is the complete flag variety, i.e. when $J = \emptyset$, the poset $\mathcal{Q}^J$ has an especially simple description. Recall that if P is a poset, then the interval poset $Int(P)$ is defined to be the poset of intervals $[x, y]$ of P, ordered by containment. In this special case, $\mathcal{Q}^J$ is simply the interval poset of the Bruhat order. Looking at Definition 5.1, we see that one can regard the poset $\mathcal{Q}^J$ as a kind of "twisted interval order."*

6. $\mathcal{Q}^J$ is thin

In this section we will study cover relations in $\mathcal{Q}^J$ as well as intervals of rank 2. In particular, we will prove that $\mathcal{Q}^J$ is graded and thin. The proofs are fairly straightforward, given what we have proved in Section 3; perhaps the most surprising aspects of these proofs are Corollary 3.9 and Proposition 3.13.

Recall that $Q_{x,u,w} \leq Q_{x',u',w'}$ if and only if there exist $u_1, u_2 \in W_J$ with $u_1 u_2 = u$ and $\ell(u) = \ell(u_1) + \ell(u_2)$, such that $x'{u'}^{-1} \leq x u_2^{-1} \leq w u_1 \leq w'$. Here $w, w' \in W^J$, $u, u', u_1, u_2 \in W_J$, and $x, x' \in W^J_{max}$. It follows that $x' \leq x$ and $w \leq w'$.

Remark 6.1. *It is easy to see that $Q_{x,u,w} \leq Q_{x',u',w}$ if and only if $x'{u'}^{-1} \leq xu^{-1}$. Similarly, using Remark 5.3, we can see that $Q_{x,u,w} \leq Q_{x,u',w'}$ if and only if $wu \leq w'u'$.*

Lemma 6.2. *Suppose that $Q_{x,u,w} < Q_{x',u',w'}$ such that $w < w'$. Then there exists some $Q_{x'',u'',w''}$ with $w \leq w'' < w'$ such that $Q_{x,u,w} \leq Q_{x'',u'',w''} \lessdot Q_{x',u',w'}$.*

Proof. By definition, there exist u_1, u_2 such that $x' \leq x u_2^{-1} \leq w u_1 \leq w'u'$. Since $w < w'$, we have $wu_1 < w'u'$. By Proposition 3.13, there exists some w'', u'' with $w'' < w$ such that $wu_1 \leq w''u'' \lessdot w'u'$. It now follows that $Q_{x,u,w} \leq Q_{x',u'',w''} \lessdot Q_{x',u',w'}$. □

Using Remark 6.1, Lemma 6.2, and the fact that the Bruhat order is graded, we deduce the following.

Proposition 6.3. *$\mathcal{Q}^J$ is graded. Furthermore, the rank of $Q_{x,u,w}$ is $\ell(wu) - \ell(x)$.*

We can also deduce a classification of the cover relations in $\mathcal{Q}^J$.

Corollary 6.4. *The cover relations in $\mathcal{Q}^J$ fall into the following three categories.*

- **Type 1:** *$Q_{x,v,w'} \lessdot Q_{x,u,w}$ such that $w' < w$. It follows that $w'v \lessdot wu$.*
- **Type 2:** *$Q_{x',v,w} \lessdot Q_{x,u,w}$ such that $x \leq x'$. It follows that $xu^{-1} \lessdot x'v^{-1}$.*
- **Type 3:** *$\hat{0} \lessdot Q_{x,u,w}$ where $Q_{x,u,w}$ is a 0-cell. It follows that $x = wu$.*

Remark 6.5. *Note that we could further subdivide the Type 2 cover relations into two cases based on whether $x < x'$ or $x = x'$; for the sake of simplicity, we won't do this.*

We will now prove that intervals of rank 2 in $\mathcal{Q}^J$ are diamonds.

Theorem 1.1. *$\mathcal{Q}^J$ is thin.*

Proof. Let us fix two elements $Q_{a,b,c} < Q_{x,u,w} \in \mathcal{Q}^J$, such that $\text{rank}(Q_{x,u,w}) - \text{rank}(Q_{a,b,c}) = 2$. It follows that $x \le a$ and $c \le w$. We must show that there are exactly two elements between $Q_{x,u,w}$ and $Q_{a,b,c}$ in $\mathcal{Q}^J$.

We will divide the proof into three cases: when $x = a$; when $c = w$; and when $x < a$ and $c < w$.

In Case 1, by Remark 6.1, we have that $cb < wu$ where $\ell(wu) - \ell(cb) = 2$. Since the Bruhat order is thin, it follows that there are precisely two elements (which we can write in the form) $t_i r_i$ where $t_i \in W^J$ and $r_i \in W_J$ such that $cb < t_i r_i < wu$. It is now easy to see that $Q_{a,b,c} < Q_{a,r_i,t_i} < Q_{a,u,w}$. Conversely, any element $Q_{a,r,t}$ between $Q_{a,b,c}$ and $Q_{a,u,w}$ must be of this form.

In Case 2, again by Remark 6.1, we have that $xu^{-1} < ab^{-1}$, where $\ell(ab^{-1}) - \ell(xu^{-1}) = 2$. It follows that there are precisely two elements (which we can write in the form) $z_i r_i^{-1}$ where $z_i \in W^J_{max}$, $r_i \in W_J$ such that $xu^{-1} < z_i r_i^{-1} < ab^{-1}$. It now follows that $Q_{a,b,c} < Q_{z_i,r_i,c} < Q_{x,u,c}$, and that any element $Q_{z,r,c}$ between $Q_{a,b,c}$ and $Q_{x,u,c}$ must be of this form.

In Case 3, it follows from our classification of the cover relations that any element between $Q_{a,b,c}$ and $Q_{x,u,w}$ must be of the form $Q_{a,r,w}$ or of the form $Q_{x,s,c}$. Since $Q_{a,b,c} < Q_{x,u,w}$, there exist $b_1, b_2 \in W_J$ with $b_1 b_2 = b$ (and the lengths add) such that $xu^{-1} \le ab_2^{-1} \le cb_1 \le w$. In fact, we must have that $xu^{-1} \lessdot ab_2^{-1}$ and $cb_1 \lessdot w$. We claim that the only elements between $Q_{a,b,c}$ and $Q_{x,u,w}$ are $Q_{a,b_2,w}$ and $Q_{x,b_1 u,c}$.

To show that (a, b_2, w) indexes an element of $\mathcal{Q}^J$, note that $cb_1 \lessdot w$ implies that $cb \lessdot wb_2$. Since $a \le cb$ we have $a \le wb_2$, and so $Q_{a,b_2,w}$ is well-defined. Since $ab_2^{-1} = ab_2^{-1} \le cb_1 \lessdot w$ and $xu^{-1} \lessdot ab_2^{-1} \le w = w$ we have that $Q_{a,b,c} \lessdot Q_{a,b_2,w} \lessdot Q_{x,u,w}$.

To show that $(x, b_1 u, c)$ indexes an element of $\mathcal{Q}^J$, we must first observe that $\ell(b_1 u) = \ell(b_1) + \ell(u)$. By applying Lemma 3.10 to the fact that $xu^{-1} \lessdot ab_2^{-1}$ (and remembering that each element of W^J_{max} has a unique expression as a product of an element of W^J and the longest element $u_0 \in W_J$), we see that we can write b_2 in the form ub' where $\ell(b_2) = \ell(u) + \ell(b')$. But now since lengths add in the product $b = b_1 b_2$, this implies that lengths add in the product $b_1 u$. Therefore $x \le cu$ implies that $x \le cb_1 u$, and hence $Q_{x,b_1 u,c}$ is well-defined.

Now $xu^{-1} \lessdot ab_2^{-1}$ together with the fact that lengths add in the product $b_1 u$ implies that $xu^{-1} b_1^{-1} \lessdot ab_2^{-1} b_1^{-1}$. Therefore $x(b_1 u)^{-1} \lessdot ab^{-1} \le c = c$ and so $Q_{a,b,c} \lessdot Q_{x,b_1 u,c}$. Finally, we have $xu^{-1} = xu^{-1} \le cb_1 \lessdot w$ and so $Q_{x,b_1 u,c} \lessdot Q_{x,u,w}$.

To complete the proof for Case 3, we need to show that the *only* two elements between $Q_{a,b,c}$ and $Q_{x,u,w}$ are $Q_{a,b_2,w}$ and $Q_{x,b_1 u,c}$. We know that elements between $Q_{a,b,c}$ and $Q_{x,u,w}$ either have the form $Q_{x,s,c}$ or $Q_{a,r,w}$. If $Q_{a,b,c} \lessdot Q_{x,s,c} \lessdot Q_{x,u,w}$ then $xu^{-1} \le xs_2^{-1} \le cs_1 \le w$ together with the fact that $c \ne w$ implies that $cs_1 \lessdot w$ and $xu^{-1} = xs_2^{-1}$. Corollary 3.9 applied to $cs_1 \lessdot w$ implies that s_1 is uniquely determined, and since $s_2 = u$, we have that s is uniquely determined. Therefore there is at most one element of the form $Q_{x,s,c}$ between $Q_{a,b,c}$ and $Q_{x,u,w}$

Similarly, if $Q_{a,b,c} \lessdot Q_{a,r,w} \lessdot Q_{x,u,w}$ then $xu^{-1} \leq ar_2^{-1} \leq wr_1 \leq w$ implies that $xu^{-1} \lessdot ar_2^{-1}$ and $r_1 = e$. Corollary 3.12 implies that r_2 is uniquely determined, and hence r is uniquely determined. Therefore there is at most one element of the form $Q_{a,r,w}$ between $Q_{a,b,c}$ and $Q_{x,u,w}$. This completes the proof for Case 2.

To complete the proof, we must address the rank 2 intervals whose least element is $\hat{0}$. Let the greatest element of such an interval be $Q_{x,u,w}$. It follows that $x \lessdot wu$. It is now easy to see that there are exactly two elements in the open interval $(\hat{0}, Q_{x,u,w})$: $Q_{wu_0,u_0,w}$ and Q_{x,u_0,xu_0}, where u_0 is the longest element in W_J. Therefore $\mathcal{Q}^J$ is thin. $\square$

7. $\mathcal{Q}^J$ is EL-shellable

In this section we will prove that $\mathcal{Q}^J$ has an EL-labeling, which implies that $\mathcal{Q}^J$ is EL-shellable [1]. In particular, this proves Alex Postnikov's conjecture [18] that the face poset of the totally nonnegative part of the (type A) Grassmannian is shellable.

Recall the three types of cover relations in $\mathcal{Q}^J$ which were described in Section 6. We now propose an edge labelling for $\mathcal{Q}^J$. See Example A.7 together with Figure 5 for the example of the Grassmannian $Gr_{2,4}(\mathbb{R})$.

Definition 7.1. *Label Type 1 edges with the element $(wu)^{-1}w'v \in T$; label Type 2 edges with the element $(x'v^{-1})^{-1}xu^{-1} \in T$; and label Type 3 edges with the symbol $\emptyset$. Choose any reflection order $\preceq$ on T such that elements of $T \cap W_J$ come last. We then choose the total order $\blacktriangleleft$ on labels of edges in $\mathcal{Q}^J$ determined by the following conditions:*

1. *If λ is any Type 1 label and μ is any Type 2 label, then $\lambda \blacktriangleleft \emptyset \blacktriangleleft \mu$.*
2. *If λ and μ are Type 1 labels then $\lambda \blacktriangleleft \mu$ if and only if $\lambda \prec \mu$.*
3. *If λ and μ are Type 2 labels then $\lambda \blacktriangleleft \mu$ if and only if $\lambda \prec \mu$.*

Remark 7.2. *Observe that the labels of Type 1 edges are never in W_J. This will be important for our arguments later.*

Theorem 1.2. *The labeling of edges of $\mathcal{Q}^J$ described above is an EL-labeling.*

Proof. Fix two elements $Q_{x,u,w}$ and $Q_{a,b,c}$ in $\mathcal{Q}^J$, such that $Q_{a,b,c} < Q_{x,u,w}$. First we will show that the lexicographically minimal chain (with respect to $\blacktriangleleft$) from $Q_{x,u,w}$ to $Q_{a,b,c}$ is increasing.

By Lemma 6.2, it is clear that the lexicographically minimal chain m will consist of a series of Type 1 edges followed by Type 2 edges; let the chain label be $(\lambda_1, \lambda_2, \ldots, \lambda_i, \mu_1, \mu_2, \ldots, \mu_j)$, where the $\lambda_k \in T$ are Type 1 edges and the $\mu_k \in T$ are Type 2 edges.

For the sake of contradiction, suppose that for some h we have $\lambda_h \blacktriangleright \lambda_{h+1}$. In other words, $\lambda_h \succ \lambda_{h+1}$, where λ_h and λ_{h+1} are the labels, respectively, for the edges of the chain $Q_{x,u_1,w_1} \gtrdot Q_{x,u_2,w_2} \gtrdot Q_{x,u_3,w_3}$. As shown in Section 6, the interval from Q_{x,u_1,w_1} to Q_{x,u_3,w_3} is a diamond. If we let $Q_{x,u_2',w_2'}$ denote the other middle element of this interval, then we know that $w_3u_3 \lessdot w_2u_2 \lessdot w_1u_1$ and also $w_3u_3 \lessdot w_2'u_2' \lessdot w_1u_1$. Since λ_h and λ_{h+1} are Type 1 labels, we know that $w_3 < w_2 <$

w_1. However, for w_2', we know only that $w_3 \leq w_2 \leq w_1$. Observe that the labels λ_h and λ_{h+1} that we've used for our elements in $\mathcal{Q}^J$ are also the labels used by Dyer for the edges of the interval $[w_3u_3, w_1u_1]$. Therefore Lemma 3.4 implies that the labels γ_h and γ_{h+1} for the chain $w_1u_1 \gtrdot w_2'u_2' \gtrdot w_3u_3$ satisfy $\gamma_h \prec \gamma_{h+1}$ and also $\gamma_h \prec \lambda_h$. We now claim that the chain $(\lambda_1, \lambda_2, \dots, \gamma_h, \gamma_{h+1}, \dots, \lambda_i, \mu_1, \mu_2, \dots, \mu_j)$ is lexicographically smaller than $(\lambda_1, \lambda_2, \dots, \lambda_i, \mu_1, \mu_2, \dots, \mu_j)$. To complete the proof of the claim, it suffices to show that γ_h is the label of a *Type 1* cover relation. It is not hard to see that γ_h is the label of a Type 1 cover relation if and only if $\gamma_h \notin W_J$. Since $w_3 < w_1$, it is clear that we cannot have both γ_h and γ_{h+1} in W_J; at least one is *not* in W_J. And now because we've chosen a reflection order in which elements of W_J come *last*, it follows that γ_h is not in W_J, and hence is the label of a Type 1 cover relation. We've now found a lexicographically smaller chain, which is a contradiction.

Therefore the lexicographically minimal chain label $(\lambda_1, \lambda_2, \dots, \lambda_i, \mu_1, \mu_2, \dots, \mu_j)$ satisfies $\lambda_1 \blacktriangleleft \lambda_2 \blacktriangleleft \dots \blacktriangleleft \lambda_i$. Let $Q_{r,s,c}$ denote the element of $\mathcal{Q}^J$ that we reach if we start at $Q_{x,u,w}$ and traverse the edges labeled by $\lambda_1, \lambda_2, \dots, \lambda_i$. We need to show that the lexicographically minimal chain from $Q_{r,s,c}$ to $Q_{a,b,c}$ is increasing. First note that all edge labels of this interval are Type 2 labels. Furthermore, by Remark 6.1, elements $Q_{d,e,c}$ in the interval $[Q_{a,b,c}, Q_{r,s,c}]$ correspond to pairs $(d, e) \in W^J_{max} \times W_J$ such that $rs^{-1} \leq de^{-1} \leq ab^{-1}$. Therefore the interval $[Q_{a,b,c}, Q_{r,s,c}]$ is isomorphic to the dual of the interval $[rs^{-1}, ab^{-1}]$. Recall that our edge labeling of $[Q_{a,b,c}, Q_{r,s,c}]$ is inherited from (the dual of) Dyer's edge labeling of $[rs^{-1}, ab^{-1}]$. It follows – using EL-shellability of intervals in Bruhat order – that the lexicographically minimal chain (with respect to $\blacktriangleleft$) from $Q_{r,s,c}$ to $Q_{a,b,c}$ is increasing. Since we've chosen an ordering in which Type 1 labels precede Type 2 labels, we have now shown that the lexicographically minimal chain from $Q_{x,u,w}$ to $Q_{a,b,c}$ is increasing.

It remains to show that this is the *unique* increasing chain from $Q_{x,u,w}$ to $Q_{a,b,c}$. Clearly the labels on any increasing chain will again consist of a series of Type 1 edge labels followed by a series of Type 2 edge labels. Suppose that there are two increasing chains m_1 and m_2 from $Q_{x,u,w}$ to $Q_{a,b,c}$; let $Q_{r,v,c}$ and $Q_{r',v',c}$ be the two intermediate elements of these chains which we obtain after starting at $Q_{x,u,w}$ and traversing the Type 1 edges of m_1 and m_2, respectively. It follows that $cv < wu$ and $cv' < wu$. Furthermore, the increasing chain labels from $Q_{x,u,w}$ to $Q_{r,v,c}$ and from $Q_{x,u,w}$ to $Q_{r',v',c}$ correspond to increasing chain labels from wu to cv and from wu to cv'. Note that by Remark 7.2, the labels on these increasing chains from wu to cv and from wu to cv' are *not* in W_J.

Now consider the interval $[c, wu]$. Clearly both cv and cv' are in this interval. By EL-shellability of the Bruhat order, we can find increasing chains from cv to c and also from cv' to c. Clearly the labels on these chains will be elements of $T \cap W_J$. And therefore by our choice of reflection ordering, the increasing chain from wu to cv extends to an increasing chain from wu to c; similarly, the increasing chain from wu to cv' extends to an increasing chain from wu to c. We have now found two increasing chains from wu to c, which contradicts the fact that reflection orders give EL-labelings of the Bruhat order. Therefore there is a *unique* increasing chain from $Q_{x,u,w}$ to $Q_{a,b,c}$.

This would complete the proof that $\mathcal{Q}^J$ is EL-shellable except that we have so far ignored the chains from $Q_{x,u,w}$ to $\hat{0}$. We will now address these chains.

Consider all maximal chains from $Q_{x,u,w}$ to rank 0 elements of $\mathcal{Q}^J$, i.e. to elements $Q_{a,b,c}$ with $cb = a$. We claim that among these, the lexicographically minimal chain m consists entirely of Type 1 edges and is increasing. First we will show that if $\mathrm{rank}(Q_{a,b,c}) > 0$ then there is a Type 1 edge from $Q_{a,b,c}$ to an element below it. Since $\mathrm{rank}(Q_{a,b,c}) > 0$, we have $a < cb$, where $a \in W^J_{max}$, i.e. $a = a'u_0$ where $a' \in W^J$ and u_0 is the longest element in W_J. It follows that $a' < c$, and so by Proposition 3.13, there exist $c' \in W^J$ and $b' \in W_J$ such that $a \le c'b' \lessdot cb$. It follows that $Q_{a,b',c'} \lessdot Q_{a,b,c}$. Therefore by induction, the lexicographically minimal chain from $Q_{x,u,w}$ to a 0-cell consists entirely of Type 1 edges. Moreover, it is increasing, by the argument that we used in the third paragraph of this proof. Therefore by adding to m the final edge to $\hat{0}$, we have found an increasing chain from $Q_{x,u,w}$ to $\hat{0}$ which is lexicographically minimal in this interval.

It remains to show that there is a *unique* increasing chain from $Q_{x,u,w}$ to $\hat{0}$. Suppose that there are two. Both of these chains end with the label $\emptyset$, so by our choice of ordering, these chains must consist of Type 1 edges followed by the $\emptyset$ edge. Therefore the two increasing chains must both pass through 0-cells of the form $Q_{x,b,c}$ and $Q_{x,b',c'}$, where $c, c' \in W^J$ and $b, b' \in W_J$. Since these are 0-cells, we must have $x = cb$ and $x = c'b'$. It follows that $c = c'$ and $b = b'$, so the increasing chain from $Q_{x,u,w}$ to $\hat{0}$ is indeed unique.

This completes the proof that $\mathcal{Q}^J$ is EL-shellable. □

Recall that $\hat{1} := Q_{u_0,u_0,w_0^J}$ is the unique maximal element of $\mathcal{Q}^J$. If we now apply Theorem 2.6 and Proposition 2.7 to the poset $P := \mathcal{Q}^J \setminus \hat{1}$, we find that $\|\mathcal{Q}^J \setminus \hat{1}\|$ is homeomorphic to a sphere. Moreover, the poset $\mathcal{Q}^J \setminus \hat{1}$ is the face poset of a regular CW complex which is homeomorphic to a sphere. Adding the element $\hat{1}$ back to the poset, we get the following results.

Corollary 7.3. *$\|\mathcal{Q}^J\|$ is homeomorphic to a ball. Furthermore, $\mathcal{Q}^J$ is the face poset of a regular CW complex which is homeomorphic to a ball.*

Now we apply Lemma 2.8 to $\mathcal{Q}^J$.

Corollary 7.4. *$\mathcal{Q}^J$ is Eulerian.*

Corollary 7.3 leads us to conjecture that the totally nonnegative part of an arbitrary flag variety together with its cell decomposition is a regular CW complex homeomorphic to a ball.

Appendix A. $\mathcal{Q}^J$ for the type A Grassmannian

In independent work, Postnikov [18] has studied the poset of cells of a natural cell decomposition of the totally nonnegative part of the (type A) Grassmannian $Gr^+_{k,n}$, and showed that this poset can be described in terms of certain tableaux (the so-called Ⅎ-diagrams) and also in terms of certain "decorated" permutations. $Gr^+_{k,n}$

is defined to be the subset of the real Grassmannian where all Plucker coordinates are non-negative, and cells are given by specifying which Plucker coordinates are positive and which are 0. It is not too hard to see that in the case of the Grassmannian, Postnikov's cell decomposition is a special case of Rietsch's cell decomposition [21]; although we will not prove that here, we will give rank-preserving bijections between the poset $\mathcal{Q}^J$ in the case of the Grassmannian, and the poset of ⅃-diagrams and the poset of decorated permutations. Additionally, we will describe in detail the case of the Grassmannian $Gr_{2,4}(\mathbb{R})$.

Recall that a *partition* $\lambda = (\lambda_1, \dots, \lambda_k)$ is a weakly decreasing sequence of non-negative numbers. For a partition λ, where $\sum \lambda_i = n$, the *Young diagram* Y_λ of shape λ is a left-justified diagram of n boxes, with λ_i boxes in the ith row. Figure 1 shows a Young diagram of shape $(4, 2, 1)$.

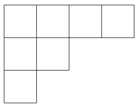

FIGURE 1. A Young diagram of shape $(4, 2, 1)$

Definition A.1. *Fix k and n. A ⅃-diagram* [1] $(\lambda, D)_{k,n}$ *is a partition λ contained in a $k \times (n-k)$ rectangle, together with a filling $D : Y_\lambda \to \{0, +\}$ which has the ⅃-property: there is no 0 which has a $+$ above it and a $+$ to its left.*

(Here, "above" means above and in the same column, and "to its left" means to the left and in the same row.) In Figure 2 we give an example of a ⅃-diagram.

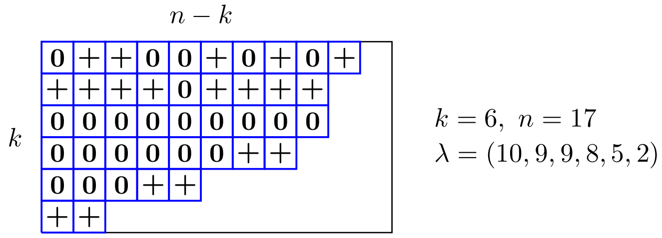

FIGURE 2. A ⅃-diagram $(\lambda, D)_{k,n}$

The *rank* of $(\lambda, D)_{k,n}$ is the number of $+$'s in the filling D.

Definition A.2. *A decorated permutation $\tilde{\pi} = (\pi, d)$ is a permutation π in the symmetric group S_n together with a coloring (decoration) d of its fixed points $\pi(i) = i$ by two colors, "clockwise" and "counterclockwise."*

We represent a decorated permutation $\tilde{\pi} = (\pi, d)$, where $\pi \in S_n$, by its *chord diagram*, constructed as follows. Put n equally spaced points around a circle, and

[1] The symbol ⅃ is meant to remind the reader of the shape of the forbidden pattern, and should be pronounced as [le], because of its relationship to the letter L.

label these points from 1 to n in clockwise order. If $\pi(i) = j$ then this is represented as a directed arrow, or chord, from i to j. If $\pi(i) = i$ then we draw a chord from i to i (i.e. a loop), and orient it either clockwise or counterclockwise, according to d.

For example, the decorated permutation $(3, 1, 5, 4, 8, 6, 7, 2)$ (written in list notation) with the fixed points 4, 6, and 7 colored in counterclockwise, clockwise, and counterclockwise, respectively, is represented by the chord diagram in Figure 3.

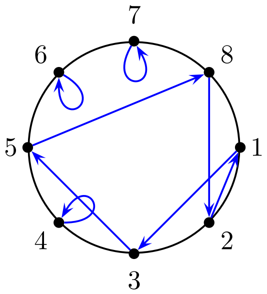

FIGURE 3. A chord diagram for a decorated permutation

Recall that i is a *weak excedence* of a permutation π if $\pi(i) \geq i$. This definition can be extended to decorated permutations as follows: i is a *weak excedence* of a decorated permutation (π, d) if either $\pi(i) > i$ or if $\pi(i)$ is a counterclockwise loop.

We will not review here the rank function on decorated permutations nor the order relations for these objects; for details, see [27]. However, we should recall Postnikov's result [18] relating the Ⅎ-diagrams and decorated permutations to $Gr^+_{k,n}$.

Theorem A.3. [18] *There is an order-preserving bijection between the poset of cells of $Gr^+_{k,n}$ and the poset of Ⅎ-diagrams $(\lambda, D)_{k,n}$. Additionally, there is an order-preserving bijection between the poset of cells of $Gr^+_{k,n}$ and the poset of decorated permutations on n letters with k weak excedences.*

We now let W be the symmetric group on n letters, $S = \{s_1, s_2, \dots, s_{n-1}\}$ be the set of adjacent transpositions, and $J = \{s_1, s_2, \dots, \hat{s}_{n-k}, \dots, s_{n-1}\}$. We will now use $\mathcal{Q}^J$ to denote the poset of cells defined in terms of this data.

Lemma A.4. *There is an order-preserving bijection Φ_1 from $\mathcal{Q}^J$ to the poset of decorated permutations in S_n with k weak excedences, which is defined as follows. Let $Q_{x,u,w} \in \mathcal{Q}^J$. Then $\Phi_1(Q_{x,u,w}) = (\pi, d)$ where $\pi = xu^{-1}w^{-1}$. To define d, we make any fixed point that occurs in one of the positions $w(1), w(2), \dots, w(n-k)$ a clockwise loop, and we make any fixed point that occurs in one of the positions $w(n-k+1), \dots, w(n)$ a counterclockwise loop.*

Additionally, there is a natural bijection between Ⅎ-diagrams $(\lambda, D)_{k,n}$ and $\mathcal{Q}^J$: we thank Postnikov for explaining this to us.

Lemma A.5. [19] *There is an order-preserving bijection Φ_2 from the set of Ⅎ-diagrams $(\lambda, D)_{k,n}$ to $\mathcal{Q}^J$, defined as follows.*

(1) *Take* $(\lambda, D)_{k,n}$ *and replace each* $+$ *with an elbow joint, and each* 0 *with a cross* $+$.
(2) *Note that the west and north borders, and the south and east borders, respectively, of* λ*, give rise to two length-*n *paths from the north-east corner to the south-east corner of the* $k \times (n-k)$ *rectangle. Label each of these paths with the numbers* 1 *through* n.
(3) *View the resulting "pipe dream" as a permutation* $w_{\lambda,D} \in S_n$*, as in A.6.*
(4) *Repeat this procedure for the* ⅃*-diagram* $(\lambda, D_0)_{k,n}$*, where* D_0 *denotes the filling of* λ *which consists entirely of* 0*'s. Denote the resulting permutation by* w_{λ,D_0}*; this permutation is in* W^J*, i.e. it is a Grassmannian permutation.*
(5) *Let* $w := w_{\lambda,D_0}$*. Since every element of* W *has a unique factorization as the product of an element in* W^J_{max} *and an element in* W_J*, define* x *and* u *by the equation* $xu^{-1} = w_{\lambda,D}$*. We now set* $\Phi_2((\lambda, D)_{k,n}) := Q_{x,u,w}$*.*

Example A.6. *Figure 4 shows a* ⅃*-diagram* $(\lambda, D)_{3,7}$ *together with the related pipe dreams. This gives rise to the permutation* $w_{\lambda,D} := (2, 1, 5, 4, 6, 3, 7)$ *(written in list notation) and the permutation* $w_{\lambda,D_0} := (2, 4, 5, 7, 1, 3, 6)$.

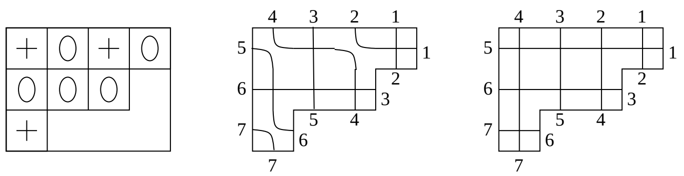

FIGURE 4. The bijection Φ_2

For a simple bijection between ⅃-diagrams and decorated permutations (which is equal to $\Phi_2 \circ \Phi_1$), see [25].

Example A.7. *We now explain the case of the Grassmannian* $Gr_{2,4}(\mathbb{R})$ *in detail. In that case, the Weyl group* W *is* S_4*, the symmetric group on 4 letters, and the set* S *of simple reflections is* $\{s_1, s_2, s_3\}$ *where* s_i *is the transposition* $(i, i+1)$ *which exchanges* i *and* $i+1$*. The subset* J *is* $\{s_1, s_3\}$ *and the parabolic subgroup* W_J *is* $\langle s_1, s_3 \rangle$*. A reflection order which puts elements of* W_J *at the end is the following:*

$$(23) \prec (24) \prec (13) \prec (14) \prec (34) \prec (12)$$

In Figure 5, we have drawn the Hasse diagram of the poset $\mathcal{Q}^J$ *for the Grassmannian* $Gr_{2,4}(\mathbb{R})$*. Elements* $Q_{x,u,w}$ *(where* $x \in W^J_{max}, u \in W_J, w \in W^J$*) are represented by* ⅃*-diagrams, and below each* ⅃*-diagram, we have listed the triple* (x, u, w) *corresponding to* $Q_{x,u,w}$*. Note that in each of these triples we have abbreviated* s_i *by* i*. Also note that we have labelled the unique increasing chain from the greatest element to the least element of* $\mathcal{Q}^J$*; every element in this chain is the totally positive part of a Schubert variety.*

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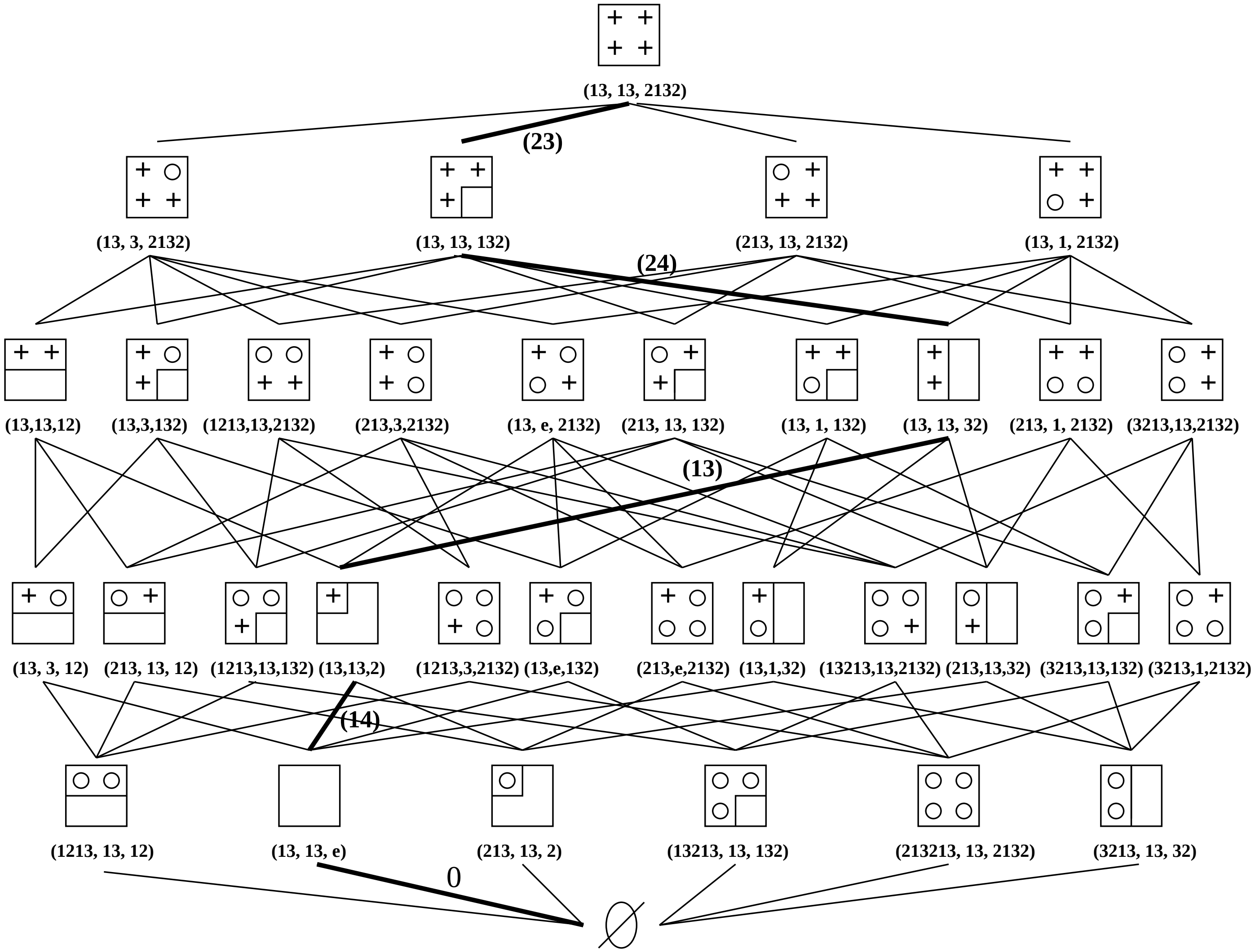

FIGURE 5. Q^J for the Grassmannian $Gr_{2,4}(\mathbb{R})$