A MARKOV CHAIN ON PERMUTATIONS WHICH PROJECTS TO THE PASEP

SYLVIE CORTEEL AND LAUREN K. WILLIAMS

Abstract. The partially asymmetric exclusion process (PASEP) is an important model from statistical mechanics which describes a system of interacting particles hopping left and right on a one-dimensional lattice of N sites. It is partially asymmetric in the sense that the probability of hopping left is q times the probability of hopping right. Additionally, particles may enter from the left with probability α and exit to the right with probability β.

It has been observed that the (unique) stationary distribution of the PASEP has remarkable connections to combinatorics – see for example the papers of Derrida *et al* [8, 9], Duchi and Schaeffer [11], Corteel [5], and Shapiro and Zeilberger [18]. Most recently we proved [7] that in fact the (normalized) probability of being in a particular state of the PASEP can be viewed as a certain weight generating function for *permutation tableaux* of a fixed shape. (This result implies the previous combinatorial results.) However, our proof relied on the *matrix ansatz* of Derrida *et al* [9], and hence did not give an intuitive explanation of why one should expect the steady state distribution of the PASEP to involve such nice combinatorics.

In this paper we define a Markov chain – which we call the PT chain – on the set of permutation tableaux which *projects* to the PASEP, in a sense which we shall make precise. This gives a new proof of the main result of [7] which bypasses the matrix ansatz altogether. Furthermore, via the bijection of [19], the PT chain can also be viewed as a Markov chain on the symmetric group. Another nice feature of the PT chain is that it possesses a certain symmetry which extends the *particle-hole symmetry* of the PASEP.

1. Introduction

The partially asymmetric exclusion process (PASEP) is an important model from statistical mechanics which is quite simple but surprisingly rich: it exhibits boundary-induced phase transitions, spontaneous symmetry breaking, and phase separation. The PASEP is regarded as a primitive model for biopolymerization [14], traffic flow [17], and formation of shocks [10]; it also appears in a kind of sequence alignment problem in computation biology [4].

In brief, the PASEP describes a system of particles hopping left and right on a one-dimensional lattice of N sites. Particles may enter the system from the left with a rate αdt and may exit the system to the right at a rate βdt. The probability of hopping left is q times the probability of hopping right.

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It has been observed that the (unique) stationary distribution of the PASEP has remarkable connections to combinatorics. Derrida *et al* [8, 9] proved a connection to Catalan and Narayana numbers in the case where $q = 0$ (TASEP) and $\alpha = \beta = 1$; Duchi and Schaeffer [11] gave a combinatorial explanation of this result by constructing a new Markov chain on "complete configurations" (enumerated by Catalan numbers) that projects to the TASEP, for general α and β. Subsequently Corteel [5] proved a connection of the PASEP to the q-Eulerian numbers of [21], thereby generalizing the result of Derrida *et al* to include the case where again $\alpha = \beta = 1$ but q is general. Shortly thereafter we proved [7] a much stronger result in the case of general α, β and q, showing that in fact the (normalized) probability of being in a particular state of the PASEP can be viewed as a certain *weight generating function* for *permutation tableaux* (certain $0-1$ tableaux) of a fixed shape – this is a Laurent polynomial in α, β, and q. However, our proof relied on the *matrix ansatz* of Derrida *et al* [9], and hence did not give an intuitive explanation of why one should expect the steady state of the PASEP to be related to such nice combinatorics.

The goal of this paper is to construct a Markov chain on permutation tableaux (which we call the PT chain) which *projects* to the PASEP: that is, after projection via a certain surjective map between the spaces of states, a walk on the state diagram of the PT chain is indistinguishable from a walk on the state diagram of the PASEP. The steady state distribution of the PT chain has the nice property that the (normalized) probability of being in a particular state (i.e. a permutation tableau) is the *weight* of that permutation tableau – this is a Laurent *monomial* in α, β, and q. Our construction generalizes the work of Duchi and Schaeffer [11] (whose work can be viewed as the $q = 0$ case of ours), and gives a new proof of the main result of [7]. Additionally by using the bijection of [19], we can view the PT chain as a Markov chain on the symmetric group. Finally, the PT chain possesses a certain symmetry which extends the *particle-hole symmetry* of the PASEP: this is a graph-automorphism on the state diagram of the PT chain which is an involution.

The structure of this paper is as follows. In Section 2 we define the PASEP. In Section 3 we define permutation tableaux, certain $0-1$ tableaux which are naturally in bijection with permutations. Section 4 defines the PT chain, introduces the notion of *projection* of Markov chains, and proves a tight relationship between the steady state distributions of two Markov chains when one projects to the other. It will be clear from the definition that the PT chain projects to the PASEP. Section 5 states and proves our main result about the steady state distribution of the PT chain, thus giving a new combinatorial proof of our main result from [7]. Section 6 recalls the definition Φ from [19], and uses it to describe both the PT chain as a Markov chain on permutations. Finally, Section 7 describes an involution on the state-diagram of the PT chain which extends the particle-hole symmetry.

It would be interesting to explore whether the PT chain has any physical significance, and whether this larger chain may shed some insight on the PASEP itself.

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their comments and encouragement. Additionally, we thank the referee for advice that enabled us to considerably simplify our arguments concerning the involution.

2. The PASEP

In the physics literature, the PASEP is defined as follows.

Definition 2.1. *We are given a one-dimensional lattice of N sites, such that each site i ($1 \leq i \leq N$) is either occupied by a particle ($\tau_i = 1$) or is empty ($\tau_i = 0$). At most one particle may occupy a given site. During each infinitesimal time interval dt, each particle in the system has a probability dt of jumping to the next site on its right (for particles on sites $1 \leq i \leq N-1$) and a probability qdt of jumping to the next site on its left (for particles on sites $2 \leq i \leq N$). Furthermore, a particle is added at site $i = 1$ with probability αdt if site 1 is empty and a particle is removed from site N with probability βdt if this site is occupied.*

Remark 2.2. *Note that we will sometimes denote a state of the PASEP as a word in $\{0,1\}^N$ and sometimes as a word in $\{\circ, \bullet\}^N$. In the latter notation, the symbol $\circ$ denotes the absence of a particle, which one can also think of as a white particle.*

It is not too hard to see [11] that our previous formulation of the PASEP is equivalent to the following discrete-time Markov chain.

Definition 2.3. *Let B_N be the set of all 2^N words in the language $\{\circ, \bullet\}^*$. The PASEP is the Markov chain on B_N with transition probabilities:*

- *If $X = A \bullet \circ B$ and $Y = A \circ \bullet B$ then $P_{X,Y} = \frac{1}{N+1}$ (particle hops right) and $P_{Y,X} = \frac{q}{N+1}$ (particle hops left).*
- *If $X = \circ B$ and $Y = \bullet B$ then $P_{X,Y} = \frac{\alpha}{N+1}$ (particle enters from left).*
- *If $X = B \bullet$ and $Y = B \circ$ then $P_{X,Y} = \frac{\beta}{N+1}$ (particle exits to the right).*
- *Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.*

See Figure 1 for an illustration of the four states, with transition probabilities, for the case $N = 2$.

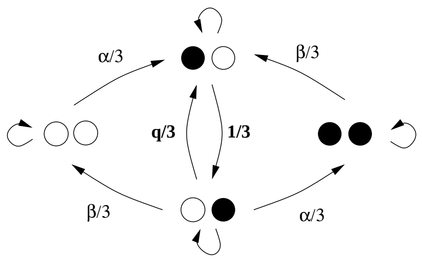

Figure 1. The state diagram of the PASEP for $N = 2$

In the long time limit, the system reaches a steady state where all the probabilities $P_N(\tau_1, \tau_2, \ldots, \tau_N)$ of finding the system in configurations $(\tau_1, \tau_2, \ldots, \tau_N)$ are stationary, i.e. satisfy

$$\frac{d}{dt} P_N(\tau_1, \ldots, \tau_N) = 0.$$

Moreover, the stationary distribution is unique [9], as shown by Derrida *et al.*

The question is now to solve for the probabilities $P_N(\tau_1, \dots, \tau_N)$. For convenience, we work with unnormalized weights $g_N(\tau_1, \dots, \tau_N)$ which are equal to the $P_N(\tau_1, \dots, \tau_N)$ up to a constant:

$$P_N(\tau_1, \dots, \tau_N) = g_N(\tau_1, \dots, \tau_N)/Z_N,$$

where Z_N is the *partition function* $\sum_\tau g_N(\tau_1, \dots, \tau_N)$. The sum defining Z_N is over all possible configurations $\tau \in \{0,1\}^N$.

Remark 2.4. *There is an obvious particle-hole symmetry* [9] *in the PASEP: since (black) particles enter at the left with probability α and exit to the right with probability β, it is equivalent to saying that holes (or white particles) are injected at the right with probability β and are removed at the left end with probability α.*

Let us define $\overline{(\tau_1, \dots, \tau_N)} = (1-\tau_N, 1-\tau_{N-1}, \dots, 1-\tau_1)$. Clearly this operation is an involution on states of the PASEP. Because of the particle-hole symmetry, one always has that $g_N^{q,\alpha,\beta}(\tau) = g_N^{q,\beta,\alpha}(\overline{\tau})$.

3. Connection with permutation tableaux

We define a *partition* $\lambda = (\lambda_1, \dots, \lambda_K)$ to be a weakly decreasing sequence of *nonnegative* integers. For a partition λ, where $\sum \lambda_i = m$, the *Young diagram* Y_λ of shape λ is a left-justified diagram of m boxes, with λ_i boxes in the ith row. We define the *half-perimeter* of λ or Y_λ to be the the number of rows plus the number of columns. The *length* of a row or column of a Young diagram is the number of boxes in that row or column. Note that we will allow a row to have length 0.

We will often identify a Young diagram Y_λ of half-perimeter t with the lattice path $p(\lambda)$ of length t which takes unit steps south and west, beginning at the north-east corner of Y_λ and ending at the south-west corner. Note that such a lattice path always begins with a step south. Figure 2 shows the path corresponding to the Young diagram of shape $(2, 1, 0)$.

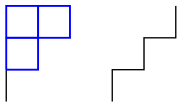

FIGURE 2. Young diagram and path for $\lambda = (2, 1, 0)$

If $\tau \in \{0,1\}^N$, we define a Young diagram $\lambda(\tau)$ of half-perimeter $N+1$ as follows. First we define a path $p = (p_1, \dots, p_{N+1}) \in \{S, W\}^{N+1}$ such that $p_1 = S$, and $p_{i+1} = S$ if and only if $\tau_i = 1$. We then define $\lambda(\tau)$ to be the partition associated to this path p. This map is clearly a bijection between the set of Young diagrams of half-perimeter $N+1$ and the set of N-tuples in $\{0,1\}^N$, and we denote the inverse map similarly: given a Young diagram λ of half-perimeter $N+1$, we define $\tau(\lambda)$ to be the corresponding N-tuple.

As in [19], we define a *permutation tableau* $\mathcal{T}$ to be a partition λ together with a filling of the boxes of Y_λ with 0's and 1's such that the following properties hold:

(1) Each column of the rectangle contains at least one 1.
(2) There is no 0 which has a 1 above it in the same column *and* a 1 to its left in the same row.

We call such a filling a *valid* filling of Y_λ.

Remark 3.1. *Permutation tableaux are closely connected to total positivity for the Grassmannian* [15, 21]. *More precisely, if we forget the requirement (1) above we recover the definition of a* Ⅎ*-diagram, an object which represents a cell in the totally nonnegative part of the Grassmannian. It would be interesting to explore whether there is a connection between total positivity and the PASEP.*

Remark 3.2. *Sometimes we will depict permutation tableaux slightly differently, replacing the* 1*'s with black dots and omitting the* 0*'s entirely, as in Figure 4.*

Note that the second requirement above can be rephrased in the following way. Read the columns of a permutation tableau $\mathcal{T}$ from right to left. If in any column we have a 0 which lies beneath some 1, then all entries to the left of 0 (which are in the same row) must also be 0's.

We now define a few statistics on permutation tableaux. A 1 in a tableau $\mathcal{T}$ is *topmost* if it has only 0's above it; a 1 in $\mathcal{T}$ is *superfluous* if it is not topmost; and a 1 is *necessary* if it is the unique 1 in its column. We define the *rank* $\mathrm{rk}(\mathcal{T})$ of a permutation tableau $\mathcal{T}$ to be the number of superfluous 1's. (Therefore $\mathrm{rk}(\mathcal{T})$ is equal to the total number of 1's in the filling minus the number of columns.) We define $f(\mathcal{T})$ to be the number of 1's in the first row of $\mathcal{T}$. We say that a zero in a permutation tableau is *restricted* if there is a one above it in the same column. And we say that a row is *unrestricted* if it does not contain a restricted entry. Define $u(\mathcal{T})$ to be the number of unrestricted rows of $\mathcal{T}$ minus 1. (We subtract 1 since the top row of a tableau is always unrestricted.)

Figure 3 gives an example of a permutation tableau $\mathcal{T}$ with rank $19 - 10 = 9$ and half-perimeter 17, such that $u(\mathcal{T}) = 3$ and $f(\mathcal{T}) = 5$.

0	1	1	0	0	1	0	1	0	1
1	1	1	1	0	1	1	1	1	
0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	1	1		
0	0	0	1	1					
1	1								

$\lambda = (10, 9, 9, 8, 5, 2, 0)$

FIGURE 3. A permutation tableau

We define the *weight* of a tableau $\mathcal{T}$ to be the monomial $\mathrm{wt}(\mathcal{T}) := q^{\mathrm{rk}(\mathcal{T})}\alpha^{-f(\mathcal{T})}\beta^{-u(\mathcal{T})}$, and we define $F_\lambda(q)$ to be the (Laurent) polynomial $\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})$, where the sum ranges over all permutation tableaux $\mathcal{T}$ of shape λ.

Our main result of [7] was the following.

Theorem 3.3. *Fix $\tau = (\tau_1, \dots, \tau_N) \in \{0,1\}^N$, and let $\lambda := \lambda(\tau)$. (Note that* half-perim$(\lambda) = N + 1$.*) The probability of finding the PASEP in configuration $(\tau_1, \dots, \tau_N)$ in the steady state is*

$$\frac{F_\lambda(q)}{Z_N}.$$

Here, $F_\lambda(q)$ is the weight-generating function for permutation tableaux of shape λ. Moreover, the partition function Z_N for the PASEP is equal to the weight-generating function for all permutation tableaux of half-perimeter $N + 1$.

However, our proof relied on the *matrix ansatz* of Derrida *et al.* In this paper we will give another proof of that result, which bypasses the matrix ansatz and gives a better explanation of the connection between permutation tableaux and the PASEP.

4. The PT chain

In this section we present our main construction, a Markov chain on permutation tableaux which projects to the PASEP. We will call this chain the *PT chain*.

Definition 4.1. *We define a projection operator* pr *which projects a state of the PT, i.e. a permutation tableau, to a state of the PASEP. If $\mathcal{T}$ is a permutation tableau of shape λ and half-perimeter $N+1$, then we define* $\mathrm{pr}(\mathcal{T}) := \tau(\lambda)$*. This is a state of the PASEP with N sites.*

Before defining the PT chain, we show an example: the state diagram of the chain for $N = 3$. In Figure 4, the $24 = 4!$ states of the PT chain are arranged into $8 = 2^3$ groups according to partition shape (four of size 1, two of size 3, two of size 7). All elements of a fixed group of tableaux project to the same state of the PASEP, depicted just above that group. We have not included the transition probabilities in Figure 4, but they are defined in the obvious manner: if there is a transition $\mathcal{S} \to T$ in the PT chain, then $\mathrm{prob}_{PT}(\mathcal{S} \to T) = \mathrm{prob}_{PASEP}(\mathrm{pr}(\mathcal{S}) \to \mathrm{pr}(T))$. Finally, observe that there is a reflective left-right symmetry in the figure.

We now define all possible transitions in the PT chain. There are four kinds of transitions, which correspond to the four kinds of transitions in the PASEP. The transition probabilities in the PT chain will be defined in accordance with the transition probabilities in the PASEP: if there is a transition $\mathcal{S} \to T$ in the PT chain, then $\mathrm{prob}_{PT}(\mathcal{S} \to T) := \mathrm{prob}_{PASEP}(\mathrm{pr}(\mathcal{S}) \to \mathrm{pr}(T))$.

In what follows, we will assume that $\mathcal{S}$ is a permutation tableau of half-perimeter $N+1$, whose shape is $\lambda = (\lambda_1, \dots, \lambda_m, \dots, \lambda_t)$ where $\lambda_1 \geq \cdots \geq \lambda_m > 0$, and $\lambda_r = 0$ for $r > m$.

4.1. **Particle enters from the left.** If the rightmost column of $\mathcal{S}$ has length 1, then there is a transition in the PT chain from $\mathcal{S}$ that corresponds to a particle entering from the left in the PASEP.

We now define a new permutation tableau $\mathcal{T}$ as follows: delete the rightmost column of $\mathcal{S}$ and add a new all-zero row of length $\lambda_1 - 1$ to $\mathcal{S}$, inserting it as far south as possible (subject to the constraint that the lengths of the rows of a

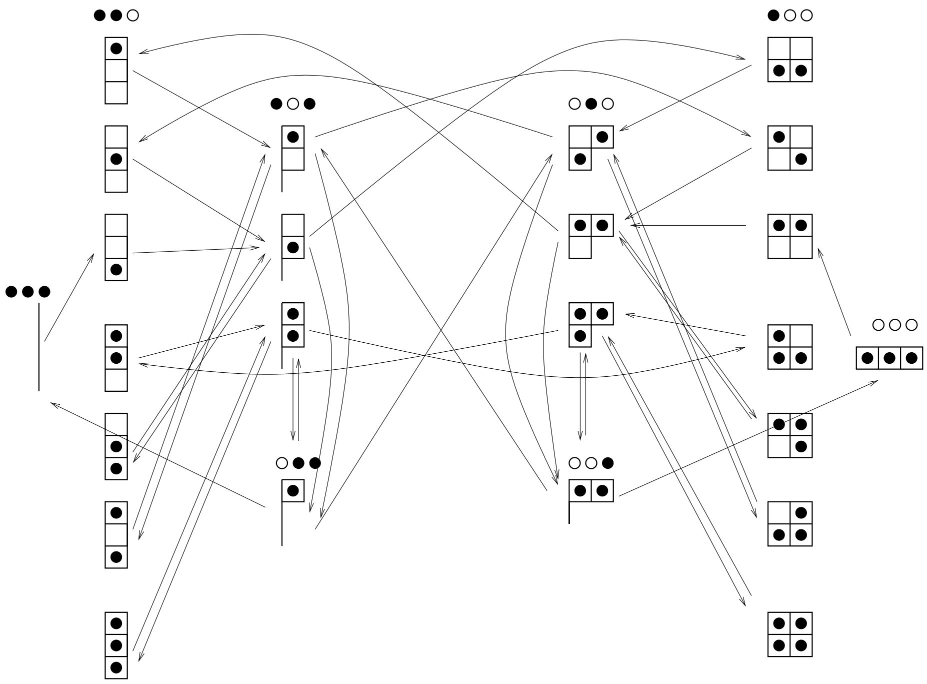

FIGURE 4. The state diagram of the PT chain for $N = 3$

permutation tableau must weakly decrease). Clearly adding an all-zero row in this way results in a new permutation tableau, since there is no way to introduce the forbidden pattern (condition (2) in the definition of permutation tableau), and each column will still contain at least one 1. See Figure 5.

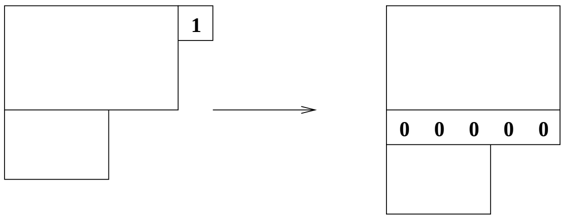

FIGURE 5

We define $\text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\alpha}{N+1}$. Since we have removed a 1 in the top row but have not affected the unrestricted rows or superfluous 1's, we have that $\text{wt}(\mathcal{T}) = \alpha \cdot \text{wt}(\mathcal{S})$, and therefore $\text{wt}(\mathcal{S}) \cdot \text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\text{wt}(\mathcal{T})}{N+1}$.

4.2. **Particle hops right.** If some row $\lambda_j > \lambda_{j+1}$ in $\mathcal{S}$ (resp. if $\lambda_t > 0$), then there is a transition in PT from $\mathcal{S}$ that corresponds to the $(j-1)$st black particle (resp. $(t-1)$st black particle) in $\text{pr}(\mathcal{S})$ hopping to the right in the PASEP.

We now define a new permutation tableau $\mathcal{T}$ as follows, based on the rightmost entry of the jth row of $\mathcal{S}$.

4.2.1. *Case 1.* Suppose that the rightmost entry of the jth row is a 0. This forces the jth row to contain only 0's. Then we define a new tableau $\mathcal{T}$ by deleting the jth row of $\mathcal{S}$ and adding a new row of $\lambda_j - 1$ 0's, inserting it as far south as possible. Clearly adding and deleting all-zero rows in this fashion results in a permutation tableau, so this operation is well-defined. See Figure 6.

We define $\text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{1}{N+1}$. If $\lambda_j > 1$ then we have not affected the number of superfluous 1's, unrestricted rows, or 1's in the top row, and so $\text{wt}(\mathcal{T}) = \text{wt}(\mathcal{S})$. It follows that $\text{wt}(\mathcal{S}) \cdot \text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\text{wt}(\mathcal{T})}{N+1}$.

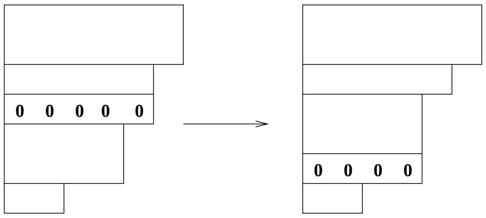

FIGURE 6

In the special case that $\lambda_j = 1$, then $\text{wt}(\mathcal{T}) = \beta^{-1}\,\text{wt}(\mathcal{S})$, and therefore $\text{wt}(\mathcal{S}) \cdot \text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\beta\,\text{wt}(\mathcal{T})}{N+1}$. Note that if this special case occurs then $\text{pr}(\mathcal{T})$ ends with a black particle. And given such a $\mathcal{T}$, there is only one such $\mathcal{S}$ with such a transition $\mathcal{S} \to \mathcal{T}$.

4.2.2. *Case 2.* Suppose that the rightmost entry of the jth row of $\mathcal{S}$ is a superfluous 1. Then we define a new tableau $\mathcal{T}$ by deleting that 1 from $\mathcal{S}$; because the column containing that 1 had at least two 1's to begin with, our new tableau will be a permutation tableau. See Figure 7.

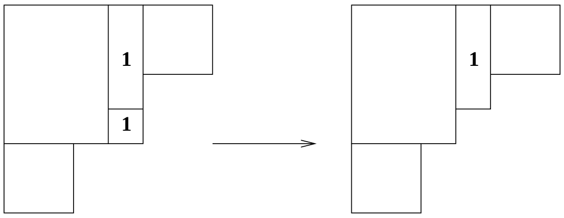

FIGURE 7

We define $\text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{1}{N+1}$. Our operation has affected only the number of superfluous 1's, so $\text{wt}(\mathcal{T}) = q^{-1}\,\text{wt}(\mathcal{S})$, and therefore $\text{wt}(\mathcal{S}) \cdot \text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{q\,\text{wt}(\mathcal{T})}{N+1}$.

4.2.3. *Case 3.* Suppose that the rightmost entry of the jth row of $\mathcal{S}$ is a necessary 1. Then we define a new tableau $\mathcal{T}$ by deleting the column containing the necessary 1 and adding a new column whose length is 1 less. That new column consists entirely of 0's except for a necessary 1 at the bottom, and it is inserted as far east a possible. Such a column cannot introduce a forbidden pattern (condition (2) in the definition of permutation tableau), so $\mathcal{T}$ is in fact a permutation tableau. See Figure 8.

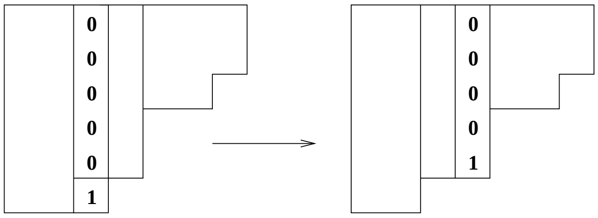

FIGURE 8

We define $\mathrm{prob}(\mathcal{S} \to \mathcal{T}) = \frac{1}{N+1}$. If the column in $\mathcal{S}$ containing the necessary 1 has length at least 2 then $\mathrm{wt}(\mathcal{T}) = \mathrm{wt}(\mathcal{S})$, because we have not changed the number of superfluous 1's, the 1's in the top row, or the number of unrestricted rows. Thus $\mathrm{wt}(\mathcal{S}) \cdot \mathrm{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\mathrm{wt}(\mathcal{T})}{N+1}$.

In the special case that the column in $\mathcal{S}$ containing the necessary 1 has length exactly 2 then $\mathrm{wt}(\mathcal{T}) = \alpha^{-1}\,\mathrm{wt}(\mathcal{S})$, because the new tableau will have a new 1 in the top row. Thus $\mathrm{wt}(\mathcal{S}) \cdot \mathrm{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\alpha\,\mathrm{wt}(\mathcal{T})}{N+1}$.

4.3. **Particle exits to the right.** If $\mathcal{S}$ contains a row of length 0 then there is a transition in PT from $\mathcal{S}$ that corresponds to a particle in $\mathrm{pr}(\mathcal{S})$ exiting the PASEP to the right. We define a new tableau $\mathcal{T}$ by deleting the tth row of $\mathcal{S}$ (which has length 0) and adding a new column of length $t-1$ which consists of $t-2$ 0's followed by a 1 (read top-to-bottom), inserting this column into the tableau as far to the right as possible. The result is clearly a permutation tableau. See Figure 9.

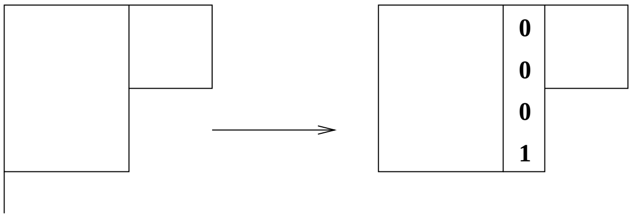

FIGURE 9

We define $\mathrm{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\beta}{N+1}$. Since we have deleted an unrestricted row (the row of length 0), we have $\mathrm{wt}(\mathcal{T}) = \beta\,\mathrm{wt}(\mathcal{S})$. Thus $\mathrm{wt}(\mathcal{S}) \cdot \mathrm{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\mathrm{wt}(\mathcal{T})}{N+1}$.

4.4. **Particle hops left.** If some row $\lambda_j > \lambda_{j+1}$ in $\mathcal{S}$ then there is a transition in PT from $\mathcal{S}$ that corresponds to the jth black particle in $\mathrm{pr}(\mathcal{S})$ hopping to the left in the PASEP. We define a new tableau $\mathcal{T}$ by increasing the length of the $(j+1)$st row by 1 and filling the extra square with a 1. The result is clearly a permutation tableau. See Figure 10.

We define $\mathrm{prob}(\mathcal{S} \to \mathcal{T}) = \frac{q}{N+1}$. Since we have added one superfluous 1, we have that $\mathrm{wt}(\mathcal{T}) = q\,\mathrm{wt}(\mathcal{S})$. Thus $\mathrm{wt}(\mathcal{S}) \cdot \mathrm{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\mathrm{wt}(\mathcal{T})}{N+1}$.

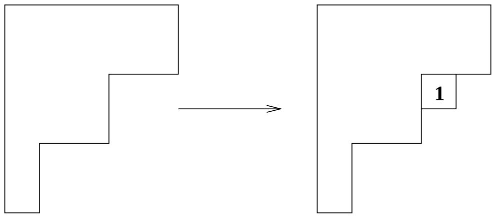

FIGURE 10

4.5. **Projection of Markov chains.** We now formulate a notion of *projection* for Markov chains. We have not been able to find this definition in the literature, but, for example, the Markov chain of Duchi and Schaeffer [11] is a projection in this sense.

Definition 4.2. *Let M and N be Markov chains on finite sets X and Y, and let F be a surjective map from X to Y. We say that M* projects *to N if the following properties hold:*

- *If x_1 and x_2 are in X such that $\text{prob}_M(x_1 \to x_2) > 0$, then $\text{prob}_M(x_1 \to x_2) = \text{prob}_N(F(x_1) \to F(x_2))$.*
- *If y_1 and y_2 are in Y and $\text{prob}_N(y_1 \to y_2) > 0$ then for each $x_1 \in X$ such that $F(x_1) = y_1$ there is a unique $x_2 \in X$ such that $F(x_2) = y_2$ and $\text{prob}_M(x_1 \to x_2) > 0$; moreover, $\text{prob}_M(x_1 \to x_2) = prob_N(y_1 \to y_2)$.*

Let $\text{prob}_M(x_0 \to x; t)$ denote the probability that if we start at state $x_0 \in M$ at time 0, then we are in state x at time t.

If M projects to N then a walk on the state diagram of M is indistinguishable from a walk on the state diagram of N in the following sense.

Proposition 4.3. *Suppose that M projects to N. Let $x_0 \in X$ and $y_0, \tilde{y} \in Y$ such that $F(x_0) = y_0$. Then*

$$\text{prob}_N(y_0 \to \tilde{y}; t) = \sum_{\tilde{x} \ s.t. \ F(\tilde{x}) = y} prob_M(x_0 \to \tilde{x}; t).$$

Proof. We use induction. The base case $t = 0$ is trivially true. Suppose that the statement is true for $t - 1$. Let $Y' \subset Y$ be the set of all states y' such that there is a transition $y' \to \tilde{y}$ with nonzero probability. Let $X' = \{x' \in X \mid F(x') \in Y'\}$. By the definition of projection, these are the only states in X which have a transition to a state $\tilde{x}$ such that $F(\tilde{x}) = \tilde{y}$. Furthermore, for each $x_1 \in X'$ there exists an $x_2 \in X$ such that $F(x_2) = \tilde{y}$ and $\text{prob}_M(x_1 \to x_2) > 0$, and $\text{prob}_M(x_1 \to x_2) = \text{prob}_N(y_1 \to y_2)$.

Clearly $\text{prob}_N(y_0 \to \tilde{y}; t) = \sum_{y' \in Y'} \text{prob}_N(y_0 \to y'; t-1) \cdot \text{prob}_N(y' \to \tilde{y})$, which by the induction hypothesis and the definition of projection is equal to

$$\sum_{y' \in Y'} \sum_{x' \text{ s.t. } F(x') = y'} \text{prob}_M(x_0 \to x'; t-1) \cdot \text{prob}_M(x' \to \tilde{x}).$$

This is equal to

$$\sum_{x' \text{ s.t. } F(x') \in Y'} \mathrm{prob}_M(x_0 \to x'; t-1) \cdot \mathrm{prob}_M(x' \to \tilde{x}).$$

But now using again the definition of projection, we see that this is equal to $\mathrm{prob}_M(x_0 \to x; t)$, as desired. □

Proposition 4.3 implies the following.

Corollary 4.4. *Suppose that M projects to N via the map F. Let $y \in Y$ and let $X' = \{x \in X \mid F(x) = y\}$. Then the steady state probability that N is in state y is equal to the steady state probabilities that M is in any of the states $x \in X'$.*

Clearly the operator pr is a surjective map from the set of permutation tableaux of half-perimeter $N+1$ to the states of the PASEP with N sites. It is clear from our definition of the PT chain that the PT chain projects to the PASEP.

Corollary 4.5. *The projection map* pr *gives a projection from the PT chain on permutation tableaux of half-perimeter $N+1$ to the PASEP with N sites.*

5. Steady state probabilities

Our main theorem is the following.

Theorem 5.1. *Consider the PT chain on permutation tableaux of half-perimeter $N+1$ and fix a permutation tableau $\mathcal{T}$ (of half-perimeter $N+1$). Then the steady state probability of finding the PT chain in state $\mathcal{T}$ is $\frac{\mathrm{wt}(\mathcal{T})}{\sum_{\mathcal{S}} \mathrm{wt}(\mathcal{S})}$. Here, the sum is over all permutation tableaux of half-perimeter $N+1$.*

Since the PT chain projects to the PASEP, Corollary 4.4 and Theorem 5.1 imply Theorem 3.3.

The goal of this section will be to prove Theorem 5.1. To do so, we will check the defining recurrences of the steady state. More precisely, it suffices to check the following. Fix a state $\mathcal{T}$, let $\mathcal{Q}$ be the collection of all states that have transitions to $\mathcal{T}$, and let $\mathcal{S}$ be the collection of all states that have transitions from $\mathcal{T}$. Then we need to prove that

$$\mathrm{wt}(\mathcal{T}) = \sum_{Q \in \mathcal{Q}} \mathrm{wt}(Q) \cdot \mathrm{prob}(Q \to \mathcal{T}) + \mathrm{wt}(\mathcal{T}) \cdot (1 - \sum_{\mathcal{S} \in \mathcal{S}} \mathrm{prob}(\mathcal{T} \to \mathcal{S})).$$

The above equation expresses the steady state probability of being in state $\mathcal{T}$ in two ways, involving two consecutive times t and $t+1$; equivalently, it encodes the condition that the transition matrix has a left eigenvector with eigenvalue 1. Combining the terms involving $\mathrm{wt}(\mathcal{T})$, we get

$$\text{(E1)} \qquad \sum_{Q \in \mathcal{Q}} \mathrm{wt}(Q) \cdot \mathrm{prob}(Q \to \mathcal{T}) = \mathrm{wt}(\mathcal{T}) \cdot \sum_{\mathcal{S} \in \mathcal{S}} \mathrm{prob}(\mathcal{T} \to \mathcal{S}).$$

In order to check this, we will divide the set of states in the PT into four different classes. In what follows, let B represent a nonempty string of black particles and let W represent a nonempty string of white particles. Then we divide the set of permutation tableaux of half-perimeter N into the four classes as follows:

(1) $\mathcal{T}$ such that $\mathrm{pr}(\mathcal{T})$ has the form BWBW...BW. ($2n$ strings)
(2) $\mathcal{T}$ such that $\mathrm{pr}(\mathcal{T})$ has the form BWBW...BWB. ($2n+1$ strings)
(3) $\mathcal{T}$ such that $\mathrm{pr}(\mathcal{T})$ has the form WBWB...WBW. ($2n+1$ strings)
(4) $\mathcal{T}$ such that $\mathrm{pr}(\mathcal{T})$ has the form WBWB...WB. ($2n$ strings)

5.1. **Analysis of transitions out of state $\mathcal{T}$.** From a state of type (1), there are n possible transitions of the form "hop right," and $n-1$ possible transitions of the form "hop left."

From a state of type (2), there are n possible "hop right" transitions, n possible "hop left" transition, and one "hop out to the right" transition.

From a state of type (3), there are n "hop right" transitions, n "hop left" transitions, and one "hop in from the left" transition.

From a state of type (4), there are $n-1$ "hop right" transitions, n "hop left" transitions, one "hop in from the left," and one "hop out from right."

We can now calculate the quantity $\sum_{\mathcal{S}\in\mathcal{S}} \mathrm{prob}(\mathcal{T}\to\mathcal{S})$ in all four cases. These quantities are as follows:

(1) $(n+q(n-1))/(N+1)$
(2) $(n+qn+\beta)/(N+1)$
(3) $(n+qn+\alpha)/(N+1)$
(4) $((n-1)+qn+\alpha+\beta)/(N+1)$

5.2. **Analysis of transitions into state $\mathcal{T}$.**

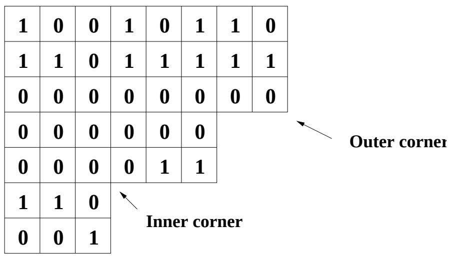

FIGURE 11. A state $\mathcal{T}$ of type (1), where $n=3, a=1, b=1, c=1$

5.2.1. *Type (1).* Fix a state $\mathcal{T}$ of type (1). The Young diagram of $\mathcal{T}$ will have n outer corners (south step followed by west step), and $n-1$ inner corners (west step followed by south step). If we say only "corner," this will mean an outer corner.

Of the entries in the n outer corners, a of them are 0's, b of them are necessary 1's, and $n-a-b$ are superfluous 1's.

For each 0 corner, there is a state Q such that $\mathrm{prob}(Q\to\mathcal{T})>0$ and such that $\mathrm{wt}(Q)\cdot\mathrm{prob}(Q\to\mathcal{T})=\frac{\mathrm{wt}(\mathcal{T})}{N+1}$. Therefore the 0 corners contribute $\frac{a\,\mathrm{wt}(\mathcal{T})}{N+1}$ to the left-hand-side of equation (E1).

For each corner which is a superfluous 1, there is a state Q such that $\mathrm{prob}(Q\to\mathcal{T})>0$ and such that $\mathrm{wt}(Q)\cdot\mathrm{prob}(Q\to\mathcal{T})=\frac{\mathrm{wt}(\mathcal{T})}{N+1}$. Therefore these superfluous 1's contribute $\frac{(n-a-b)\,\mathrm{wt}(\mathcal{T})}{N+1}$ to the left-hand-side of equation (E1).

For each corner which is a necessary 1, there is a state Q such that $\mathrm{prob}(Q \to \mathcal{T}) > 0$ and such that $\mathrm{wt}(Q) \cdot \mathrm{prob}(Q \to \mathcal{T}) = \frac{\mathrm{wt}(\mathcal{T})}{N+1}$. Therefore these necessary 1's contribute $\frac{b\,\mathrm{wt}(\mathcal{T})}{N+1}$ to the left-hand-side of equation (E1).

For each inner corner, there is a state Q such that $\mathrm{prob}(Q \to \mathcal{T}) > 0$ and such that $\mathrm{wt}(Q) \cdot \mathrm{prob}(Q \to \mathcal{T}) = \frac{q\,\mathrm{wt}(\mathcal{T})}{N+1}$. Therefore these inner corners contribute $\frac{q(n-1)\,\mathrm{wt}(\mathcal{T})}{N+1}$ to the left-hand-side of equation (E1).

The sum of all of these contributions is $\frac{\mathrm{wt}(\mathcal{T})}{N+1}(n+q(n-1))$. Comparing this with the results of Subsection 5.1, we see that equation (E1) holds for states of Type (1).

1	0	0	1	0	1	1	0
1	1	0	1	1	1	1	1
0	0	0	0	0	0	0	0
0	0	0	0	0	0		
0	0	0	0	1	1		
1	1	0					
0	0	1					

FIGURE 12. A state $\mathcal{T}$ of type (2), where $n = 3$

5.2.2. *Type (2).* Fix a state $\mathcal{T}$ of type (2). The Young diagram of $\mathcal{T}$ will have n outer corners, and n inner corners.

Of the entries in the n outer corners, a of them are 0's, b of them are necessary 1's, and $n-a-b$ are superfluous 1's.

As before, the 0 corners contribute $\frac{a\,\mathrm{wt}(\mathcal{T})}{N+1}$ to the left-hand-side of equation (E1).

Similarly, the corners containing superfluous 1's contribute $\frac{(n-a-b)\,\mathrm{wt}(\mathcal{T})}{N+1}$ to the left-hand-side of equation (E1).

The corners containing necessary 1's contribute $\frac{b\,\mathrm{wt}(\mathcal{T})}{N+1}$ to the left-hand-side of equation (E1).

The n inner corners contribute $\frac{qn\,\mathrm{wt}(\mathcal{T})}{N+1}$ to the left-hand-side of equation (E1).

The final empty row of the Young diagram contributes $\frac{\beta\,\mathrm{wt}(\mathcal{T})}{N+1}$ to equation (E1).

The sum of all of these contributions is $\frac{\mathrm{wt}(\mathcal{T})}{N+1}(n+qn+\beta)$. Comparing this with what we found in Subsection 5.1, we see that equation (E1) holds for states of Type (2).

5.2.3. *Type (3).* Fix a state $\mathcal{T}$ of type (3). The Young diagram of $\mathcal{T}$ will have n outer corners (not counting the corner formed by the rightmost column of length 1), and n inner corners.

Of the entries in the n outer corners, a of them are 0's, b of them are necessary 1's, and $n-a-b$ are superfluous 1's.

As before, the 0 corners contribute $\frac{a\,\mathrm{wt}(\mathcal{T})}{N+1}$ to the left-hand-side of equation (E1).

The corners containing superfluous 1's contribute $\frac{(n-a-b)\,\mathrm{wt}(\mathcal{T})}{N+1}$ to the left-hand-side of equation (E1).

1	0	0	1	0	1	1	0	1	1
1	1	0	1	1	1	1	1		
0	0	0	0	0	0	0	0		
0	0	0	0	0	0				
0	0	0	0	1	1				
1	1	0							
0	0	1							

FIGURE 13. A state $\mathcal{T}$ of type (3), where $n = 3$

The corners containing necessary 1's contribute $\frac{b\,\mathrm{wt}(\mathcal{T})}{N+1}$ to the left-hand-side of equation (E1).

The n inner corners contribute $\frac{qn\,\mathrm{wt}(\mathcal{T})}{N+1}$ to the left-hand-side of equation (E1).

The (leftmost) column of length 1 contributes $\frac{\alpha\,\mathrm{wt}(\mathcal{T})}{N+1}$ to equation (E1).

The sum of all of these contributions is $\frac{\mathrm{wt}(\mathcal{T})}{N+1}(n + qn + \alpha)$. Comparing this with Subsection 5.1, we see that equation (E1) holds for states of Type (3).

1	0	0	1	0	1	1	0	1	1
1	1	0	1	1	1	1	1		
0	0	0	0	0	0	0	0		
0	0	0	0	0	0				
0	0	0	0	1	1				
1	1	0							
0	0	1							

FIGURE 14. A state $\mathcal{T}$ of type (4), where $n = 4$

5.2.4. *Type (4).* Fix a state $\mathcal{T}$ of type (4). The Young diagram of $\mathcal{T}$ will have $n-1$ outer corners (not counting the corner formed by the rightmost column of length 1), and n inner corners.

Of the entries in the $n-1$ outer corners, a of them are 0's, b of them are necessary 1's, and $n-1-a-b$ are superfluous 1's.

As before, the 0 corners contribute $\frac{a\,\mathrm{wt}(\mathcal{T})}{N+1}$ to the left-hand-side of equation (E1).

The corners containing superfluous 1's contribute $\frac{(n-1-a-b)\,\mathrm{wt}(\mathcal{T})}{N+1}$ to the left-hand-side of equation (E1).

The corners containing necessary 1's contribute $\frac{b\,\mathrm{wt}(\mathcal{T})}{N+1}$ to the left-hand-side of equation (E1).

The n inner corners contribute $\frac{qn\,\mathrm{wt}(\mathcal{T})}{N+1}$ to the left-hand-side of equation (E1).

The (leftmost) column of length 1 contributes $\frac{\alpha\,\mathrm{wt}(\mathcal{T})}{N+1}$ to equation (E1).

The final empty row of the Young diagram contributes $\frac{\beta\,\mathrm{wt}(\mathcal{T})}{N+1}$ to equation (E1).

The sum of all of these contributions is $\frac{\mathrm{wt}(\mathcal{T})}{N+1}((n-1) + qn + \alpha + \beta)$. Comparing this with Subsection 5.1, we see that equation (E1) holds for states of Type (4).

This completes the proof of the main theorem.

6. The PT chain as a Markov chain on permutations

There is a bijection Φ between the set of permutation tableaux of half-perimeter n and the permutations in S_n [19], which translates various statistics on permutation tableaux into statistics on permutations. This bijection allows us to interpret the PT chain as a Markov chain on permutations.

In this section we will describe the PT chain in terms of permutations. First we need to recall the definition of the bijection from [19].

6.1. The bijection from permutation tableaux to permutations.

Before defining Φ, it is necessary to introduce some notation. A *weak excedance* of a permutation π is an index i such that $\pi(i) \geq i$. A non-excedance is an index i such that $\pi(i) < i$.

We define the *diagram* $D(\mathcal{T})$ associated with a permutation tableaux $\mathcal{T}$ of shape λ as follows. Recall that $p(\lambda)$ is the lattice path which cuts out the south-east border of λ. From north-east to south-west, label each of the (unit) steps in this path with a number from 1 to n. Then, remove the 0's from $\mathcal{T}$ and replace each 1 in $\mathcal{T}$ with a vertex. Finally, from each vertex v, draw an edge to the east and an edge to the south; each such edge should connect v to either a closest vertex in the same row or column, or to one of the labels from 1 to n. The resulting picture is the *diagram* $D(\mathcal{T})$. See Figure 15.

We now define the permutation $\pi = \Phi(\mathcal{T})$ via the following procedure. For each $i \in \{1, \dots, n\}$, find the corresponding position on $D(\mathcal{T})$ which is labeled by i. If the label i is on a vertical step of P, start from this position and travel straight west as far as possible on edges of $D(\mathcal{T})$. Then, take a "zig-zag" path southeast, by traveling on edges of $D(\mathcal{T})$ south and east and turning at each opportunity (i.e. at each new vertex). This path will terminate at some label $j \geq i$, and we define $\pi(i) = j$. If i is not connected to any edge (equivalently, if there are no vertices in the row of i) then we set $\pi(i) = i$. Similarly, if the label i is on a horizontal step of P, start from this position and travel north as far as possible on edges of $D(\mathcal{T})$. Then, as before, take a zig-zag path south-east, by traveling on edges of $D(\mathcal{T})$ east and south, and turning at each opportunity. This path will terminate at some label $j < i$, and we let $\pi(i) = j$.

See Figure 16 for a picture of the path taken by i.

Example 6.1. *If $\mathcal{T}$ is the permutation tableau whose diagram is given in Figures 15 and 16, then* $\Phi(\mathcal{T}) = 74836215$.

Various properties of Φ were proved in [19]; we now recall those that will be useful to us.

Proposition 6.2. [19]

- *In $\Phi(\mathcal{T})$, the letter i is a fixed point if and only if there is an entire row in $\mathcal{T}$ that has no 1's and whose right hand edge is labeled by i.*

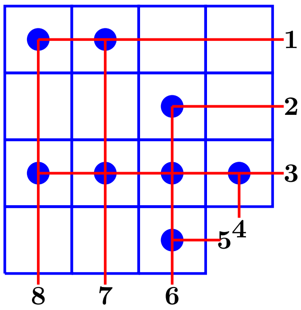

FIGURE 15. The diagram of a tableau.

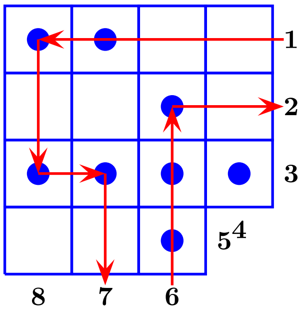

FIGURE 16. The paths taken by 1 and 6: $\pi(1) = 7$, $\pi(6) = 2$.

- *The weak excedances of $\pi = \Phi(\mathcal{T})$ are precisely the labels on the vertical edges of P. The non-excedances of π are precisely the labels on the horizontal edges of P. In particular, $\Phi(\mathcal{T})$ is a permutation in S_n with precisely k weak excedances.*

6.2. **The PT chain on permutations.** Now that we have defined Φ, it is a straightforward exercise to translate the PT chain on permutation-tableaux into a Markov chain on permutations.

For convenience, we will first make two definitions. Consider a permutation π on a set S. Let i be in S such that $\pi(i) \neq i$. We define the *collapse of π at i* to be the permutation σ on the set $S \backslash \{i\}$ defined by $\sigma(\pi^{-1}(i)) := \pi(i)$ and $\sigma(j) = \pi(j)$ for $j \neq \pi^{-1}(i)$. For example if $S = \{3, 4, 6, 7\}$ and $\pi = (6, 7, 4, 3)$, then after collapsing π at 7 we get $\sigma = (6, 3, 4)$, a permutation on the set $\{3, 4, 6\}$.

We also want the notion of normalizing a permutation. If π is a permutation on an ordered set $S = \{s_1 < s_2 < ... < s_n\}$ of cardinality n, then the *normalization of* π $\mathrm{Norm}(\pi)$ is the permutation on $\{1, 2, ..., n\}$ that we get by replacing s_i with i.

We are now ready to describe the PT chain in terms of permutations. An example is shown in Figure 17, which is simply the result of applying Φ to Figure 4.

In what follows, we will use the notation $\{a, \widehat{b}, c\}$ to denote the set $\{a, c\}$.

Proposition 6.3. *After applying Φ, the PT chain is the Markov chain on the permutations S_{N+1} with the following transitions. In all cases, $\pi \in S_{N+1}$.*

- *"Particle enters from the left"*

Suppose that $\pi(2) = 1$. Let $i+1$ be the minimal number greater than 2 which is a non-excedance in π. Define σ to be the permutation on $\{1, \widehat{2}, 3, \ldots, N+1\} \cup \{i + \frac{1}{2}\}$ obtained by collapsing π at 2, and adding a fixed point at $i + \frac{1}{2}$. In this case there is a transition $\pi \to \mathrm{Norm}(\sigma)$ with probability $\mathrm{prob}(\pi \to \mathrm{Norm}(\sigma)) = \frac{\alpha}{N+1}$.

- *"Particle hops right"*

Suppose that for some $i \geq 2$, $\pi(i) \geq i$ and $\pi(i+1) < i+1$. Define σ as follows:

Case 1: If $\pi(i) = i$ then let $j+1$ be the smallest non-excedance of π such that $j > i$. Define σ to be the permutation on $\{1, 2, \ldots, i-1, \widehat{i}, i+1, \ldots, N+1\} \cup \{i + \frac{1}{2}\}$ obtained by collapsing π at i and inserting a new fixed point at $(j + 1/2)$.

Case 2: If $\pi(i) > i$ and $\pi(i+1) < i$, then define σ by $\sigma(i) = \pi(i+1)$, $\sigma(i+1) = \pi(i)$, and $\sigma(j) = \pi(j)$ for $j \neq i, i+1$.

Case 3: If $\pi(i) > i$ and $\pi(i+1) = i$, then let j be the greatest number less than i which is a weak excedance (it exists since 1 is a weak excedance). Let $b = \sigma^{-1}(j)$. Let σ be the permutation on $\{1, 2, \ldots, \widehat{i+1}, \ldots, N+1\} \cup \{j + \frac{1}{2}\}$ obtained by collapsing π at $i+1$, and replacing $\pi(b) = i$ by $\sigma(b) = i + \frac{1}{2}$ and $\sigma(i + \frac{1}{2}) = i$.

Then there is a transition $\pi \to \mathrm{Norm}(\sigma)$ with probability $\frac{1}{N+1}$.

- *"Particle exits to the right"*

If $\pi(N+1) = N+1$, then let i be the maximal number less than $N+1$ such that $\pi(i) \geq i$, and let $a = \pi^{-1}(i)$. Define σ to be the permutation on $\{1, 2, \ldots, N\} \cup \{i + \frac{1}{2}\}$ obtained by collapsing π at $N+1$ and replacing $\pi(a) = i$ by $\sigma(a) = i + \frac{1}{2}$ and $\sigma(i + \frac{1}{2}) = i$. Then there is a transition $\pi \to \mathrm{Norm}(\sigma)$ such that $\mathrm{prob}(\pi \to \mathrm{Norm}(\sigma)) = \frac{\beta}{N+1}$.

- *"Particle hops left"*

If $\pi(i) < i$ and $\pi(i+1) \geq i+1$ then define σ to be the permutation on $\{1, 2, \ldots, N+1\}$ defined by $\sigma(i) = \pi(i+1)$, $\sigma(i+1) = \pi(i)$ and $\sigma(j) = \pi(j)$ for $j \neq i, i+1$. Then there is a transition $\pi \to \sigma$ such that $\mathrm{prob}(\pi \to \sigma) = \frac{q}{N+1}$.

Proof. This proof follows easily from the definition of the PT chain and the bijection Φ. Clearly inserting a new all-zero row into a tableau corresponds to inserting a new fixed point into a permutation, i.e. inserting a "minimal weak excedance" $i \to i$. Inserting a new column which consists from top to bottom of zeros and a single one corresponds to adding a new "minimal non-excedance" $i+1 \to i$. Finally, adding or removing an outer corner to a tableau corresponds to switching the values of $\pi(i)$ and $\pi(i+1)$. $\square$

Now that we have defined the PT chain in terms of permutations, we need to define the maps and statistics on permutations that are relevant to the PASEP.

We define a surjective map pr from S_{N+1} to states of the PASEP with N sites as follows: if $\pi \in S_{N+1}$ has W as its set of weak-excedances, then $\mathrm{pr}(\pi) = (a_1, \ldots, a_N) \in \{0,1\}^N$ is defined by $a_i = 1$ if $i+1 \in W$ and $a_i = 0$ otherwise. Clearly the map pr on permutation tableaux corresponds via Φ to this map pr on permutations.

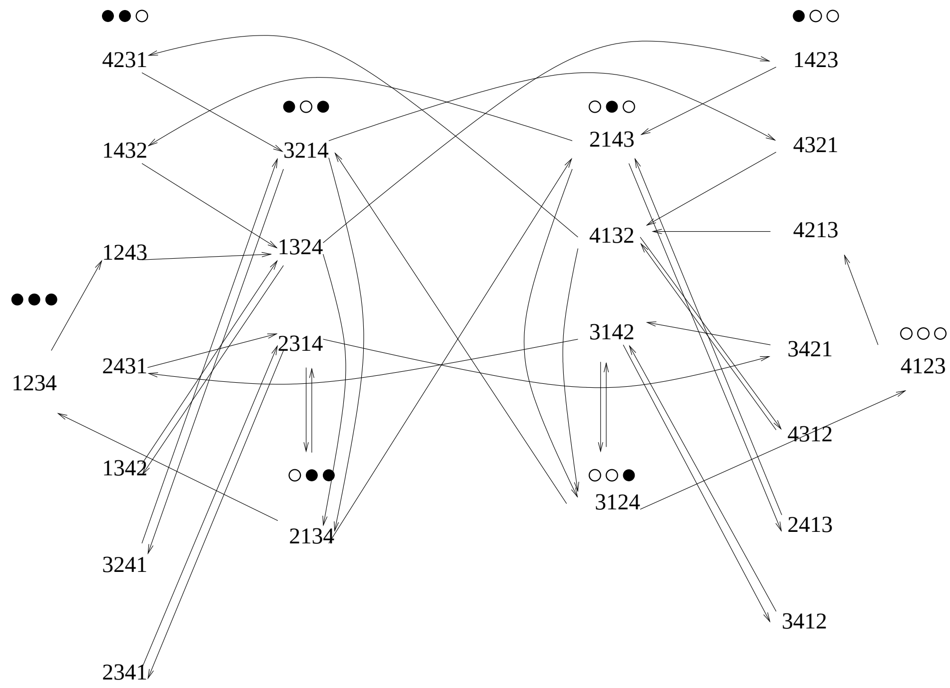

FIGURE 17. The state diagram of the PT chain, depicted in terms of permutations

We define a *crossing* of a permutation π to be a pair of indices i and j such that either $i < j \leq \pi(i) < \pi(j)$ or $\pi(i) < \pi(j) < i < j$. In [19] it was shown that the superfluous ones in a permutation tableaux $\mathcal{T}$ are in bijection with the crossings of $\Phi(\mathcal{T})$. Therefore we define $\mathrm{rk}(\pi)$ to be the number of crossings of π.

We define a *LR-maximum* of π to be an index i such that $\pi(i) > \pi(j)$ for $j < i$, and we call it *special* if in addition, $\pi(i) > \pi(1)$. Similarly we define a *RL-minimum* of π to be an index i such that $\pi(i) < \pi(j)$ for $j > i$, and we call it *special* if in addition, $\pi(i) < \pi(1)$. Abusing notation, we let $f(\pi)$ denote the number of special RL-minima, and we let $u(\pi)$ denote the number of special LR-maxima of π. We show in [6] that if $\mathcal{T}$ is a permutation tableau, then $f(\Phi(\mathcal{T})) = f(\mathcal{T})$ and $u(\Phi(\mathcal{T})) = u(\mathcal{T})$.

In analogy with our weight function on tableaux, we now define $\mathrm{wt}^{q,\alpha,\beta}(\pi) = q^{rk(\pi)}\alpha^{-f(\pi)}\beta^{-u(\mathcal{T})}$.

We can now translate Theorem 5.1 into the language of permutations.

Corollary 6.4. *Consider the PT chain on permutations in S_{N+1} and fix a permutation $\pi \in S_{N+1}$. Then the steady state probability of finding the PT chain in state π is $\frac{\mathrm{wt}(\pi)}{\sum_{\pi'} \mathrm{wt}(\pi')}$. Here, the sum is over all permutations $\pi' \in S_{N+1}$.*

7. The involution

The goal of this section is to introduce an involution I on permutations (equivalently, on permutation tableaux) which generalizes the particle-hole symmetry of the PASEP, and reveals a symmetry in the PT chain. This symmetry is a graph automorphism of the state diagram – this is depicted as a reflective symmetry from left to right in Figure 4. Using the same notation that we did for the particle-hole symmetry on states of the PASEP, we will use $I(\pi)$ or $\overline{\pi}$, or $I(\mathcal{T})$ or $\overline{\mathcal{T}}$ to denote the image of π or $\mathcal{T}$ under the involution. Because in this case it is easier to work with permutations, we will give all proofs in terms of permutations, and only *define* the analogous involution on permutation tableaux. Philippe Duchon has informed us that he independently discovered such an involution [12].

Definition 7.1. *Let* $\pi = (\pi(1), \dots, \pi(N+1))$ *be an element of* S_{N+1}. *We define the permutation* $\overline{\pi} = (\overline{\pi}(1), \dots, \overline{\pi}(N+1))$ *as follows:*

$$\begin{aligned} \overline{\pi}(1) &= N+2-\pi(1) \\ \overline{\pi}(i) &= N+2-\pi(N+3-i), \quad 2 \le i \le N+1. \end{aligned}$$

We will prove the following result.

Theorem 7.2. *The map* I *on permutations in* S_{N+1} *which sends* π *to* $\overline{\pi}$ *has the following properties:*

1. $\mathrm{pr}(\pi) = \overline{\mathrm{pr}(\overline{\pi})}$. *In other words,* $\mathrm{pr}(\pi)$ *and* $\mathrm{pr}(\overline{\pi})$ *are related via the particle-hole symmetry.*
2. $\mathrm{rk}(\pi) = \mathrm{rk}(\overline{\pi})$.
3. $u(\pi) = f(\overline{\pi})$.
4. $f(\pi) = u(\overline{\pi})$.

Theorem 7.2 and Corollary 6.4 then immediately imply the following result.

Corollary 7.3. *Conider the PT chain on permutations in* S_{N+1} *and fix* $\pi \in S_{N+1}$. *Then* $\mathrm{wt}(\pi, q, \alpha, \beta) = \mathrm{wt}(\overline{\pi}, q, \beta, \alpha)$. *Moreover, the steady state probability of finding the PT chain in state* π *is equal to the steady state probability of finding the PT chain in state* $\overline{\pi}$.

This can be seen as an extension of the particle-hole symmetry that was mentioned in Remark 2.4. Indeed our Theorem 3.1 states that the probability to be in state τ is the (normalized) weight-generating function $\frac{F_{\lambda(\tau)}(q)}{Z_N}$ for all permutation tableaux of shape $\lambda(\tau)$. After translating this into the corresponding statement for permutations, Corollary 7.3 immediately implies that $F_{\lambda(\tau)}(q) = F_{\lambda(\overline{\tau})}(q)$.

Additionally, the following result reveals a symmetry in the state diagram of the PT chain.

Theorem 7.4. *There is a transition in the PT chain from* π *to* σ *if and only if there is a transition from* $\overline{\pi}$ *to* $\overline{\sigma}$. *Furthermore, the transition probabilities are related as follows:*

1. $\mathrm{prob}(\pi \to \sigma) = \frac{\alpha}{N+1}$ *if and only if* $\mathrm{prob}(\overline{\pi} \to \overline{\sigma}) = \frac{\beta}{N+1}$.
2. $\mathrm{prob}(\pi \to \sigma) = \frac{\beta}{N+1}$ *if and only if* $\mathrm{prob}(\overline{\pi} \to \overline{\sigma}) = \frac{\alpha}{N+1}$.
3. $\mathrm{prob}(\pi \to \sigma) = \frac{1}{N+1}$ *if and only if* $\mathrm{prob}(\overline{\pi} \to \overline{\sigma}) = \frac{1}{N+1}$.
4. $\mathrm{prob}(\pi \to \sigma) = \frac{q}{N+1}$ *if and only if* $\mathrm{prob}(\overline{\pi} \to \overline{\sigma}) = \frac{q}{N+1}$.

7.1. **Proof of Theorem 7.2.** We present a series of Lemmas that give a refined version of Theorem 7.2.

Lemma 7.5. *I is an involution.*

Proof. We have $\overline{\overline{\pi}}(1) = N+2-(N+2-\pi(1)) = \pi(1)$, and for $i \geq 2$, $\overline{\overline{\pi}}(i) = N+2-\overline{\pi}(N+3-i) = N+2-(N+2-\pi(N+3-(N+3-i))) = \pi(i)$. □

Lemma 7.6. *Let $\pi \in S_{N+1}$. Then for $i \geq 2$, i is a weak excedance of π if and only if $N+3-i$ is not a weak excedance of $\overline{\pi}$. It follows that $\mathrm{pr}(\pi) = \overline{\mathrm{pr}(\overline{\pi})}$.*

Proof. By definition, $\overline{\pi}(N+3-i) = N+2-\pi(i)$. Note that $\pi(i) \geq i$ if and only if $\overline{\pi}(N+3-i) \leq N+2-i$, i.e. $N+3-i$ is *not* a weak excedance of $\overline{\pi}$. Since the projection operator from permutations to states of the PASEP forgets about $\pi(1)$ and maps the permutation to a state based on whether the other $\pi(i)$ are weak excedances, what we have proved implies that $\mathrm{pr}(\pi) = \overline{\mathrm{pr}(\overline{\pi})}$. □

Lemma 7.7. *For i and j not equal to 1, there is a crossing in positions i and j in π if and only if there is a crossing in positions $N+3-i$ and $N+3-j$ in $\overline{\pi}$. Additionally, the number of crossings involving position 1 in π is equal to the number of crossings involving position 1 in $\overline{\pi}$. Therefore $\mathrm{rk}(\mathcal{T}) = \mathrm{rk}(\overline{\mathcal{T}})$.*

Proof. Consider a crossing in positions i and j in π such that neither i nor j is 1. Without loss of generality, assume that $i < j \leq \pi(i) < \pi(j)$. Then it follows that $N+2-\pi(j) < N+2-\pi(i) < N+3-j < N+3-i$. But since $\overline{\pi}(N+3-i) = N+2-\pi(i)$ and $\overline{\pi}(N+3-j) = N+2-\pi(j)$, we have that $\overline{\pi}(N+3-j) < \overline{\pi}(N+3-i) < N+3-j < N+3-i$ which implies that $N+3-i$ and $N+3-j$ are positions of a crossing in $\overline{\pi}$.

The number of crossings involving position 1 in π is equal to the number of indices k such that $1 < k \leq \pi(1) < \pi(k)$. Therefore the number of crossings involving position 1 in $\overline{\pi}$ is equal to

$$\begin{aligned}
\#\{k \mid k \leq \overline{\pi}(1) < \overline{\pi}(k)\} &= \#\{k \mid k \leq N+2-\pi(1) < N+2-\pi(N+3-k)\} \\
&= \#\{k \mid k > \pi(1) > \pi(k)\} \\
&= \#\{k \mid k \leq \pi(1) < \pi(k)\}.
\end{aligned}$$

The first step comes from the definition of $\overline{\pi}$. The second step comes from replacing k by $N+3-k$, decreasing by $N+2$ and negating. The last step comes from the fact that π is a permutation. □

Lemma 7.8. *The index i is a special RL-minimum of π if and only if the index $N+3-i$ is a special LR-maximum of $\overline{\pi}$. And the index i is a special LR-maximum of π if and only if the index $N+3-i$ is a special RL-minimum of $\overline{\pi}$.*

Proof. We just prove the first part. The second part follows thanks to the Lemma 7.5. Suppose that the index i is a special RL-minimum of π. This means that $\pi(i) < \pi(1)$ and $\pi(i) < \pi(j)$ for $j > i$. By definition, $\overline{\pi}(1) = N+2-\pi(1)$ and $\overline{\pi}(N+3-i) = N+2-\pi(N+3-(N+3-i)) = N+2-\pi(i)$. So $\pi(i) < \pi(1)$ implies that

$\overline{\pi}(N+3-i) = N+2-\pi(i) > N+2-\pi(1) = \overline{\pi}(1)$. Now note that since $\pi(i) < \pi(j)$ for $j > i$ we have $\overline{\pi}(N+3-i) = N+2-\pi(i) > N+2-\pi(j) = \overline{\pi}(N+3-j)$, which completes the proof that $N+3-i$ is a special LR-maximum of $\overline{\pi}$. □

The combination of Lemmas 7.5-7.8 therefore implies Theorem 7.2.

7.2. **Symmetry in the PT chain.** In this section we will prove Theorem 7.4.

Proof of Theorem 7.4. Let π be a permutation in S_{N+1}. We will analyze in turn the various kinds of transitions from π to σ in the PT chain.

- "Particle enters from the left" and "exits to the right":
  Note that $\pi(2) = 1$ if and only if $\overline{\pi}(N+1) = N+1$, so there is a transition of type "particle enters from the left" out of π if and only if there is a transition of type "particle exits to the right" out of $\overline{\pi}$.
- "Particle hops right, Cases 1 and 3":
  If $\pi(i) = i$ and $\pi(i+1) < i+1$ then $\overline{\pi}(N+2-i) = N+2-\pi(i+1) > N+1-i$ and $\overline{\pi}(N+3-i) = N+2-\pi(i) = N+2-i$. Conversely, $\pi(i) > i$ and $\pi(i+1) = i$ then $\overline{\pi}(N+2-i) = N+2-\pi(i+1) = N+2-i$ and $\overline{\pi}(N+3-i) = N+2-\pi(i) < N+2-i$ which implies that $\overline{\pi}(N+3-i) < N+3-i$. So there is a Case 1 transition out of π at index i if and only if there is a Case 3 transition out of $\overline{\pi}$ at $N+2-i$, and vice-versa.
- "Particle hops right, Case 2":
  If $\pi(i) > i$ and $\pi(i+1) < i$ then $\overline{\pi}(N+3-i) = N+2-\pi(i) < N+2-i$ and $\overline{\pi}(N+2-i) = N+2-\pi(i+1) > N+2-i$. So there is a "particle hops right Case 2" transition out of π if and only if there is a "particle hops right Case 2" transition out of $\overline{\pi}$.
- "Particle hops left":
  If $\pi(i) < i$ and $\pi(i+1) \geq i+1$ then $\overline{\pi}(N+3-i) = N+2-\pi(i) > N+2-i$ implies that $\overline{\pi}(N+3-i) \geq N+3-i$ and $\overline{\pi}(N+2-i) = N+2-\pi(i+1) \geq N+1-i$ implies that $\overline{\pi}(N+2-i) < N+2-i$. So there is a "particle hops left" transition out of π at index i if and only if there is a "particle hops left" transition out of $\overline{\pi}$ at index $N+2-i$.

We also need to show that the transition probabilities $\mathrm{prob}(\pi \to \sigma)$ and $\mathrm{prob}(\overline{\pi} \to \overline{\sigma})$ are related as specified in Theorem 7.4. These calculations are straightforward but tedious so we will show only the first case.

Suppose $\pi(2) = 1$ and let σ be the permutation obtained after a transition of the form "particle enters from the left." Let $i+1$ be the minimum number greater than 2 which is a nonexcedance. So σ is the permutation on $\{1, \hat{2}, 3, \dots, N+1\} \cup \{i+\frac{1}{2}\}$ obtained by collapsing π at 2 and adding a fixed point at $i+\frac{1}{2}$.

Let μ be the permutation obtained from $\overline{\pi}$ after a transition of the form "particle exits to the right." Let j be the maximum number less than $N+1$ such that $\overline{\pi}(j) \geq j$. So $j = N+2-i$. Let $a = \overline{\pi}^{-1}(N+2-i)$. Then $a = N+3-\pi^{-1}(i)$. Then μ is the permutation on $\{1, 2, \dots, N\} \cup \{N+2\frac{1}{2}-i\}$ obtained by collapsing at $N+1$ and replacing $\overline{\pi}(a) = N+2-i$ by $\mu(a) = N+2\frac{1}{2}-i$ and $\mu(N+2\frac{1}{2}-i) = N+2-i$. It is now straightforward to see that in fact $\mu = \overline{\sigma}$.

□

We now define the involution in terms of permutation tableaux.

Let $\mathcal{T}$ be a permutation tableau with K rows and $N+1-K$ columns and shape $\lambda = (\lambda_1, \dots, \lambda_K)$. Numbering the rows from top to bottom and the columns from left to right, let $\mathcal{T}(i,j)$ denote the filling of the cell (i,j) of $\mathcal{T}$. The *conjugate* of $\mathcal{T}$, which we shall denote by $\mathcal{T}'$, is the tableau of shape λ' such that $\mathcal{T}'(i,j) = \mathcal{T}(j,i)$ for all i,j. Here $\lambda' = (\lambda'_1, \dots, \lambda'_{N+1-K})$ is the conjugate partition, i.e. the partition formed by the *columns* of λ. If $a \in \{0,1\}$, let a^c denote $1-a$.

Let $\mathcal{T}$ be a permutation tableau with shape $\lambda = (\lambda_1, \dots, \lambda_K)$ and conjugate shape $\lambda' = (\lambda'_1, \dots, \lambda'_{N+1-K})$. We define $\overline{\mathcal{T}}$ to be the tableau of shape $(K-1, \lambda'_1 - 1, \lambda'_2 - 1, \dots, \lambda'_{N+1-K} - 1)$ whose entries are as follows.

(1) $\overline{\mathcal{T}}(1,j) = 1$ if row $j+1$ of $\mathcal{T}$ is unrestricted and 0 otherwise for $1 \le j \le K-1$.
(2) $\overline{\mathcal{T}}(i,j) = \mathcal{T}(j+1, i-1)^c$ if cell $(j+1, i-1)$ of $\mathcal{T}$ contains a topmost one or a rightmost restricted zero, and $\mathcal{T}(j+1, i-1)$ otherwise.

One can check that $\pi = \Phi(\mathcal{T})$ if and only if $\bar{\pi} = \Phi(\bar{\mathcal{T}})$.

References

[1] E. Babson, E. Steingrímsson, Generalized permutation patterns and a classification of the Mahonian statistics, Sém. Lothar. Combin., B44b 2000, 18 pp.
[2] R. Brak, J. Essam, Asymmetric exclusion model and weighted lattice paths, J. Phys. A 37 (2004), no. 14, 4183 – 4217.
[3] R. Brak, S. Corteel, J. Essam, R. Parviainen and A. Rechnitzer, A combinatorial derivation of the PASEP stationary state, Elec. Jal of Combinatorics, no 1, R108.
[4] R. Bundschuh, Phys. Rev. E volume 65 031911, 2002.
[5] S. Corteel: Crossings and alignments of permutations, Adv. in Appl. Math. 38 (2007), no 2, 149-163.
[6] S. Corteel, E. Steingrímsson, L. Williams, in preparation, 2007.
[7] S. Corteel, L. Williams, Permutation tableaux and the asymmetric exclusion process, to appear in Adv. in Appl. Math (2007).
[8] B. Derrida, E. Domany, D. Mukamel, An exact solution of a one-dimensional asymmetrix exclusion model with open boundaries, J. Stat. Phys, Vol 69, 1992.
[9] B. Derrida, M. Evans, V. Hakim, V. Pasquier, Exact solution of a 1D asymmetrix exclusion model using a matrix formulation, J. Phys. A: Math. Gen. 26 (1993), 1493–1517.
[10] B. Derrida, J. Lebowitz, E. Speer, J. Stat. Phys. volume 89 number 135, 1997.
[11] E. Duchi, G. Schaeffer, A combinatorial approach to jumping particles, J. Combin. Theory Ser. A 110 (2005), 1–29.
[12] P. Duchon, Private communication, 2006.
[13] D. Foata, D. Zeilberger, Denert's permutation statistic is indeed Euler-Mahonian, Stud. Appl. Math. 83 (1990), no. 1, 31–59.
[14] J. MacDonald, J. Gibbs, A. Pipkin, Biopolymers, volume 6 issue 1, 1968.
[15] A. Postnikov, Total positivity, Grassmannians, and networks, preprint (2006) arXiv:math.CO/0609764
[16] T. Sasamoto, One-dimensional partially asymmetric simple exclusion process with open boundaries: orthogonal polynomials approach, J. Phys. A: Math. Gen. 32 (1999), 7109 – 7131.
[17] M. Schreckenberg, D. Wolf, Traffic and Granular Flow '97 (Singapore: Springer), 1998.

[18] L. Shapiro, D. Zeilberger, a Markov chain occurring in Enzyme kinetics, J. Math. Biology (1982) 15, 351 – 357.

[19] E. Steingrímsson, L. Williams, Permutation tableaux and permutation patterns, J. of Combin. Theory Series A, 114 (2007), no 2, 211-234. arXiv:math.CO/0507149.

[20] M. Uchiyama, T. Sasamoto, M. Wadati, Asymmetric simple exclusion process with open boundaries and Askey-Wilson polynomials, J. Phys. A: Math. Gen. 37 (2004) 4985–5002.

[21] L. Williams: Enumeration of totally positive Grassmann cells, Advances in Math, **190** (2005), 319–342.

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