THE FULL KOSTANT-TODA HIERARCHY ON THE POSITIVE FLAG VARIETY

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Abstract. We study some geometric and combinatorial aspects of the solution to the full Kostant-Toda (f-KT) hierarchy, when the initial data is given by an arbitrary point on the totally non-negative (tnn) flag variety of $\mathrm{SL}_n(\mathbb{R})$. The f-KT flows on the tnn flag variety are complete, and we show that their asymptotics are completely determined by the cell decomposition of the tnn flag variety given by Rietsch [Rie98]. Our results represent the first results on the asymptotics of the f-KT hierarchy (and even the f-KT lattice); moreover, our results are not confined to the generic flow, but cover non-generic flows as well. We define the f-KT flow on the weight space via the moment map, and show that the closure of each f-KT flow forms an interesting convex polytope which we call a *Bruhat interval polytope*. In particular, the Bruhat interval polytope for the generic flow is the permutohedron of the symmetric group $\mathfrak{S}_n$. We also prove analogous results for the full symmetric Toda hierarchy, by mapping our f-KT solutions to those of the full symmetric Toda hierarchy. In the Appendix we show that Bruhat interval polytopes are *generalized permutohedra*, in the sense of Postnikov [Pos09].

Contents

1. Introduction

The Toda lattice, introduced by Toda in 1967 (see [Tod89] for a comprehensive treatment), is an integrable Hamiltonian system representing the dynamics of n particles of unit mass, moving on a line under the influence of exponential repulsive forces. The dynamics can be encoded by a matrix equation called the *Lax equation*

$$\frac{dL}{dt} = [\pi_{\mathfrak{so}}(L), L], \tag{1.1}$$

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where L is a tridiagonal symmetric matrix and $\pi_{\mathfrak{so}}(L)$ is the *skew-symmetric projection* of L. More specifically,

$$L = L(t) = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & a_2 & b_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{n-1} & b_n \end{pmatrix} \qquad \text{and} \quad \pi_{\mathfrak{so}}(L) := (L)_{>0} - (L)_{<0},$$

where $(L)_{>0}$ (respectively $(L)_{<0}$) is the strictly upper (resp. lower) triangular part of L, so that $\pi_{\mathfrak{so}}(L)$ represents a skew-symmetrization of the matrix L. The entries a_i and b_j of L are functions of t.

The Toda lattice gives an *iso-spectral deformation* of the eigenvalue problem of L, that is, the eigenvalues of $L(t)$ are independent of t. It is an immediate consequence of the Lax equation that for any positive integer k, the trace $\mathrm{tr}(L^k)$ of L^k is a constant of motion (it is invariant under the Toda flow). These invariants are the power sum symmetric functions of the eigenvalues, and are sometimes referred to as *Chevalley invariants*. Note that assuming $\mathrm{tr}(L) = 0$ (i.e. $L \in \mathfrak{sl}_n(\mathbb{R})$), we have $n-1$ independent Chevalley invariants, $H_k := \mathrm{tr}(L^{k+1})$ for $k = 1, \ldots, n-1$.

One remarkable property of the Toda lattice is that for generic initial data, i.e. L has distinct eigenvalues

$$\lambda_1 \; < \; \lambda_2 \; < \; \cdots \; < \; \lambda_n$$

and $a_k(0) \neq 0$ for all k, the asymptotic form of the Lax matrix is given by

$$L(t) \quad \longrightarrow \quad \begin{cases} \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) & \text{as} \quad t \to -\infty \\ \mathrm{diag}(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1) & \text{as} \quad t \to \infty. \end{cases} \tag{1.2}$$

In other words, all off-diagonal elements $a_i(t)$ approach 0, and the time evolution of the Toda lattice sorts the eigenvalues of L (see [DNT83, Sym82]). This property is referred to as the *sorting property*, which has important applications to matrix eigenvalue algorithms. It is also known that if we let L range over all tridiagonal matrices with fixed eigenvalues $\lambda_1 < \cdots < \lambda_n$, the set of fixed points of the Toda lattice – i.e. those points L such that $dL/dt = 0$ – are precisely the diagonal matrices. Therefore there are $|\mathfrak{S}_n| = n!$ fixed points of the Toda lattice, where $\mathfrak{S}_n$ is the symmetric group on n letters.

The *full symmetric Toda lattice* is a generalization of the Toda lattice: it is defined using the Lax equation (1.1), but now L can be any symmetric matrix. For generic L, the full symmetric Toda lattice is again an integrable Hamiltonian system [DLNT86] and it has the same asymptotic behavior from (1.2) [KM96]. Recently the non-generic flows were studied in [CSS12], including their asymptotics as $t \to \pm\infty$.

In this paper we consider a different generalization of the Toda lattice called the *full Kostant-Toda lattice*, or *f-KT lattice*, first studied in [EFS93]. Like the Toda lattice and the full symmetric Toda lattice, the f-KT lattice is an integrable Hamiltonian system, whose constants of motion are given by the so-called *chop integrals* [EFS93].[1] It is defined by the Lax equation

$$\frac{dL}{dt} = [(L)_{\geq 0}, L], \tag{1.3}$$

[1] Note, however, that we do not use the chop integrals in our study of the f-KT flows.

where L is a *Hessenberg matrix*, i.e. any matrix of the form

$$L = \begin{pmatrix} a_{1,1} & 1 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & \cdots & 1 \\ a_{n,1} & a_{n,2} & \cdots & \cdots & a_{n,n} \end{pmatrix}, \tag{1.4}$$

and $(L)_{\geq 0}$ denotes the weakly upper triangular part of L. In terms of the entries $a_{i,j} = a_{i,j}(t)$, the f-KT lattice is defined by the system of equations

$$\frac{da_{\ell+k,k}}{dt} = a_{\ell+k+1,k} - a_{\ell+k,k-1} + (a_{\ell+k,\ell+k} - a_{k,k})a_{\ell+k,k} \tag{1.5}$$

for $k = 1, \ldots, n-\ell$ and $\ell = 0, 1, \ldots, n-1$. Here we use the convention that $a_{i,j} = 0$ if $j = 0$ or $i = n+1$. Note that the index ℓ represents the ℓth subdiagonal of the matrix L, that is, $\ell = 0$ corresponds to the diagonal elements $a_{k,k}$ and $\ell = 1$ corresponds to the elements $a_{k+1,k}$ of the first subdiagonal, etc. The f-KT lattice also gives an iso-spectral deformation of the matrix L.

Both the full symmetric Toda lattice and the full Kostant-Toda lattice have the same number of free parameters, namely $\frac{n(n+1)}{2}$. One should note, however, that when working over $\mathbb{R}$, one can map each full symmetric matrix to a Hessenberg matrix, but not vice-versa in general. For example, the symmetric matrix $\begin{pmatrix} b_1 & a \\ a & b_2 \end{pmatrix}$ can be mapped to the Hessenberg matrix $\begin{pmatrix} b_1 & 1 \\ a^2 & b_1 \end{pmatrix}$. However, the Hessenberg matrix $\begin{pmatrix} \beta_1 & 1 \\ \alpha & \beta_2 \end{pmatrix}$ *cannot* be mapped to a real symmetric matrix if $\alpha < 0$. In this sense, the f-KT lattice may be considered to be more general than the full symmetric Toda lattice.

We also consider the f-KT hierarchy which consists of the symmetries of the f-KT lattice generated by the Chevalley invariants $H_k = \mathrm{tr}(L^{k+1})$ for $k = 1, \ldots, n-1$, see e.g. [KS08]. Each symmetry is given by the equation

$$\frac{\partial L}{\partial t_k} = [(L^k)_{\geq 0}, L], \qquad \text{for} \quad k = 1, 2, \ldots, n-1. \tag{1.6}$$

We let $\mathbf{t} := (t_1, \ldots, t_{n-1})$ denote the multi-time variables representing the flow parameters. Note that the flows commute with each other, and the equation for $k = 1$ corresponds to the f-KT lattice, with $t = t_1$.

It is well-known that the solution space of the Toda lattice (and its generalizations) can be described by the flag variety G/B^+ – see e.g. [KS08] for some basic information on the f-KT lattice and the full symmetric Toda lattice. In this paper we use as a main tool the *totally non-negative part* $(G/B^+)_{\geq 0}$ of the flag variety (which we often abbreviate as the *tnn flag variety*). As shown by Rietsch [Rie98] and Marsh-Rietsch [MR04], $(G/B^+)_{\geq 0}$ has a decomposition into cells $\mathcal{R}_{v,w}^{>0}$ which are indexed by pairs (v, w) of permutations in $\mathfrak{S}_n$, where $v \leq w$ in Bruhat order.

Our first main result concerns the asymptotics of f-KT flows associated to points in the tnn flag variety. More specifically, in Proposition 5.3, we associate to each point gB^+ of $(G/B^+)_{\geq 0}$ an initial Hessenberg matrix L^0 for the f-KT lattice. The corresponding f-KT flow is complete for such initial matrices. In Theorem 5.13 we prove a generalization of the sorting property: we show that if $gB^+ \in \mathcal{R}_{v,w}^{>0}$, then as $t \to \pm\infty$, the diagonal of $L(t)$ contains the eigenvalues, sorted according to v and w, respectively. This result was previously unknown, except in the special case that L is tridiagonal and $(v, w) = (e, w_0)$, where $e = (1, 2, \ldots, n)$ is the identity permutation and $w_0 = (n, n-1, \ldots, 1)$ is the longest permutation. In that case, we recover the classical sorting property for the generic flow in the Toda lattice.

Our second main result concerns the moment map images of the flows of the f-KT hierarchy. In Theorem 6.10 we show that if $gB^+ \in \mathcal{R}_{v,w}^{>0}$, and we apply the moment map to the corresponding f-KT flow, then its closure is a convex polytope $\mathsf{P}_{v,w}$ which generalizes the permutohedron. More specifically, $\mathsf{P}_{v,w}$ is the convex hull of the permutation vectors z such that $v \leq z \leq w$. Note that $\mathsf{P}_{v,w}$ is precisely the permutohedron when $v = e$ and $w = w_0$. In the appendix we study combinatorial properties of these *Bruhat interval polytopes*: they are Minkowski sums of matroid polytopes and hence *generalized permutohedra* in the sense of Postnikov [Pos09].

Finally, we show that one can map the f-KT flows $L(t)$ studied above (i.e. those coming from points gB^+ of the tnn flag variety) to full symmetric Toda flows $\mathcal{L}(t)$. This allows us to deduce analogues of our previous results for the full symmetric Toda lattice. More specifically, Theorem 7.9 proves that if $gB^+ \in \mathcal{R}_{v,w}^{>0}$, then as $t \to \pm\infty$, $\mathcal{L}(t)$ tends to a diagonal matrix with eigenvalues sorted according to v and w, respectively. And Theorem 7.13 proves that the closure of the image of the moment map applied to such a symmetric Toda flow is the Bruhat interval polytope $\mathsf{P}_{v,w}$.

The structure of this paper is as follows. In Sections 2 and 3 we provide some background on the flag variety, the Grassmannian, and their non-negative parts. In Section 4 we introduce the full Kostant-Toda lattice (and hierarchy), and define the so-called *τ-functions* for the hierarchy. We then explain how to express the solution $L(t)$ using the LU-factorization. In Section 5 we associate to each point gB^+ of the tnn flag variety an initial matrix L^0, and we analyze the asymptotics of the corresponding flow as $t \to \pm\infty$. In Section 6 we study the moment map images of the f-KT flows associated to points in $(G/B^+)_{\geq 0}$. In Section 7 we map our f-KT solutions (which come from the tnn flag variety) to the symmetric Toda lattice, which allows us to transfer our results on the asymptotics and moment polytopes to the symmetric Toda lattice. We end this paper with a self-contained appendix which explores combinatorial properties of Bruhat interval polytopes.

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2. The flag variety and its tnn part

In this section we provide some background on the flag variety and its totally non-negative part, the *tnn flag variety*. In subsequent sections we use the geometry of the flag variety (and the Grassmannian) to describe combinatorial aspects of the solutions to the full Kostant-Toda lattice.

2.1. The flag variety.

Throughout this paper, we consider the group $G = \mathrm{SL}_n(\mathbb{R})$ and the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$. Let B^+ and B^- be the Borel subgroups of upper and lower-triangular matrices. Let U^+ and U^- be the unipotent radicals of B^+ and B^-; these are the subgroups of upper and lower-triangular matrices with 1's on the diagonals. We let H denote the Cartan subgroup of G, which is the subgroup of diagonal matrices. We let $\mathfrak{b}^\pm, \mathfrak{u}^\pm$ and $\mathfrak{h}$ denote their Lie algebras.

For each $1 \leq i \leq n-1$ we have a homomorphism $\phi_i : \mathrm{SL}_2 \to \mathrm{SL}_n$ such that

$$\phi_i \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & a & b & & \\ & & c & d & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \in \mathrm{SL}_n,$$

that is, ϕ_i replaces a 2×2 block of the identity matrix with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a is at the (i,i)-entry. We have 1-parameter subgroups of G defined by

$$x_i(m) = \phi_i \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \text{ and } y_i(m) = \phi_i \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}, \quad \text{where } m \in \mathbb{R}.$$

Let W denote the Weyl group $N_G(T)/T$, where $N_G(T)$ is the normalizer of the maximal torus T. The simple reflections $s_i \in W$ are given by $s_i := \dot{s}_i T$ where $\dot{s}_i := \phi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and any $w \in W$ can be expressed as a product $w = s_{i_1} s_{i_2} \dots s_{i_\ell}$ with $\ell = \ell(w)$ factors. Here $\ell(w)$ denotes the length of w. We set $\dot{w} = \dot{s}_{i_1} \dot{s}_{i_2} \dots \dot{s}_{i_\ell}$. In our setting W is isomorphic to $\mathfrak{S}_n$, the symmetric group on n letters, and s_i corresponds to the transposition exchanging i and $i+1$. We let $\leq$ denote the (strong) Bruhat order on W, as in [Hum90, Section 5.9].

Definition 2.1. The *real flag variety* is the variety of all *flags*

$$\{V_\bullet = V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n \mid \dim V_i = i\}$$

of vector subspaces of $\mathbb{R}^n$. As we explain below, it can be identified with the homogeneous space G/B^+.

Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$. The *standard flag* is $E_\bullet = E_1 \subset E_2 \subset \cdots \subset E_n$ where E_i is the span of $\{e_1, \dots, e_i\}$. Note that the group G acts on flags; the Borel subgroup B^+ is the stabilizer of the standard flag $E_\bullet$. Any flag $V_\bullet$ may be written as $gE_\bullet$ for some $g \in G$. Note that $gE_\bullet = g'E_\bullet$ if and only if $gB^+ = g'B^+$. In this way we may identify the flag variety with G/B^+.

We have two opposite Bruhat decompositions of G/B^+:

$$G/B^+ = \bigsqcup_{w \in W} B^+ \dot{w} B^+/B^+ = \bigsqcup_{v \in W} B^- \dot{v} B^+/B^+.$$

We define the intersection of opposite Bruhat cells

$$\mathcal{R}_{v,w} := (B^+ \dot{w} B^+/B^+) \cap (B^- \dot{v} B^+/B^+),$$

which is nonempty precisely when $v \leq w$. The strata $\mathcal{R}_{v,w}$ are often called *Richardson varieties.*

2.2. **The tnn part of the flag variety.**

Definition 2.2. [Lus94] The *tnn part* $U^-_{\geq 0}$ *of* U^- is defined to be the semigroup in U^- generated by the $y_i(p)$ for $p \in \mathbb{R}_{\geq 0}$. The *tnn part* $(G/B^+)_{\geq 0}$ *of* G/B^+ is defined by

$$(G/B^+)_{\geq 0} := \overline{\{\, uB^+ \mid u \in U^-_{\geq 0} \,\}},$$

where the closure is taken inside G/B^+ in its real topology. We sometimes refer to $(G/B^+)_{\geq 0}$ as the *tnn flag variety.*

Lusztig [Lus94, Lus98] introduced a natural decomposition of $(G/B^+)_{\geq 0}$.

Definition 2.3. [Lus94] For $v, w \in W$ with $v \leq w$, let

$$\mathcal{R}^{>0}_{v,w} := \mathcal{R}_{v,w} \cap (G/B^+)_{\geq 0}.$$

Then the tnn part of the flag variety G/B^+ has the decomposition,

$$(G/B^+)_{\geq 0} = \bigsqcup_{w \in \mathfrak{S}_n} \left(\bigsqcup_{v \leq w} \mathcal{R}^{>0}_{v,w} \right). \tag{2.1}$$

Lusztig conjectured and Rietsch proved [Rie98] that $\mathcal{R}_{v,w}^{>0}$ is a semi-algebraic cell of dimension $\ell(w)-\ell(v)$. Subsequently Marsh-Rietsch [MR04] provided an explicit parameterization of each cell. To state their result, we first review the notion of positive distinguished subexpression, as in [Deo85] and [MR04].

Let $\mathbf{w} := s_{i_1} \dots s_{i_m}$ be a reduced expression for $w \in W$. A *subexpression* $\mathbf{v}$ of $\mathbf{w}$ is a word obtained from the reduced expression $\mathbf{w}$ by replacing some of the factors with 1. For example, consider a reduced expression in the symmetric group $\mathfrak{S}_4$, say $s_3 s_2 s_1 s_3 s_2 s_3$. Then $1\, s_2\, 1\, 1\, s_2\, s_3$ is a subexpression of $s_3 s_2 s_1 s_3 s_2 s_3$. Given a subexpression $\mathbf{v}$, we set $v_{(k)}$ to be the product of the leftmost k factors of $\mathbf{v}$, if $k \geq 1$, and $v_{(0)} = 1$.

Definition 2.4. [Deo85, MR04] Given a subexpression $\mathbf{v}$ of $\mathbf{w} = s_{i_1} s_{i_2} \dots s_{i_m}$, we define

$$\begin{aligned} J_{\mathbf{v}}^{\circ} &:= \{k \in \{1,\dots,m\} \mid v_{(k-1)} < v_{(k)}\}, \\ J_{\mathbf{v}}^{+} &:= \{k \in \{1,\dots,m\} \mid v_{(k-1)} = v_{(k)}\}, \\ J_{\mathbf{v}}^{\bullet} &:= \{k \in \{1,\dots,m\} \mid v_{(k-1)} > v_{(k)}\}. \end{aligned}$$

The subexpression $\mathbf{v}$ is called *non-decreasing* if $v_{(j-1)} \leq v_{(j)}$ for all $j = 1,\dots,m$, e.g. if $J_{\mathbf{v}}^{\bullet} = \emptyset$. It is called *distinguished* if we have $v_{(j)} \leq v_{(j-1)}\, s_{i_j}$ for all $j \in \{1,\dots,m\}$. In other words, if right multiplication by s_{i_j} decreases the length of $v_{(j-1)}$, then in a distinguished subexpression we must have $v_{(j)} = v_{(j-1)} s_{i_j}$. Finally, $\mathbf{v}$ is called a *positive distinguished subexpression* (or a PDS for short) if $v_{(j-1)} < v_{(j-1)} s_{i_j}$ for all $j \in \{1,\dots,m\}$. In other words, it is distinguished and non-decreasing.

Lemma 2.5. [MR04] Given $v \leq w$ and a reduced expression $\mathbf{w}$ for w, there is a unique PDS $\mathbf{v}_+$ for v contained in $\mathbf{w}$.

Theorem 2.6. [MR04, Proposition 5.2, Theorem 11.3] Choose a reduced expression $\mathbf{w} = s_{i_1} \dots s_{i_m}$ for w with $\ell(w) = m$. To $v \leq w$ we associate the unique PDS $\mathbf{v}_+$ for v in $\mathbf{w}$. Then $J_{\mathbf{v}_+}^{\bullet} = \emptyset$. We define

$$G_{\mathbf{v}_+,\mathbf{w}}^{>0} := \left\{ g = g_1 g_2 \cdots g_m \;\middle|\; \begin{array}{ll} g_\ell = y_{i_\ell}(p_\ell) & \text{if } \ell \in J_{\mathbf{v}}^{+}, \\ g_\ell = \dot{s}_{i_\ell} & \text{if } \ell \in J_{\mathbf{v}}^{\circ}, \end{array} \right\}, \tag{2.2}$$

where each p_ℓ ranges over $\mathbb{R}_{>0}$. The set $G_{\mathbf{v}_+,\mathbf{w}}^{>0}$ lies in $U^- \dot{v} \cap B^+ \dot{w} B^+$, $G_{\mathbf{v}_+,\mathbf{w}}^{>0} \cong \mathbb{R}_{>0}^{\ell(w)-\ell(v)}$, and the map $g \mapsto gB^+$ defines an isomorphism

$$G_{\mathbf{v}_+,\mathbf{w}}^{>0} \xrightarrow{\sim} \mathcal{R}_{v,w}^{>0}.$$

Example 2.7. Consider the reduced decomposition $\mathbf{w} = s_2 s_3 s_1 s_4 s_5 s_3 s_2$ for $w \in W = S_6$. Let $v = s_3 s_4 s_2 \leq w$. Then the PDS $\mathbf{v}_+$ for v in $\mathbf{w}$ is $\mathbf{v}_+ = 1 s_3 1 s_4 1 1 s_2$. The set $G_{\mathbf{v}_+,\mathbf{w}}^{>0}$ consists of all elements of the form

$$y_2(p_1)\dot{s}_3 y_1(p_3)\dot{s}_4 y_5(p_5) y_3(p_6)\dot{s}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ p_3 & 0 & -1 & 0 & 0 & 0 \\ p_1 p_3 & 0 & -p_1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & p_6 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_5 & 1 \end{pmatrix} \quad \text{where each } p_i \in \mathbb{R}_{>0}.$$

3. The Grassmannian and its tnn part

3.1. **The Grassmannian.** The *real Grassmannian* $Gr_{k,n}$ is the space of all k-dimensional subspaces of $\mathbb{R}^n$. An element of $Gr_{k,n}$ can be viewed as a full-rank $k \times n$ matrix A modulo left multiplication by nonsingular $k \times k$ matrices. In other words, two $k \times n$ matrices are equivalent, i.e. they represent the same point in $Gr_{k,n}$, if and only if they can be obtained from each other by row operations.

Let $\binom{[n]}{k}$ be the set of all k-element subsets of $[n] := \{1, \ldots, n\}$. For $I \in \binom{[n]}{k}$, let $\Delta_I(A)$ be the *Plücker coordinate*, that is, the maximal minor of the $k \times n$ matrix A located in the column set I. The map $A \mapsto (\Delta_I(A))$, where I ranges over $\binom{[n]}{k}$, induces the *Plücker embedding* $Gr_{k,n} \hookrightarrow \mathbb{RP}^{\binom{n}{k}-1}$.

Just as for the flag variety, one may identify the Grassmannian with a homogeneous space. Let P_k be the parabolic subgroup which fixes the k-dimensional subspace spanned by $e_1, \ldots, e_k$. (This is a block upper-triangular matrix containing B^+.) Then we may identify $Gr_{k,n}$ with the space of cosets G/P_k.

There is a natural projection $\pi_k : G/B^+ \to Gr_{k,n}$ such that $\pi_k(V_1 \subset \cdots \subset V_n) = V_k$. One may equivalently express this projection as the map $\pi_k : G/B^+ \to G/P_k$, where $\pi_k(gB^+) = gP_k$. Abusing notation, we simply write $\pi_k(g) = A_k$ with $A_k \in Gr_{k,n} \cong G/P_k$ instead of $\pi_k(gB^+) = gP_k$.

Concretely, for $g \in G$, $\pi_k(g)$ is represented by the $k \times n$ matrix A_k consisting of the leftmost k columns of g, i.e.

$$g = \begin{pmatrix} g_{1,1} & \cdots & g_{1,k} & \cdots & g_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ g_{k,1} & \cdots & g_{k,k} & \cdots & g_{k,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n,1} & \cdots & g_{n,k} & \cdots & g_{n,n} \end{pmatrix} \quad \longmapsto \quad A_k = \begin{pmatrix} g_{1,1} & \cdots & g_{k,1} & \cdots & g_{n,1} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ g_{1,k} & \cdots & g_{k,k} & \cdots & g_{n,k} \end{pmatrix}. \tag{3.1}$$

This is equivalent to the following formula using the Plücker embedding into the projectivization of the wedge product space $\mathbb{P}(\bigwedge^k \mathbb{R}^n) \cong \mathbb{RP}^{\binom{n}{k}-1}$ with the standard basis $\{e_i : i = 1, \ldots, n\}$,

$$g \cdot e_1 \wedge \cdots \wedge e_k = \sum_{1 \le i_1 < \cdots < i_k \le n} \Delta_{i_1,\ldots,i_k}(A_k)\, e_{i_1} \wedge \cdots \wedge e_{i_k}. \tag{3.2}$$

The Plücker coordinates $\Delta_{i_1,\ldots,i_k}(A_k)$ are then given by

$$\Delta_{i_1,\ldots,i_k}(A_k) = \langle e_{i_1} \wedge \cdots \wedge e_{i_k},\ g \cdot e_1 \wedge \cdots \wedge e_k \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\bigwedge^k \mathbb{R}^n$.

Remark 3.1. There is a variant of the projection π_k that will be useful to us later. Let $\{u_i : i = 1, \ldots, n\}$ be an ordered basis of $\mathbb{R}^n$. Then one can define the map $\tilde{\pi}_k(g) = \tilde{A}_k$ by

$$g \cdot e_1 \wedge \cdots \wedge e_k = \sum_{1 \le i_1 < \cdots < i_k \le n} \Delta_{i_1,\ldots,i_k}(\tilde{A}_k)\, u_{i_1} \wedge \cdots \wedge u_{i_k}. \tag{3.3}$$

Using the relation $u_{i_1} \wedge \cdots \wedge u_{i_k} = U \cdot e_{i_1} \wedge \cdots \wedge e_{i_k}$ with the $n \times n$ matrix $U = (u_1, \ldots, u_n)$, the Plücker coordinates $\Delta_{i_1,\ldots,i_k}(\tilde{A}_k)$ are then given by

$$\Delta_{i_1,\ldots,i_k}(\tilde{A}_k) = \langle e_{i_1} \wedge \cdots \wedge e_{i_k},\ U^{-1} g \cdot e_1 \wedge \cdots \wedge e_k \rangle.$$

This implies that $\tilde{\pi}_k(g) = \tilde{A}_k = \pi_k(U^{-1}g)$ and hence in general, A_k and $\tilde{A}_k$ represent different elements of $Gr_{k,n}$. The map $\tilde{\pi}_k$ will be useful in Section 6 where we consider the moment map (see Lemma 6.5).

3.2. **The tnn part of the Grassmannian.** One may define the tnn part of the Grassmannian $(Gr_{k,n})_{\ge 0}$ either as the projection of $(G/B^+)_{\ge 0}$, following Lusztig [Lus98], or equivalently in terms of Plücker coordinates, following Postnikov [Pos].

Definition 3.2. The *tnn part of the Grassmannian* $(Gr_{k,n})_{\ge 0}$ is the image $\pi_k((G/B^+)_{\ge 0})$. Equivalently, $(Gr_{k,n})_{\ge 0}$ is the subset of $Gr_{k,n}$ such that all Plücker coordinates are non-negative.

Let $W_k = \langle s_1, \ldots, \hat{s}_k, \ldots, s_{n-1} \rangle$ be a parabolic subgroup of $W = \mathfrak{S}_n$ obtained by deleting the transposition s_k from the generating set. Let W^k denote the set of minimal-length coset representatives of W/W_k. Recall that a *descent* of a permutation z is a position j such that $z(j) > z(j+1)$. Then W^k is the subset of permutations which have at most one descent, and if it exists, that descent must be in position k.

Definition 3.3. For each $w \in W^k$ and $v \le w$, define $\mathcal{P}_{v,w}^{>0} = \pi_k(\mathcal{R}_{v,w}^{>0})$.

Theorem 3.4. [Rie98] We have a decomposition

$$(Gr_{k,n})_{\ge 0} = \bigsqcup_{w \in W^k} \bigsqcup_{v \le w} \mathcal{P}_{v,w}^{>0}$$

of $(Gr_{k,n})_{\ge 0}$ into cells.

Remark 3.5. If $w \in W^k$, then the projection $\pi_k : G/B^+ \to Gr_{k,n}$ is an isomorphism when restricted to $\mathcal{R}_{v,w}$, and hence is an isomorphism from $\mathcal{R}_{v,w}^{>0}$ to the corresponding cell $\mathcal{P}_{v,w}^{>0}$ of $(Gr_{k,n})_{\ge 0}$. More generally, for w not necessarily in W^k, π_k is a surjective map taking cells of $(G/B^+)_{\ge 0}$ to cells of $(Gr_{k,n})_{\ge 0}$.

There is another description of the cell decomposition from Theorem 3.4 which is due to Postnikov [Pos], who discovered it independently. To give Postnikov's description, we first review the matroid stratification of the Grassmannian.

Definition 3.6. Given $A \in Gr_{k,n}$, let $\mathcal{M}(A)$ be the collection of k-element subsets I of $[n]$ such that $\Delta_I(A) \ne 0$. Such a collection is the set of *bases* of a *(realizable) matroid*. Now given a collection $\mathcal{M}$ of k-element subsets of $[n]$, we define the *matroid stratum* $S_{\mathcal{M}}$ to be

$$S_{\mathcal{M}} = \{A \in Gr_{k,n} \mid \mathcal{M}(A) = \mathcal{M}\}.$$

Note that $S_{\mathcal{M}}$ may be empty. Letting $\mathcal{M}$ vary over all subsets of $\binom{[n]}{k}$, we have the *matroid stratification* of the Grassmannian $Gr_{k,n} = \sqcup_{\mathcal{M}} S_{\mathcal{M}}$.

Theorem 3.7. [Pos, Theorem 3.5] The intersection of each matroid stratum $S_{\mathcal{M}}$ with $(Gr_{k,n})_{\ge 0}$ is either empty or is a cell. This gives a cell decomposition of $(Gr_{k,n})_{\ge 0}$.

3.3. **Flag minors of the tnn flag variety.**

Definition 3.8. Let M be an $n \times n$ matrix with real entries. Any determinant of a $k \times k$ submatrix (for $1 \le k \le n$) is called a *flag minor* if its set of columns is precisely $\{1, 2, \dots, k\}$, the leftmost k columns of M. And we say that M is *flag non-negative* if all of its flag minors are non-negative.

Remark 3.9. Note that the flag minors of $g \in G$ are precisely the Plücker coordinates of the projections of gB^+ to the various Grassmannians $\pi_k(gB^+)$ for $1 \le k \le n$.

Lemma 3.10. Let $v \le w$ be elements of W. Choose a reduced expression $\mathbf{w}$ for w and let $\mathbf{v}_+$ be the PDS for v within $\mathbf{w}$. Then $G_{\mathbf{v}_+,\mathbf{w}}^{>0}$ consists of flag non-negative matrices, i.e. for any $g \in G_{\mathbf{v}_+,\mathbf{w}}^{>0}$, all flag minors of g are non-negative.

Proof. Choose any k such that $1 \le k \le n$. Recall from Definition 3.2 that the projection π_k maps the tnn flag variety $(G/B^+)_{\ge 0}$ to the tnn Grassmannian $(Gr_{k,n})_{\ge 0}$, i.e. the subset of the real Grassmannian such that all $k \times k$ minors are non-negative. When we identify $G_{\mathbf{v}_+,\mathbf{w}}^{>0}$ with the cell $\mathcal{R}_{v,w}^{>0}$, this projection maps a matrix $g \in G_{\mathbf{v}_+,\mathbf{w}}^{>0}$ to the span of the leftmost k columns of g. It follows that the $k \times k$ flag minors of g are non-negative. □

We now give a sufficient condition for a flag minor of $G_{\mathbf{v}_+,\mathbf{w}}^{>0}$ to be positive.

Given an element $g \in G_{\mathbf{v}_+,\mathbf{w}}^{>0}$ and a k-element set $I_k = \{i_1 < i_2 < \cdots < i_k\} \in \binom{[n]}{k}$, we let $\Delta_{I_k}^k(g)$ denote the minor of the matrix g given by

$$\Delta_{I_k}^k(g) := \langle e_{i_1} \wedge \cdots \wedge e_{i_k},\, g \cdot e_1 \wedge \cdots \wedge e_k \rangle.$$

In other words, $\Delta_{I_k}^k(g) = \Delta_{I_k}(A_k)$ is the *flag minor* of g which uses the leftmost k columns of g and the rows indexed by I_k.

Let $[k] := \{1, \dots, k\}$. For any $z \in W$ we define the ordered set $z \cdot [k] = \{z(1), \dots, z(k)\}$. (By ordered set, we mean that we sort the elements of $z \cdot [k]$ according to their value.) Then we have the following.

Lemma 3.11. Let $v \le w$ be elements in $W = \mathfrak{S}_n$, and choose $z \in \mathfrak{S}_n$ arbitrarily. Choose a reduced subexpression $\mathbf{w}$ for w; this determines the PDS $\mathbf{v}_+$ for v in $\mathbf{w}$. Choose any $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$. Then we have

$$\Delta^k_{z\cdot[k]}(g) > 0 \qquad \text{for} \quad 1 \le k \le n$$

if and only if

$$v \le z \le w.$$

The classical *tableau criterion* for Bruhat order on $\mathfrak{S}_n$ will be useful for the proof.

Lemma 3.12. [Ehr34] [BB05, Theorem 2.6.3] We have that $x \le y$ in Bruhat order on $\mathfrak{S}_n$ if and only if $x_{i,k} \le y_{i,k}$ for all $1 \le i \le k \le n-1$, where $x_{i,k}$ is the ith entry in the increasing rearrangement of $x_1, x_2, \dots, x_k$, and similarly for $y_{i,k}$.

We now prove Lemma 3.11.

Proof. First suppose that z satisfies $v \le z \le w$. Then by [Rie06], the cell $\mathcal{R}^{>0}_{z,w}$ is contained in the closure $\overline{\mathcal{R}^{>0}_{v,w}}$, and hence $\pi_k(\mathcal{R}^{>0}_{z,w}) \subset \pi_k(\overline{\mathcal{R}^{>0}_{v,w}})$. Recall from Section 3 that the projection π_k of any cell of $(G/B^+)_{\ge 0}$ is a cell of $(Gr_{k,n})_{\ge 0}$, and that cells of $(Gr_{k,n})_{\ge 0}$ are defined by specifying that certain Plücker coordinates are 0 and all others are strictly positive. Therefore if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two cells of $(Gr_{k,n})_{\ge 0}$ such that $\mathcal{C}_1 \subset \overline{\mathcal{C}_2}$, and the Plücker coordinate Δ_J is positive on all points of C_1, then it must also be positive on all points of $\mathcal{C}_2$.

Recall that $\mathcal{R}^{>0}_{z,w} \subset \mathcal{R}_{z,w} = (B^+\dot{w}B^+/B^+) \cap (B^-\dot{z}B^+/B^+)$. Then $\pi_k(\mathcal{R}^{>0}_{z,w}) \subset \pi_k(B^-\dot{z}B^+/B^+)$. It is well known (see e.g. [Pos, Section 2.3]) that the lexicographically minimal nonvanishing Plücker coordinate on $\pi_k(B^-\dot{z}B^+/B^+)$ is $\Delta_{z\cdot[k]}$. Therefore $\Delta_{z\cdot[k]}$ must be strictly positive on any point of $\pi_k(\mathcal{R}^{>0}_{z,w})$.

And now since $\pi_k(\mathcal{R}^{>0}_{z,w}) \subset \pi_k(\overline{\mathcal{R}^{>0}_{v,w}})$, it follows that the Plücker coordinate $\Delta_{z\cdot[k]}$ must be positive on any point in $\pi_k(\mathcal{R}^{>0}_{v,w})$. Finally recall that the projection π_k from G/B^+ to $Gr_{k,n}$ maps an element $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$ to the span of its leftmost k columns, i.e. $\pi_k(g) = A_k$. Therefore we must have that $\Delta_{z\cdot[k]}(A_k) = \Delta^k_{z\cdot[k]}(g) > 0$.

Now suppose that $\Delta^k_{z\cdot[k]}(g) > 0$ for all k such that $1 \le k \le n$. As before, $\pi_k(g)$ represents an element of $\pi_k(\mathcal{R}_{v,w})$, and the lexicographically minimal nonvanishing Plücker coordinate on $\pi_k(\mathcal{R}_{v,w})$ is $\Delta_{v\cdot[k]}(A_k)$. Therefore we must have that the subset $z\cdot[k]$ is lexicographically greater than $v\cdot[k]$. But since this is true for all k, Lemma 3.12 implies that $v \le z \le w$. ☐

4. The full Kostant-Toda lattice

The $\mathfrak{sl}_n(\mathbb{R})$-*full Kostant-Toda (f-KT) lattice* is defined as follows. Let $L \in \mathfrak{sl}_n(\mathbb{R})$ be given by

$$L \in \epsilon + \mathfrak{b}^-,$$

where ϵ is the matrix with 1's on the superdiagonal and zeros elsewhere. We use the coordinates (1.4) for L and refer to it as a *Lax matrix*. Then the f-KT lattice is defined by the *Lax equation* (1.3).

One of the goals of this paper is to describe the behavior of the solution $L(t)$ of the f-KT lattice when the initial matrix $L(0) = L^0$ is associated to an arbitrary point on the tnn part $(G/B^+)_{\ge 0}$ of the flag variety G/B^+. In this section we will give background on the f-KT lattice, including a characterization of the fixed points, and the method of finding solutions via the LU-factorization.

4.1. **The fixed points of the f-KT lattice.** Let $\mathcal{F}_\Lambda$ be the *isospectral variety* consisting of the Hessenburg matrices of (1.4) with fixed eigenvalues $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$,[2] i.e.

$$\mathcal{F}_\Lambda := \{L \in \epsilon + \mathfrak{b}^- \mid L \text{ has eigenvalues } \Lambda\}.$$

[2]Note that we will also occasionally use Λ to denote the matrix $\mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$.

Throughout this paper, we assume that all the eigenvalues are real and distinct, and have the ordering

$$\lambda_1 \;<\; \lambda_2 \;<\; \cdots \;<\; \lambda_n.$$

Note that since $L \in \mathfrak{sl}_n(\mathbb{R})$, we have $\sum_{i=1}^{n} \lambda_i = 0$.

We let E denote the Vandermonde matrix in the λ_i's:

$$E := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}. \tag{4.1}$$

Definition 4.1. A Lax matrix L is a *fixed point* or *stationary point* of the f-KT lattice if

$$\frac{dL}{dt} = 0, \qquad \text{or equivalently,} \qquad [(L)_{\geq 0}, L] = 0.$$

Lemma 4.2. Let L be an $n \times n$ matrix with distinct eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ and let F be an $n \times n$ matrix. Then

$$[F, L] = 0 \qquad \text{implies} \qquad F = \sum_{k=0}^{n-1} c_k L^k$$

for some constants c_k, i.e. F is a polynomial of L.

Proof. Since L is diagonalizable, we can write $L = M\Lambda M^{-1}$ for some invertible matrix M. Then $[F, L] = 0$ implies that $F = MDM^{-1}$ for some diagonal matrix D. Let $D = \mathrm{diag}(\mu_1, \ldots, \mu_n)$; the μ_j's are the eigenvalues of F. Then each μ_j can be expressed as

$$\mu_j = \sum_{i=0}^{n-1} a_i \lambda_j^i \qquad \text{for} \quad j = 1, \ldots, n.$$

Note that this equation is $\mu = a\,E$ with $\mu := (\mu_1, \ldots, \mu_n)$ and $a := (a_0, \ldots, a_{n-1})$. Since the Vandermonde matrix E is invertible, a is uniquely determined for given μ. We can equivalently write this equation as $D = \sum_{i=1}^{n-1} a_i \Lambda^i$ with $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, and hence we have

$$FM = MD = M\left(\sum_{i=0}^{n-1} a_i \Lambda^i\right) = \left(\sum_{i=0}^{n-1} a_i L^i\right) M,$$

where we have used $L = M\Lambda M^{-1}$. Since M is invertible, this implies that $F = \sum_{i=0}^{n-1} a_i L^i$. ∎

Lemma 4.3 allows us to characterize the fixed points of the f-KT lattice.

Lemma 4.3. If L has distinct eigenvalues and $[(L)_{\geq 0},\ L] = 0$, then $L = (L)_{\geq 0}$, that is, $(L)_{<0} = 0$.

Proof. The decomposition $L = (L)_{\geq 0} + (L)_{<0}$ gives

$$[(L)_{\geq 0},\ L] = -[(L)_{<0},\ L] = 0.$$

Then Lemma 4.2 implies that we have

$$(L)_{<0} = \sum_{k=0}^{n-1} c_k L^k.$$

Diagonalizing both sides, we have the row vector equation $0 = c\,E$, where $c = (c_0, \ldots, c_{n-1})$ and the Vandermonde matrix E is given by (4.1). Since E is invertible, we have $c = 0$. ∎

Then we have the following result.

Proposition 4.4. Let L be in $\mathcal{F}_\Lambda$. Then L is a fixed point of the f-KT lattice if and only if

$$L = (L)_{\geq 0}.$$

Moreover, the diagonal part of a fixed point L is given by

$$\text{diag}(L) = \text{diag}(\lambda_{\pi(1)}, \lambda_{\pi(2)}, \ldots, \lambda_{\pi(n)}) \qquad \text{for some} \quad \pi \in \mathfrak{S}_n.$$

Proof. The fact that a matrix of the form $L = (L)_{\geq 0}$ is a fixed point of the f-KT lattice follows directly from the definitions. The other direction is just a corollary of Lemma 4.3.

Now suppose L is a fixed point and hence $L = (L)_{\geq 0}$. Since L has the eigenvalues Λ, the diagonal part of L consists of the eigenvalues. Any arrangement of the eigenvalues can be expressed as $(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(n)})$ with some $\pi \in \mathfrak{S}_n$. □

4.2. **Matrix factorization and the τ-functions of the f-KT lattice.** To find the solution $L(t)$ of the f-KT lattice in terms of the initial matrix $L(0) = L^0$, it is standard to consider the *LU-factorization* of the matrix $\exp(tL^0)$:

$$\exp(tL^0) = u(t)b(t) \qquad \text{with} \quad u(t) \in U^-,\ b(t) \in B^+. \tag{4.2}$$

Note here that $u(0) = b(0) = I$, the identity matrix. It is known and easy to show that this factorization exists if and only if the principal minors are all nonzero. Under this assumption (which applies throughout this section), we let $\tau_k(t)$ denote the k-th principal minor, i.e.

$$\tau_k(t) := [\exp(tL^0)]_k \qquad \text{for} \quad k = 1, 2, \ldots, n. \tag{4.3}$$

These functions, which are called *τ-functions*, play a key role in the method for solving the f-KT lattice, as we will explain below. We first have the following result (see e.g. [Sym82] which deals with the original Toda lattice).

Proposition 4.5. The solution $L(t)$ is given by

$$L(t) = u^{-1}(t)L^0u(t) = b(t)L^0b(t)^{-1}.$$

Proof. Taking the derivative of (4.2), we have

$$\frac{d}{dt}\exp(tL^0) = L^0ub = ubL^0 = \dot{u}b + u\dot{b},$$

where $\dot{x}$ means the derivative of $x(t)$. This equation can also be written as

$$u^{-1}L^0u = bL^0b^{-1} = u^{-1}\dot{u} + \dot{b}b^{-1}.$$

We denote this as $\tilde{L}$, and show $\tilde{L} = L$. Using $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{b}^+$, we decompose $\tilde{L}$ as

$$u^{-1}\dot{u} = (\tilde{L})_{<0} \qquad \text{and} \qquad \dot{b}b^{-1} = (\tilde{L})_{\geq 0}.$$

To show $\tilde{L} = L$, we first show that $\tilde{L}$ also satisfies the f-KT lattice. Differentiating $\tilde{L} = u^{-1}L^0u$, we have

$$\frac{d\tilde{L}}{dt} = -u^{-1}\dot{u}\tilde{L} + \tilde{L}u^{-1}\dot{u} = [-u^{-1}\dot{u}, \tilde{L}].$$

Here we have used $\frac{d}{dt}u^{-1} = -u^{-1}\dot{u}u^{-1}$. Writing $u^{-1}\dot{u} = (\tilde{L})_{<0} = \tilde{L} - (\tilde{L})_{\geq 0}$, we obtain

$$\frac{d\tilde{L}}{dt} = [(\tilde{L})_{\geq 0}, \tilde{L}].$$

Since $\tilde{L}(0) = L^0 = L(0)$, i.e. the initial data are the same, the uniqueness theorem of the differential equation implies that $\tilde{L}(t) = L(t)$. This completes the proof. □

We also have an explicit formula for the diagonal elements of $L(t)$.

Proposition 4.6. The diagonal elements of the matrix $L = L(t)$ can be expressed by

$$a_{k,k}(t) = \frac{d}{dt} \ln \frac{\tau_k(t)}{\tau_{k-1}(t)}, \tag{4.4}$$

where $\tau_k(t) = [\exp(tL^0)]_k$ is the k-th τ-function as defined in (4.3).

Proof. First recall from the proof of Proposition 4.5 that $\dot{b}b^{-1} = (L)_{\geq 0}$. Then the diagonal elements $b_{k,k}$ of the matrix b satisfy the equation,

$$\frac{db_{k,k}}{dt} = a_{k,k}b_{k,k} \qquad \text{for} \quad k = 1, \ldots, n,$$

which gives $a_{k,k} = \frac{d}{dt} \ln b_{k,k}$. From the decomposition $\exp(tL^0) = ub$, we have

$$\tau_k = [ub]_k = \prod_{i=1}^{k} b_{i,i}.$$

This implies that $b_{k,k} = \dfrac{\tau_k}{\tau_{k-1}}$. This completes the proof. □

If M is a matrix, we define the notation with multi-time variables $\mathbf{t} = (t_1, \ldots, t_{n-1})$

$$\Theta_M(\mathbf{t}) := \sum_{j=1}^{n-1} M^j t_j. \tag{4.5}$$

The proof of Proposition 4.5 can be easily extended to give the following solution to the f-KT hierarchy.

Proposition 4.7. Consider the LU-factorization

$$\exp(\Theta_{L^0}(\mathbf{t})) = u(\mathbf{t})b(\mathbf{t}) \qquad \text{with} \quad u(\mathbf{t}) \in U^-, \ b(\mathbf{t}) \in B^+. \tag{4.6}$$

The solution $L(\mathbf{t})$ of the f-KT hierarchy is then given by

$$L(\mathbf{t}) = u(\mathbf{t})^{-1} L^0 u(\mathbf{t}) = b(\mathbf{t}) L^0 b(\mathbf{t})^{-1}.$$

This leads to the following definition of the τ-functions for the f-KT hierarchy.

$$\tau_k(\mathbf{t}) = [\exp(\Theta_{L^\mathbf{0}}(\mathbf{t}))]_k \qquad \text{with} \quad \mathbf{t} = (t_1, \ldots, t_{n-1}), \tag{4.7}$$

where $L^\mathbf{0} = L(\mathbf{0})$, the initial matrix of $L(\mathbf{t})$. As before, the LU-factorization of Proposition 4.7 exists if and only if each τ-function $\tau_k(\mathbf{t})$ is nonzero.

Remark 4.8. An explicit formula for each entry $a_{i,j}(\mathbf{t})$ of $L(\mathbf{t})$ has been obtained in [AvM99, KY96] in terms of the τ-functions and their derivatives with respect to t_j's. However, in this paper we need only the formula for the diagonal elements given in (4.4) with $t = t_1$. We included a direct proof of this formula in order to keep the paper self-contained.

In Section 5, we will associate an initial matrix $L^\mathbf{0}$ to each point on the tnn flag variety. We obtain in this way a large family of regular solutions of the f-KT hierarchy.

5. The solution of the f-KT hierarchy from the tnn flag variety

In this section we discuss the behavior of solutions of the f-KT hierarchy when the initial point is associated to a point on the tnn flag variety. By using the decomposition (2.1) of the tnn flag variety $(G/B^+)_{\geq 0}$, we determine the asymptotic form of the matrix $L(\mathbf{t})$ when the first time variable $t = t_1$ goes to $\pm\infty$. Each asymptotic form is a particular fixed point of the f-KT lattice. We then extend the asymptotic analysis to the case with the multi-times: one can reach any fixed point by sending the multi-time variables to infinity in a particular direction.

We first illustrate how to embed the f-KT flow into the flag variety.

5.1. **The companion embedding of the f-KT flow into the flag variety.** Let us first recall the companion embedding of the iso-spectral variety $\mathcal{F}_\Lambda$ into the flag variety G/B^+ defined in [FH91]

$$\begin{aligned} c_\Lambda : \mathcal{F}_\Lambda &\longrightarrow G/B^+ \\ L &\longmapsto u\,B^+ \end{aligned} \tag{5.1}$$

where $u \in U^-$ is the unique element given by the decomposition $L = u^{-1}C_\Lambda u$ with the companion matrix

$$C_\Lambda = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ \pm\sigma_n & \mp\sigma_{n-1} & \cdots & \sigma_2 & 0 \end{pmatrix} \in \ \epsilon + \mathfrak{b}^-. \tag{5.2}$$

Here the σ_i's are obtained from the characteristic polynomial $\det(\lambda I - L) = \sum_{i=0}^{n}(-1)^i\sigma_i\lambda^{n-i}$ with $\sigma_0 = 1$, that is, the σ_i's are the elementary symmetric polynomials of the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$:

$$\sigma_1 = \sum_{j=1}^{n}\lambda_j = 0, \quad \sigma_2 = \sum_{i<j}\lambda_i\lambda_j, \quad \sigma_3 = \sum_{i<j<k}\lambda_i\lambda_j\lambda_k, \qquad \cdots \qquad \sigma_n = \prod_{i=1}^{n}\lambda_i.$$

By Proposition 4.7, each f-KT flow is represented by the map

$$Ad_{u(\mathbf{t})^{-1}} \ : \ L^0 \longrightarrow L(\mathbf{t}).$$

We then have the following (see also [FH91, CK02, KS08]).

Proposition 5.1. Each f-KT flow maps to the flag variety as

$$\begin{array}{ccc} L^{\mathbf{0}} & \xrightarrow{c_\Lambda} & u_0\,B^+ \\ {\scriptstyle Ad_{u(\mathbf{t})^{-1}}}\big\downarrow & & \big\downarrow \\ L(\mathbf{t}) & \xrightarrow{c_\Lambda} & \left\{\begin{array}{l} u_0\,u(\mathbf{t})\,B^+ \\ = u_0\exp(\Theta_{L^{\mathbf{0}}}(\mathbf{t}))\,B^+ \\ = \exp(\Theta_{C_\Lambda}(\mathbf{t}))\,u_0\,B^+ \end{array}\right. \end{array} \tag{5.3}$$

where $L^{\mathbf{0}} = u_0^{-1}C_\Lambda u_0$. That is, the initial matrix $L^{\mathbf{0}}$ determines the element $u_0 \in U^-$, and each f-KT flow corresponds to an $\exp(\Theta_{C_\Lambda}(\mathbf{t}))$-orbit on the flag variety with the initial point u_0B^+.

Remark 5.2. By Proposition 4.4, a fixed point of the f-KT lattice has the form $L = (L)_{\geq 0}$. This implies that if L is a fixed point, then $\exp(\Theta_L(\mathbf{t})) \in B^+$, hence $u(\mathbf{t})$ is the identity matrix. That is, a fixed point of the f-KT flow in a space $\mathcal{F}_\Lambda$ is the fixed point of the $\exp(\Theta_{C_\Lambda}(\mathbf{t}))$-action in the flag variety G/B^+.

5.2. **The full Kostant-Toda flow on $\mathcal{R}_{v,w}^{>0}$.** In this section we associate to each matrix $g \in G_{\mathbf{v}_+,\mathbf{w}}^{>0}$ (representing a point of $\mathcal{R}_{v,w}^{>0}$) an initial matrix $L^{\mathbf{0}}$ for the f-KT hierarchy. We then express the τ-functions of the f-KT hierarchy with the initial matrix $L^{\mathbf{0}}$ in terms of g. Since the solution of the f-KT hierarchy can be given in terms of the τ-functions in (4.7) (recall Remark 4.8), this allows one to express the solution in terms of g.

Recall the definition of the Vandermonde matrix E from (4.1). Also recall that $[A]_k$ denotes the kth principal minor of the matrix A. The main result of this section is the following.

Proposition 5.3. To each matrix $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$ we can associate an initial matrix $L^{\mathbf{0}} \in \mathcal{F}_\Lambda$, defined by $L^{\mathbf{0}} = u_0^{-1} C_\Lambda u_0$, where C_Λ is given by (5.2), and $u_0 \in U^-$ and $b_0 \in B^+$ are uniquely determined by the equation $Eg = u_0 b_0$. For this choice of g and $L^{\mathbf{0}}$, the τ-functions for the f-KT hierarchy with initial matrix $L^{\mathbf{0}}$ are given by

$$\tau_k(\mathbf{t}) = [\exp(\Theta_{C_\Lambda}(\mathbf{t}))u_0]_k = d_k \left[E \exp\left(\Theta_\Lambda(\mathbf{t})\right) g\right]_k, \tag{5.4}$$

where $d_k = [b_0^{-1}]_k$.

Remark 5.4. Note that if we use Proposition 5.3 to associate L^0 to g, and we subsequently apply the companion embedding to L^0, then we will obtain the point $u_0 B^+ = E g b_0^{-1} B^+ = E g B^+$ of the flag variety. This is actually a point on the *totally positive part* $\mathcal{R}^{>0}_{e,w_0}$ of the flag variety.

The following lemma implies that the construction of $L^{\mathbf{0}}$ in Proposition 5.3 is well-defined.

Lemma 5.5. For each $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$, the product Eg has the LU-factorization, that is, there exist unique $u_0 \in U^-$ and $b_0 \in B^+$ such that $Eg = u_0\, b_0$.

Proof. We calculate the principal minors $[E\,g]_k$ of Eg for $k = 1, \ldots, n$. We have that

$$[E\,g]_k = \left| \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_n^{k-1} \end{pmatrix} \begin{pmatrix} g_{1,1} & \cdots & g_{1,k} \\ g_{2,1} & \cdots & g_{2,k} \\ \vdots & \ddots & \vdots \\ g_{n,1} & \cdots & g_{n,k} \end{pmatrix} \right|.$$

Since $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$, Lemma 3.10 implies that all the $k \times k$ minors of the leftmost k columns are non-negative (and since $g \in G/B^+$, at least one of them is positive). Also since $\lambda_1 < \cdots < \lambda_n$, all the $k \times k$ minors of the top k rows of the Vandermonde matrix are positive. Therefore the Binet-Cauchy lemma implies that $[E\,g]_k > 0$ for all $k = 1 \ldots n$, and hence Eg has the LU-factorization. □

We now complete the proof of Proposition 5.3.

Proof. Our initial matrix is given by

$$L^{\mathbf{0}} = u_0^{-1} C_\Lambda u_0 = u_0^{-1} E \Lambda E^{-1} u_0,$$

where we have used the diagonalization, $C_\Lambda = E\,\Lambda\,E^{-1}$. We then have

$$\begin{aligned} \exp\left(\Theta_{L^{\mathbf{0}}}(\mathbf{t})\right) &= u_0^{-1} \exp(\Theta_{C_\Lambda}(\mathbf{t}))\, u_0 \\ &= u_0^{-1} E\, \exp\left(\Theta_\Lambda(\mathbf{t})\right)\, E^{-1}\, u_0 \\ &= u_0^{-1} E\, \exp\left(\Theta_\Lambda(\mathbf{t})\right)\, g\, b_0^{-1}. \end{aligned}$$

Therefore by (4.7), the τ-functions of the f-KT hierarchy are given by

$$\begin{aligned} \tau_k(\mathbf{t}) &= \left[\exp\left(\Theta_{L^{\mathbf{0}}}(\mathbf{t})\right)\right]_k = \left[u_0^{-1} E \exp\left(\Theta_\Lambda(\mathbf{t})\right) g b_0^{-1}\right]_k \\ &= \left[E \exp\left(\Theta_\Lambda(\mathbf{t})\right) g b_0^{-1}\right]_k = d_k \left[E \exp\left(\Theta_\Lambda(\mathbf{t})\right) g\right]_k, \end{aligned}$$

where $d_k = [b_0^{-1}]_k$. □

Remark 5.6. From the proof above we see that $\exp(\Theta_{C_\Lambda}(\mathbf{t}))u_0 = E \exp(\Theta_\Lambda(\mathbf{t})) g b_0^{-1}$. This implies that the f-KT flow gives a (non-compact) torus action on the flag variety. More precisely, the torus $(\mathbb{R}_{>0})^n$ acts by $\exp(\Theta_\Lambda(\mathbf{t}))$ on the basis vectors consisting of the columns of the Vandermonde matrix E, that is, we have $\exp(\Theta_{L^0}(\mathbf{t}))u_0 B^+ = E \exp(\Theta_\Lambda(\mathbf{t})) g B^+$. Note here that the torus $\exp(\Theta_\Lambda(\mathbf{t}))$ acts on the point gB^+.

Definition 5.7. For $I = \{i_1 < i_2 < \ldots < i_k\}$, we set $E_{i_j}(\mathbf{t}) := \exp\theta_{i_j}(\mathbf{t})$ with $\theta_{i_j}(\mathbf{t}) = \sum_{m=1}^{n-1}\lambda_{i_j}^m t_m$, and define

$$E_I(\mathbf{t}) := \prod_{\ell<m}(\lambda_{i_m} - \lambda_{i_\ell})\, e^{\theta_I(\mathbf{t})} \quad \text{with} \quad \theta_I(\mathbf{t}) = \sum_{j=1}^{k}\theta_{i_j}(\mathbf{t}).$$

Note that $E_I(\mathbf{t})$ is always positive.

Corollary 5.8. Use the notation of Proposition 5.3. Recall from (3.1) that $A_k = \pi_k(g)$ consists of the leftmost k columns of g. Then we can write the τ-function as

$$\tau_k(\mathbf{t}) = d_k \sum_{I\in\binom{[n]}{k}} \Delta_I(A_k)E_I(\mathbf{t}). \tag{5.5}$$

Proof. By Proposition 5.3, we have that

$$\begin{aligned}\tau_k(\mathbf{t}) &= d_k\left[E\exp\left(\Theta_\Lambda(\mathbf{t})\right)g\right]_k \\ &= d_k\left[\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1}\end{pmatrix}\begin{pmatrix} e^{\theta_1(\mathbf{t})} & & & \\ & e^{\theta_2(\mathbf{t})} & & \\ & & \ddots & \\ & & & e^{\theta_n(\mathbf{t})}\end{pmatrix} g\right]_k \\ &= d_k\left|\begin{pmatrix} e^{\theta_1(\mathbf{t})} & e^{\theta_2(\mathbf{t})} & \cdots & e^{\theta_n(\mathbf{t})} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1}e^{\theta_1(\mathbf{t})} & \lambda_2^{k-1}e^{\theta_2(\mathbf{t})} & \cdots & \lambda_1^{k-1}e^{\theta_n(\mathbf{t})}\end{pmatrix}\begin{pmatrix} g_{11} & \cdots & g_{1k} \\ g_{21} & \cdots & g_{2k} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nk}\end{pmatrix}\right| \\ &= d_k\sum_{I\in\binom{[n]}{k}} E_I(\mathbf{t})\Delta_I(A_k),\end{aligned}$$

where in the last step, we have used the Binet-Cauchy lemma. □

When the initial matrix L^0 comes from a point in $G^{>0}_{\mathbf{v}_+,\mathbf{w}}$, the f-KT flow is complete.

Proposition 5.9. Let $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$, and $Eg = u_0b_0$ as in Proposition 5.3. Let $L(0) = L^0 = u_0^{-1}C_\Lambda u_0$. Then the solution $L(\mathbf{t})$ of the f-KT hierarchy is regular for all $\mathbf{t} = (t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1}$.

Proof. Recall that by Proposition 4.7, the solution of the f-KT lattice is given by $L(\mathbf{t}) = u^{-1}(\mathbf{t})L^0u(\mathbf{t})$, where $u(\mathbf{t}) \in U^-$ is obtained from the LU-factorization of the matrix $\exp(\Theta_{L^0}(\mathbf{t})) = u(\mathbf{t})b(\mathbf{t})$ for $b(t) \in B^+$. The τ-functions are the principal minors of $\exp(\Theta_{L^0}(\mathbf{t}))$, and by Corollary 5.8 and Lemma 3.10 they are all positive. Therefore the LU-factorization exists and is unique for each $\mathbf{t} \in \mathbb{R}^{n-1}$. This implies that $L(\mathbf{t})$ is regular for all $\mathbf{t} \in \mathbb{R}^{n-1}$, that is, the f-KT flow is complete. □

Remark 5.10. If a τ-function vanishes at a fixed multi-time $\mathbf{t} = \mathbf{t}_*$, then the LU-factorization of $\exp(\Theta_{L^0}(\mathbf{t}_*))$ fails, i.e. $\exp(\Theta_{L^0}(\mathbf{t}_*)) \in U^-\dot{z}B^+$ for some $z \in \mathfrak{S}_n$ such that $z \neq e$. This means that the f-KT flow becomes singular and hits the boundary of the top Schubert cell, see [CK02, FH91]. The set of times $\mathbf{t}_*$ where the τ-function vanishes is called the *Painlevé divisor*. It is then quite interesting to identify z in terms of the initial matrix L^0 given in the general form of (1.4). This will be discussed in a future work.

Remark 5.11. The τ-function in (5.5) has the Wronskian structure, that is, if we define the functions $\{f_1, \ldots, f_k\}$ by

$$(f_1(\mathbf{t}), \ldots, f_k(\mathbf{t})) := (E_1(\mathbf{t}), \ldots, E_n(\mathbf{t}))\,A_k^T,$$

then we have

$$\tau_k(\mathbf{t}) = d_k\,\mathrm{Wr}(f_1(\mathbf{t}), \ldots, f_k(\mathbf{t})),$$

where the Wronskian is for the t_1-variable. Furthermore, if we identify the first three variables as $t_1 = x,\ t_2 = y$ and $t_3 = t$ in (5.5), then we obtain the τ-function for the KP equation which gives rise to soliton solutions of the KP equation from the Grassmannian $Gr_{k,n}$ [KW13]. That is, τ_k is associated with a point of the Grassmannian $Gr_{k,n}$. Then the set of τ-functions $(\tau_1, \ldots, \tau_{n-1})$ is associated with a point of the flag variety, and the solution space of the f-KT hierarchy is naturally given by the complete flag variety.

5.3. **Asymptotic behavior of the f-KT lattice.** In this section we consider the asymptotics of the solution $L(t)$ to the f-KT lattice as $t = t_1$ goes to $\pm\infty$, and where $L(0) = L^0$ is the initial matrix from Proposition 5.3. To analyze the asymptotics, we use the τ-functions $\tau_k(\mathbf{t})$ from (5.5) with all other times t_j being constants for $j \geq 2$. In Section 5.4, we extend the asymptotic results of this section to the case of the f-KT hierarchy in the $\mathbf{t}$-space.

Recall that we have a fixed order $\lambda_1 < \cdots < \lambda_n$ on the eigenvalues, and that $z \cdot [k]$ denotes the ordered set $\{z(1), z(2), \ldots, z(k)\}$.

Lemma 5.12. Let $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$. Then each $\tau_k(t)$-function from (5.5) has the following asymptotic behavior:

$$\tau_k(t) \ \longrightarrow \ \begin{cases} E_{w\cdot[k]}(t) & \text{as} \quad t \to \infty \\ E_{v\cdot[k]}(t) & \text{as} \quad t \to -\infty \end{cases}$$

Note that in Lemma 5.12 we are ignoring the coefficients of the exponentials $E_{w\cdot[k]}(t)$ and $E_{v\cdot[k]}(t)$.

Proof. Recall that

$$\mathcal{M}(A_k) = \left\{ I \in \binom{[n]}{k} \ \Big| \ \Delta_I(A_k) \neq 0 \right\}.$$

Since $A_k = \pi_k(g)$ and $g \in U^-\dot{v} \cap B^+\dot{w}B^+$ (by Theorem 2.6), the lexicographically maximal and minimal elements in $\mathcal{M}(A_k)$ are respectively given by $w \cdot [k]$ and $v \cdot [k]$.

Recall from Definition 5.7 that $E_j(t) = \exp(\theta_j(t))$. Below, we write E_j for $E_j(t)$. Since $\lambda_1 < \cdots < \lambda_n$, we have

$$E_1 \ll E_2 \ll \cdots \ll E_n, \quad \text{as} \quad t \to \infty,$$
$$E_1 \gg E_2 \gg \cdots \gg E_n, \quad \text{as} \quad t \to -\infty,$$

which implies the lemma. □

Our main theorem on the asymptotic behavior of the matrix $L(\mathbf{t})$ is the following.

Theorem 5.13. Let $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$ and let $L^0 \in \mathcal{F}_\Lambda$ be the initial matrix as given by Proposition 5.3. Then the diagonal elements $a_{k,k} = a_{k,k}(t)$ of $L(t)$ satisfy

$$a_{k,k} \ \longrightarrow \ \begin{cases} \lambda_{w(k)} & \text{as} \quad t \to \infty \\ \lambda_{v(k)} & \text{as} \quad t \to -\infty \end{cases}$$

Furthermore, $L(t)$ approaches a fixed point of the f-KT flow as $t \to \pm\infty$: we have $(L)_{<0} \to 0$ and

$$L(t) \ \longrightarrow \ \begin{cases} \epsilon + \operatorname{diag}(\lambda_{w(1)}, \lambda_{w(2)}, \ldots, \lambda_{w(n)}) & \text{as} \quad t \to \infty \\ \epsilon + \operatorname{diag}(\lambda_{v(1)}, \lambda_{v(2)}, \ldots, \lambda_{v(n)}) & \text{as} \quad t \to -\infty \end{cases}$$

Proof. We consider only the case for $t \to \infty$; the proof is the same in the other case.

First recall from (4.4) that

$$a_{k,k} = \frac{d}{dt} \ln \frac{\tau_k}{\tau_{k-1}}.$$

Then from Lemma 5.12, we have

$$a_{k,k} \ \longrightarrow \ \frac{d}{dt} \ln \frac{E_{w\cdot[k]}}{E_{w\cdot[k-1]}} = \frac{d}{dt} \ln E_{w(k)} = \lambda_{w(k)} \quad \text{as } t \to \infty.$$

Note that $u_{k,k} := a_{k,k} - \lambda_{w(k)}$ decays *exponentially* to zero as $t \to \infty$, i.e. we have $\frac{da_{k,k}}{dt} \to 0$ exponentially. This decay property of the diagonal elements is key for proving the convergence $(L)_{<0} \to 0$.

Now we show $(L)_{<0} \to 0$. From (1.5) with $\ell = 0$, we have

$$\frac{da_{k,k}}{dt} = a_{k+1,k} - a_{k,k-1}, \qquad \text{for} \quad k = 1, \dots, n,$$

where by convention we have $a_{i,j} = 0$ if $i = n+1$ or $j = 0$. Since the left hand side decays exponentially, the case $k = 1$ gives $a_{2,1} \to 0$. Now by induction on k we find that all the elements of the first sub-diagonal decay exponentially, that is, we have

$$a_{k+1,k} \longrightarrow 0 \quad \text{for} \quad k = 1, \dots, n-1.$$

From (1.5) with $\ell = 1$, we have

$$\frac{da_{k+1,k}}{dt} = a_{k+2,k} - a_{k+1,k-1} + (a_{k+1,k+1} - a_{k,k})a_{k+1,k}, \qquad \text{for} \quad k = 1, \dots, n-1,$$

and hence the elements of the second sub-diagonal of $(L)_{<0}$ are given by

$$a_{k+2,k} = a_{k+1,k-1} + \frac{da_{k+1,k}}{dt} - (a_{k+1,k+1} - a_{k,k})a_{k+1,k}.$$

Now using induction on k, all the functions on the right-hand side decay exponentially, and hence the elements $a_{k+2,k}$ for $k = 1, \dots, n-2$ also decay exponentially. (The base case with $k = 1$ shows the decay of $a_{3,1}$.)

Continuing this argument for $\ell > 1$ yields $(L)_{<0} \to 0$. This completes the proof. □

Example 5.14. Consider the $\mathfrak{sl}_5$ f-KT lattice. Let $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$ where $\mathbf{v}_+$ and $\mathbf{w}$ are given by

$$\mathbf{w} = s_2 s_3 s_1 s_4 s_3 s_2 \qquad \text{and} \qquad \mathbf{v}_+ = s_2\, 1\, 1\, s_4 s_3\, 1.$$

Since $w \cdot (1,2,3,4,5) = (3,5,1,4,2)$ and $v \cdot (1,2,3,4,5) = (1,3,5,2,4)$, we have the following asymptotic matrices,

$$L(t) \;\longrightarrow\; \begin{cases} \epsilon + \operatorname{diag}(\lambda_1, \lambda_3, \lambda_5, \lambda_2, \lambda_4) & \text{as} \quad t \to -\infty \\ \epsilon + \operatorname{diag}(\lambda_3, \lambda_5, \lambda_1, \lambda_4, \lambda_2) & \text{as} \quad t \to \infty \end{cases}$$

Definition 5.15. We say that a flow $L(t)$ *has the asymptotic form* (v,w) for $(v,w) \in \mathfrak{S}_n \times \mathfrak{S}_n$ if when $t \to -\infty$ (resp. $t \to \infty$) the Lax matrix $L(t)$ tends to $\epsilon + \operatorname{diag}(\lambda_{v(1)}, \dots, \lambda_{v(n)})$ (resp. $\epsilon + \operatorname{diag}(\lambda_{w(1)}, \dots, \lambda_{w(n)})$).

Then Theorem 5.13 implies the following.

Corollary 5.16. We consider f-KT flows coming from points of $(G/B^+)_{\geq 0}$. If $v \leq w$ are comparable elements in $\mathfrak{S}_n$, then there exists a regular f-KT flow whose asymptotic form is (v,w). Moreover, the dimension of the space of such flows is $\ell(w) - \ell(v)$.

Proof. By Proposition 5.9, a flow coming from $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$ is regular. Theorem 2.6 implies that if we consider all flows $L(t)$ with initial matrix L^0 coming from some g representing a point of $(G/B^+)_{\geq 0}$, the dimension of the space of flows with asymptotic form (v,w) is precisely $\ell(w) - \ell(v)$. □

Remark 5.17. The fact that the space of flows with the asymptotic form (v,w) has dimension $\ell(w)-\ell(v)$ is similar to [CSS12, Corollary 3.3] for the case of the full symmetric Toda lattice. We will discuss the relation between the f-KT lattice and the full symmetric Toda lattice in Section 7.

5.4. **Asymptotic behavior of the f-KT flow in the t-space.** We now consider the solution $L(\mathbf{t})$ to the f-KT hierarchy with the multi-time variables $\mathbf{t} = (t_1, \dots, t_{n-1})$. The main result in this section is the following.

Proposition 5.18. Let $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$, and let $z \in \mathfrak{S}_n$. Assume that $v \le z \le w$, or equivalently (by Lemma 3.11) $\Delta^k_{z\cdot[k]}(g) > 0$ for all $k = 1, \dots, n-1$. Then there exists a direction $\mathbf{t}(s) = s\mathbf{c}$ with a constant vector $\mathbf{c} = (c_1, \dots, c_{n-1})$ such that

$$L(\mathbf{t}(s)) \;\longrightarrow\; \epsilon + \mathrm{diag}(\lambda_{z(1)}, \dots, \lambda_{z(n)}) \quad \text{as} \quad s \to \infty. \tag{5.6}$$

In other words, when we take such a limit, $L(\mathbf{t})$ approaches the fixed point $\epsilon + \mathrm{diag}(\lambda_{z(1)}, \dots, \lambda_{z(n)})$.

To prove the proposition, we use the following lemma.

Lemma 5.19. For any permutation $z \in \mathfrak{S}_n$, one can find a multi-time $\mathbf{c} = (c_1, \dots, c_{n-1}) \in \mathbb{R}^{n-1}$ such that $E_{z(1)}(\mathbf{c}) > E_{z(2)}(\mathbf{c}) > \cdots > E_{z(n)}(\mathbf{c})$.

Proof. Recall that $E_i(\mathbf{t}) = \exp \theta_i(\mathbf{t})$ with $\theta_i(\mathbf{t}) = \sum_{m=1}^{n-1} \lambda_i^m t_m$. Define the function $\ell_i : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$\ell_i(t_0, \mathbf{t}) = t_0 + \lambda_i t_1 + \lambda_i^2 t_2 + \cdots + \lambda_i^{n-1} t_{n-1} = t_0 + \theta_i(\mathbf{t}) = (t_0, t_1, \dots, t_{n-1}) \cdot E_i^0,$$

where E_i^0 is the i-th column vector of the Vandermonde matrix E. To prove the lemma, it suffices to find a point $(t_0, \mathbf{t})$ such that $\ell_{z(1)}(t_0, \mathbf{t}) > \ell_{z(2)}(t_0, \mathbf{t}) > \cdots > \ell_{z(n)}(t_0, \mathbf{t})$, which also implies that $\theta_{z(1)}(\mathbf{t}) > \theta_{z(2)}(\mathbf{t}) > \cdots > \theta_{z(n)}(\mathbf{t})$, and hence $E_{z(1)}(\mathbf{t}) > E_{z(2)}(\mathbf{t}) > \cdots > E_{z(n)}(\mathbf{t})$.

Let $(r_1, \dots, r_n) \in \mathbb{R}^n$ be any point such that $r_{z(1)} > \cdots > r_{z(n)}$. Then since E is an invertible matrix, we can solve the equation $(t_0, t_1, \dots, t_{n-1}) \cdot E = (r_1, \dots, r_n)$. This proves the lemma. ∎

Remark 5.20. In the proof of Lemma 5.19, note that E_i^0 is the normal vector to the plane given by $\ell_i(t_0, \mathbf{t}) =$ constant. Since the λ_i's are distinct, the E_i^0's are linearly independent. Therefore the hyperplane arrangement

$$\ell_i(t_0, \mathbf{t}) - \ell_j(t_0, \mathbf{t}) = 0 \quad \text{for} \quad 1 \le i < j \le n \tag{5.7}$$

is a *braid arrangement* which has $n!$ regions in its complement, indexed by the permutations $z \in \mathfrak{S}_n$ (the permutation specifies the total order of the values that the functions ℓ_i take on that region). Since the arrangement (5.7) does not depend on the t_0 variable, the $(n-1)$-dimensional $\mathbf{t}$-space is divided into $n!$ convex polyhedral cones, each labeled by some $z \in \mathfrak{S}_n$ specifying the total order of the $\theta_i(\mathbf{t})$'s. See Figure 1 and Example 5.21 below.

We now prove Proposition 5.18.

Proof. Choose a point $\mathbf{c}$ in the $\mathbf{t}$-space such that $E_{z(1)}(\mathbf{c}) > \cdots > E_{z(n)}(\mathbf{c})$; by Lemma 5.19, one exists. Recall that $E_I(\mathbf{t}) = \prod_{\ell<m}(\lambda_{i_m} - \lambda_{i_\ell}) \prod_{j=1}^k E_{i_j}(\mathbf{t})$ for $I = \{i_1, \dots, i_k\}$. This implies that $E_{z\cdot[k]}(\mathbf{c})$ dominates the other exponentials in the τ_k-function (5.5) at the point $\mathbf{c}$. Then we consider the direction $\mathbf{t}(s) = s\mathbf{c}$, and take the limit $s \to \infty$. This gives

$$\tau_k(\mathbf{t}(s)) \;\approx\; d_k \Delta^k_{z\cdot[k]}(g) E_{z\cdot[k]}(\mathbf{t}(s)) \quad \text{as} \quad s \to \infty.$$

Now using the formula for $a_{k,k}(\mathbf{t})$ in (4.4) with $t = t_1$, one can follow the proof of Theorem 5.13 to show that $a_{k,k}(\mathbf{t}(s)) \to \lambda_{z(k)}$ as $s \to \infty$. We also have that $\left.\dfrac{\partial a_{k,k}}{\partial t_1}\right|_{\mathbf{t}=\mathbf{t}(s)} \to 0$ as $s \to \infty$, which allows us to use the proof of Theorem 5.13 to show that $(L)_{<0} \to 0$, i.e. $L(\mathbf{t}(s))$ approaches the fixed point as $s \to \infty$. ∎

Example 5.21. Consider the case of the $\mathfrak{sl}_3(\mathbb{R})$ f-KT lattice, shown in Figure 1. The $\mathbf{t}$-space is divided into $|\mathfrak{S}_3| = 6$ convex polyhedral cones by the braid arrangement $\ell_i - \ell_j = 0$ for $1 \le i < j \le 3$. The cone defined by $\ell_i > \ell_j > \ell_k$ is labeled by the permutation $z \in \mathfrak{S}_3$ such that $z(1) = i$, $z(2) = j$, and $z(3) = k$. In this example, we have the following limits of t_2 with t_1=constant:

$$L(t_1,t_2) \longrightarrow \begin{cases} \epsilon + \mathrm{diag}(\lambda_3,\lambda_1,\lambda_2) & \text{as} \quad t_2 \to \infty \\ \epsilon + \mathrm{diag}(\lambda_2,\lambda_1,\lambda_3) & \text{as} \quad t_2 \to -\infty \end{cases}$$

Thus each cone corresponds to a fixed point of the f-KT hierarchy, and the polygon connecting those fixed points is a hexagon (see the right of Figure 1). This polygon is the so-called *permutohedron* of $\mathfrak{S}_3$. We will discuss the permutohedron and a generalization of it in Section 6.

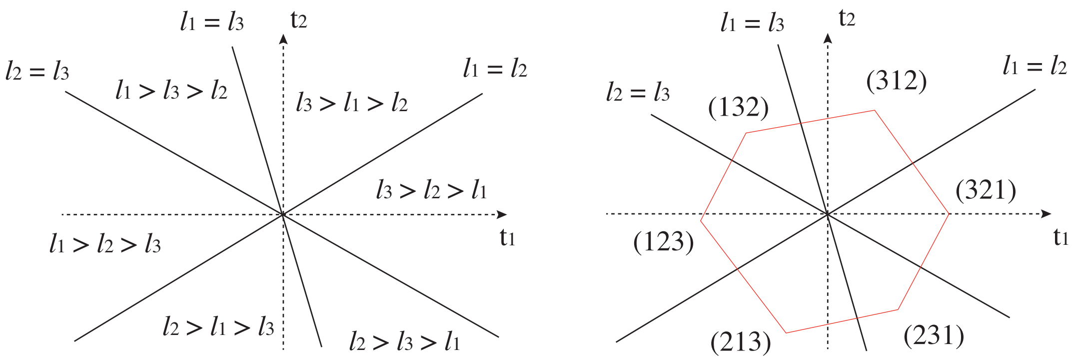

FIGURE 1. The braid arrangement for the $\mathfrak{sl}_3(\mathbb{R})$ f-KT lattice. Here $\ell_i - \ell_j = (\lambda_i - \lambda_j)t_1 + (\lambda_i^2 - \lambda_j^2)t_2$ with $\lambda_1 < \lambda_2 < \lambda_3$. There are six cones, and each cone has the ordering $\ell_{z(1)} > \ell_{z(2)} > \ell_{z(3)}$ with some $z \in \mathfrak{S}_3$. Each vertex (i,j,k) of the polytope corresponds to the cone with $\ell_i > \ell_j > \ell_k$.

6. The moment polytope of the f-KT lattice

In this section we define the moment map for the flag variety, and study the image of the moment map on the f-KT flows coming from the tnn flag variety. In Theorem 6.10 we show that we obtain in this way certain convex polytopes which generalize the permutohedron.

6.1. **The moment map for Grassmannians and flag varieties.** Let L_i denote a weight of the standard representation of $\mathfrak{sl}_n$, i.e. for $h = \mathrm{diag}(h_1,\dots,h_n) \in \mathfrak{h}$, $\mathsf{L}_i(h) = h_i$. Then let $\mathfrak{h}_{\mathbb{R}}^*$ denote the dual of the Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}$, i.e.

$$\mathfrak{h}_{\mathbb{R}}^* := \mathrm{Span}_{\mathbb{R}} \left\{ \mathsf{L}_1,\dots,\mathsf{L}_n \;\middle|\; \sum_{j=1}^n \mathsf{L}_j = 0 \right\} \cong \mathbb{R}^{n-1}.$$

For $I = \{i_1,\dots,i_k\}$, we set $\mathsf{L}(I) = \mathsf{L}_{i_1} + \mathsf{L}_{i_2} + \dots + \mathsf{L}_{i_k} \in \mathfrak{h}_{\mathbb{R}}^*$. The *moment map* for the Grassmannian $\mu_k : Gr_{k,n} \to \mathfrak{h}_{\mathbb{R}}^*$ is defined by

$$\mu_k(A_k) := \frac{\sum_{I\in\mathcal{M}(A_k)} |\Delta_I(A_k)|^2\,\mathsf{L}(I)}{\sum_{I\in\mathcal{M}(A_k)} |\Delta_I(A_k)|^2}, \tag{6.1}$$

see e.g. [GS87, GGMS87, Shi02].

We recall the following fundamental result of Gelfand-Goresky-MacPherson-Serganova [GGMS87] on the moment map for the Grassmannian (which in turn uses the convexity theorem of Atiyah [Ati82] and Guillemin-Sternberg [GS82]).

Theorem 6.1. [GGMS87, Section 2] If $A_k \in Gr_{k,n}$ and we consider the action of the torus $(\mathbb{C}^*)^n$ on $Gr_{k,n}$ (which rescales columns of the matrix representing A_k), then the closure of the image of the moment map applied to the torus orbit of A_k is a convex polytope

$$\Gamma_{\mathcal{M}(A_k)} = \mathrm{Conv}\{\mathsf{L}(I) \ \mid \ \Delta_I(A_k) \neq 0 \text{ i.e. } I \in \mathcal{M}(A_k)\} \tag{6.2}$$

called a *matroid polytope*, whose vertices correspond to the fixed points of the action of the torus.

Remark 6.2. In representation theory, this polytope is a weight polytope of the fundamental representation of $\mathfrak{sl}_n$ on $\bigwedge^k V$, where V is the standard representation.

Corollary 6.3. If $A_k \in Gr_{k,n}$ and we consider the action of the *positive* torus $(\mathbb{R}_{>0})^n$ on $Gr_{k,n}$, the conclusion of Theorem 6.1 still holds.

Proof. If $T_u := \mathrm{diag}(u_1, \dots, u_n)$ with $u_i \in \mathbb{C}^*$ then for $I = \{i_1, \dots, i_k\}$, we have that $\Delta_I(A_k T_u) = u_{i_1} u_{i_2} \dots u_{i_k} \Delta_I(A_k)$. It's clear from the definition that $\mu_k(A_k T_u)$ depends only on the magnitudes of each of the u_i's. Therefore the image of the moment map applied to the torus orbit of A_k is the same if we restrict to the positive torus. □

One may extend the moment map from the Grassmannian to the flag variety G/B^+ [GS87]. First recall that the flag variety G/B^+ has a projective embedding in $\mathbb{P}(V) \times \mathbb{P}(\bigwedge^2 V) \times \cdots \times \mathbb{P}(\bigwedge^{n-1} V)$ which is the image of G acting on the highest weight vector $[e_1 \otimes (e_1 \wedge e_2) \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{n-1})]$ (which has weight $n\mathsf{L}_1 + (n-1)\mathsf{L}_2 + \cdots + \mathsf{L}_n = \sum_{k=1}^{n-1} \mathsf{L}(I_k)$ with $I_k := \{1, \dots, k\}$).

The *moment map* for the flag variety $\mu : G/B^+ \to \mathfrak{h}^*_{\mathbb{R}}$ is defined by

$$\mu(g) := \sum_{k=1}^{n-1} \mu_k(A_k), \qquad \text{where} \qquad A_k = \pi_k(g).$$

If we let gB^+ vary over all points in G/B^+, we obtain the *moment polytope*, which is also called the *permutohedron* Perm_n. For $(i_1, i_2, \dots, i_n) \in \mathfrak{S}_n$, define the weight

$$\mathsf{L}_{i_1, i_2, \dots, i_n} := n\mathsf{L}_{i_1} + (n-1)\mathsf{L}_{i_2} + \cdots + \mathsf{L}_{i_n} = \sum_{k=1}^{n} \mathsf{L}(I_k),$$

where $I_k := \{i_1, \dots, i_k\}$. (Note $\mathsf{L}(I_n) = \mathsf{L}_1 + \cdots + \mathsf{L}_n = 0$.) By definition, the permutohedron Perm_n is the convex hull of the weights $\mathsf{L}_\pi := \mathsf{L}_{i_1, \dots, i_n}$ as $\pi(1, \dots, n) = (i_1, \dots, i_n)$ varies over $\mathfrak{S}_n$, i.e.

$$\mathrm{Perm}_n = \mathrm{Conv}\{\mathsf{L}_\pi \in \mathfrak{h}^*_{\mathbb{R}} \mid \pi \in \mathfrak{S}_n\}.$$

Example 6.4. Consider the case of $G = \mathrm{SL}_3(\mathbb{R})$. The permutohedron Perm_3 is given by Figure 2.

6.2. **The moment polytope of the f-KT flows.** Recall from Section 5.2 that to each point of the tnn flag variety (represented by some point $g \in G^{>0}_{\mathbf{v}_+, \mathbf{w}}$), one can associate a unique element $u_0 \in U^-$, which in turn determines an initial point L^0 of a flow. Throughout this section we fix g, u_0, and L^0 accordingly. Furthermore we let $A_k = \pi_k(g)$ be the span of the leftmost k columns of g, as in (3.1). Our main goal here is to compute the image of the moment map $\mu : G/B^+ \to \mathfrak{h}^*_{\mathbb{R}}$ when applied to the f-KT flow $\exp(\Theta_{C_\Lambda}(\mathbf{t}))$ on the point $u_0 B^+$ of the flag variety described in (5.3).

The following lemma will help us compute the image of the moment map.

Lemma 6.5. Let $\mathcal{E} := \{E_i^0 : i = 1, \dots, n\}$ be the ordered set of column vectors of E, i.e. $E_i^0 = (1, \lambda_i, \dots, \lambda_i^{n-1})^T$. Then the projection $\tilde{\pi}_k$ of (3.3) on $\mathbb{P}(\bigwedge^k \mathbb{R}^n)$ with the basis $\mathcal{E}$ is given by

$$\tilde{\pi}_k(\exp(\Theta_{C_\Lambda}(\mathbf{t})) u_0) = A_k e^{\Theta_\Lambda(\mathbf{t})}.$$

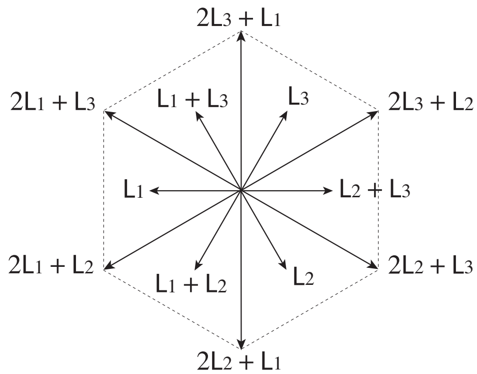

FIGURE 2. The permutohedron Perm_3, which is a hexagon. Each vertex is given by the weight $\mathsf{L}_z := \mathsf{L}_{z(1),z(2),z(3)} = 2\mathsf{L}_{z(1)} + \mathsf{L}_{z(2)} = 3\mathsf{L}_{z(1)} + 2\mathsf{L}_{z(2)} + \mathsf{L}_{z(3)}$ for some $z \in \mathfrak{S}_3$.

Proof. We compute $\exp(\Theta_{C_\Lambda}(\mathbf{t}))u_0 \cdot e_1 \wedge \cdots \wedge e_k$ on $\bigwedge^k \mathbb{R}^n$ with the basis $\mathcal{E}$. The first equality below comes from Remark 5.6, and recall that $d_k = [b_0^{-1}]_k$.

$$\begin{aligned}
&\exp(\Theta_{C_\Lambda}(\mathbf{t}))u_0 \cdot e_1 \wedge \cdots \wedge e_k = E\, e^{\Theta_\Lambda(\mathbf{t})}\, g b_0^{-1} \cdot e_1 \wedge \cdots \wedge e_k \\
=& \sum_{1\le i_1<\cdots<i_k\le n} E e^{\Theta_\Lambda(\mathbf{t})} e_{i_1} \wedge \cdots \wedge e_{i_k} \langle e_{i_1} \wedge \cdots \wedge e_{i_k}, g b_0^{-1} \cdot e_1 \wedge \cdots \wedge e_k\rangle \\
=& d_k \sum_{1\le i_1<\cdots<i_k\le n} E e^{\Theta_\Lambda(\mathbf{t})} e_{i_1} \wedge \cdots \wedge e_{i_k} \langle e_{i_1} \wedge \cdots \wedge e_{i_k}, g \cdot e_1 \wedge \cdots \wedge e_k\rangle \\
=& d_k \sum_{1\le i_1<\cdots<i_k\le n} \Delta_{i_1,\ldots,i_k}(A_k) E e^{\Theta_\Lambda(\mathbf{t})} e_{i_1} \wedge \cdots \wedge e_{i_k} \\
=& d_k \sum_{1\le i_1<\cdots<i_k\le n} \Delta_{i_1,\ldots,i_k}(A_k) e^{\theta_{i_1,\ldots,i_k}(\mathbf{t})} E \cdot e_{i_1} \wedge \cdots \wedge e_{i_k} \\
=& d_k \sum_{1\le i_1<\cdots<i_k\le n} \Delta_{i_1,\ldots,i_k}(A_k e^{\Theta_\Lambda(\mathbf{t})}) E^0_{i_1} \wedge \cdots \wedge E^0_{i_k},
\end{aligned}$$

where $\theta_{i_1,\ldots,i_k}(\mathbf{t}) = \sum_{j=1}^{k} \theta_{i_j}(\mathbf{t})$ with $\theta_j(\mathbf{t}) = \sum_{m=1}^{n-1} \lambda_j^m t_m$. This completes the proof. □

Lemma 6.5 implies that the Plücker coordinates for the f-KT flow are very simple when computed in the basis $\mathcal{E} = \{E^0_1, \ldots, E^0_n\}$.

We then define the moment map μ in this basis as follows. First write $\varphi(\mathbf{t}; g) := \mu(\exp(\Theta_{C_\Lambda}(\mathbf{t}))u_0)$ and $\varphi_k(\mathbf{t}; g) := \mu_k(\tilde{\pi}_k(\exp(\Theta_{C_\Lambda}(\mathbf{t}))u_0)) = \mu_k(A_k e^{\Theta_\Lambda(\mathbf{t})})$ with $\pi_k(g) = A_k$. Then we have

$$\varphi(\mathbf{t}; g) = \sum_{k=1}^{n-1} \varphi_k(\mathbf{t}; g) \qquad \text{with} \quad \varphi_k(\mathbf{t}; g) = \sum_{I\in\mathcal{M}(A_k)} \alpha_I^k(\mathbf{t}; g)\, \mathsf{L}(I), \tag{6.3}$$

$$\text{and} \qquad \alpha_I^k(\mathbf{t}; g) = \frac{\left(\Delta_I(A_k e^{\Theta_\Lambda(\mathbf{t})})\right)^2}{\sum_{J\in\mathcal{M}(A_k)} \left(\Delta_J(A_k e^{\Theta_\Lambda(\mathbf{t})})\right)^2}.$$

Note here that $0 < \alpha_I^k(\mathbf{t}; g) < 1$ and $\sum_{I\in\mathcal{M}(A_k)} \alpha_I^k(\mathbf{t}; g) = 1$ for each k.

Definition 6.6. We define the *moment map image of the f-KT flow* for $g \in G^{>0}_{\mathbf{v},\mathbf{w}}$ to be the set

$$\mathsf{Q}_g = \overline{\{\varphi(\mathbf{t};g) \mid \mathbf{t} \in \mathbb{R}^{n-1}\}} := \overline{\bigcup_{\mathbf{t}\in\mathbb{R}^{n-1}} \varphi(\mathbf{t};g)}.$$

Here the closure is taken using the usual topology of the Euclidian norm on $\mathfrak{h}^*_{\mathbb{R}} \cong \mathbb{R}^{n-1}$.

Proposition 6.7. For each k, define $\mathsf{Q}^k_g := \overline{\{\varphi_k(\mathbf{t};g) \mid \mathbf{t} \in \mathbb{R}^{n-1}\}}$. Then Q^k_g equals the corresponding matroid polytope from (6.2), i.e.

$$\mathsf{Q}^k_g = \Gamma_{\mathcal{M}(A_k)} \quad \text{where} \quad A_k = \pi_k(g).$$

Proof. Recall that $\varphi_k(\mathbf{t};g) = \mu_k(A_k e^{\Theta_\Lambda(\mathbf{t})})$. Since the λ_i's are distinct, as $\mathbf{t}$ varies over $(\mathbb{R})^{n-1}$, the diagonal matrices $T_\Lambda := e^{\Theta_\Lambda(\mathbf{t})}$ cover all points in the positive torus. The result now follows from Corollary 6.3. □

Corollary 6.8. Let $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$. Then the moment map image Q_g of the f-KT flow for g is a Minkowski sum of matroid polytopes. More specifically, for $A_k = \pi_k(g)$, $k = 1,\dots,n-1$, we have

$$\mathsf{Q}_g = \sum_{k=1}^{n-1} \Gamma_{\mathcal{M}(A_k)}.$$

Proof. This follows from (6.3) and Proposition 6.7. □

We also define a certain polytope which sits inside the permutohedron.

Definition 6.9. Let v and w be two permutations in $\mathfrak{S}_n$ such that $v \le w$. We define the *Bruhat interval polytope* associated to (v,w) to be the following convex hull:

$$\mathsf{P}_{v,w} := \mathrm{Conv}\{\mathsf{L}_z \in \mathfrak{h}^*_{\mathbb{R}} \mid v \le z \le w\}.$$

In other words, this is the convex hull of all permutation vectors corresponding to permutations z lying in the Bruhat interval $[v,w]$. In particular, if $w = w_0$ and $v = e$, then we have $\mathsf{P}_{e,w_0} = \mathrm{Perm}_n$.

The main result of this section is the following.

Theorem 6.10. Let $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$. Then the moment map image of the f-KT flow for g is the convex polytope $\mathsf{P}_{v,w}$, i.e.

$$\mathsf{Q}_g = \mathsf{P}_{v,w}.$$

Proof. Recall from Corollary 6.8 that Q_g is a Minkowski sum of the matroid polytopes $\Gamma_{\mathcal{M}(A_k)}$ for $1 \le k \le n-1$. Therefore Q_g is a convex polytope.

Now consider an arbitrary vertex V of Q_g. We claim that V has the form L_z for some $z \in \mathfrak{S}_n$. Since Q_g is the Minkowski sum of the polytopes $\Gamma_{\mathcal{M}(A_k)}$, V can be written as a sum of vertices of those polytopes, i.e. for $I_k \in \mathcal{M}(A_k)$, $k = 1,\dots,n-1$, the vertex V has the form

$$\mathsf{L}(I_1) + \mathsf{L}(I_2) + \cdots + \mathsf{L}(I_{n-1}). \tag{6.4}$$

To prove the claim, it suffices to show that $I_1 \subset I_2 \subset \cdots \subset I_{n-1}$.

By (6.3), any point of $\{\varphi(\mathbf{t};g) \mid \mathbf{t} \in \mathbb{R}^{n-1}\}$ has the form

$$\sum_{k=1}^{n-1} \sum_{I\in\mathcal{M}(A_k)} \alpha^k_I\, \mathsf{L}(I) \tag{6.5}$$

where $0 < \alpha^k_I < 1$ and $\sum_{I\in\mathcal{M}(A_k)} \alpha^k_I = 1$ for each k. Because each A_k is the projection $\pi_k(g)$ of the same element g, it follows that whenever the coefficient of $\mathsf{L}(I)$ is nonzero (i.e. $I \in \mathcal{M}(A_k)$), for each $j < k$ there exists some j-element subset $J \subset I$ such that the coefficient of $\mathsf{L}(J)$ is nonzero (i.e. $J \in \mathcal{M}(A_j)$). We call the latter property the *flag* property of points in $\{\varphi(\mathbf{t};g) \mid \mathbf{t} \in \mathbb{R}^{n-1}\}$.

But now the vertex V is a limit of points of the form (6.5) which have the flag property, and also V has the form (6.4). In particular, for each k, precisely one coefficient α_I^k equals 1 and the rest are zero. It follows that $I_1 \subset I_2 \subset \cdots \subset I_{n-1}$. This proves the claim.

Now suppose that z is a permutation such that $v \le z \le w$. We will show that $\mathsf{L}_z \in \mathsf{Q}_g$. By Lemma 3.11, we have $\Delta^k_{z\cdot[k]}(g) \neq 0$ for each k, and hence $z \cdot [k] \in \mathcal{M}(A_k)$. It follows that $\sum_{i=1}^{k} \mathsf{L}_{z(i)} = \mathsf{L}(z\cdot[k])$ is a vertex of $\Gamma_{\mathcal{M}(A_k)}$, and hence that $\sum_{k=1}^{n}(n-k)\,\mathsf{L}_{z(k)} = \mathsf{L}_z$ is a point of Q_g. (Recall that $\sum_{j=1}^{n} L_j = 0$.) This shows that if $v \le z \le w$ then $\mathsf{L}_z \in \mathsf{Q}_g$.

Conversely, suppose that for some permutation z we have $\mathsf{L}_z \in \mathsf{Q}_g$. Then since $\mathsf{L}_z = \sum_{k=1}^{n}(n-k)\mathsf{L}_{z(k)}$,

$$\varphi(\mathbf{t};g) = \sum_{k=1}^{n-1}\sum_{I\in\mathcal{M}(A_k)} \alpha_I^k(\mathbf{t};g)\,\mathsf{L}(I),$$

and $0 < \alpha_I^k(\mathbf{t};g) < 1$, it follows that $z\cdot[k] \in \mathcal{M}(A_k)$ for $1 \le k \le n-1$. Therefore $\Delta^k_{z\cdot[k]}(g) \neq 0$ for $1 \le k \le n-1$, and hence by Lemma 3.11, we have that $v \le z \le w$. We have shown that $\mathsf{L}_z \in \mathsf{Q}_g$ if and only if $v \le z \le w$. Therefore Q_g is precisely the convex hull of the points L_z where $v \le z \le w$. ∎

Corollary 6.11. The Bruhat interval polytope $\mathsf{P}_{v,w}$ is a Minkowski sum of matroid polytopes

$$\mathsf{P}_{v,w} = \sum_{k=1}^{n-1}\Gamma_{\mathcal{M}_k}.$$

Here $\mathcal{M}_k$ is the matroid defining the cell of $(Gr_{k,n})_{\ge 0}$ that we obtain by projecting the cell $\mathcal{R}^{>0}_{v,w}$ of $(G/B^+)_{\ge 0}$ to $(Gr_{k,n})_{\ge 0}$.

Proof. This follows from Corollary 6.8 and Theorem 6.10. ∎

By Proposition 5.18, each weight vector $\mathsf{L}_{i_1,\ldots,i_n}$ can be associated to the ordered set of eigenvalues,

$$\mathsf{L}_{i_1,\ldots,i_n} \quad\Longleftrightarrow\quad (\lambda_{i_1},\lambda_{i_2},\ldots,\lambda_{i_n}).$$

For example, the highest weight for the permutohedron is given by

$$\mathsf{L}_{1,2,\ldots,n} = \sum_{k=1}^{n}(n-k)\mathsf{L}_k \quad\Longleftrightarrow\quad (\lambda_1,\lambda_2,\ldots,\lambda_n).$$

which corresponds to the asymptotic form of $\mathrm{diag}(L)$ with $v = e$ for $t \to -\infty$.

Example 6.12. Consider the $\mathfrak{sl}_4(\mathbb{R})$ f-KT hierarchy. We take

$$w = s_2s_3s_2s_1 \text{ and } v = s_3,$$

which gives

$$w\cdot(1,2,3,4) = (4,1,3,2) \text{ and } v\cdot(1,2,3,4) = (1,2,4,3).$$

There are eight permutations z satisfying $v \le z \le w$, i.e.

$$v = s_3,\quad s_3s_2,\quad s_2s_3,\quad s_3s_1,\quad s_3s_2s_1,\quad s_2s_3s_1,\quad s_2s_3s_2,\quad w = s_2s_3s_2s_1.$$

We illustrate the moment polytopes in Figure 3. The vertices are labeled by the index set "$i_1i_2i_3i_4$" of the eigenvalues $(\lambda_{i_1},\lambda_{i_2},\lambda_{i_3},\lambda_{i_4})$. The vertex with the white circle indicates the asymptotic form of $\mathrm{diag}(L)$ for $t \to -\infty$ (i.e. $(1,2,4,3) = v\cdot(1,2,3,4)$ in the right figure), and the black one indicates the asymptotic form for $t \to \infty$ (i.e. $(4,1,3,2) = w\cdot(1,2,3,4)$ in the right figure).

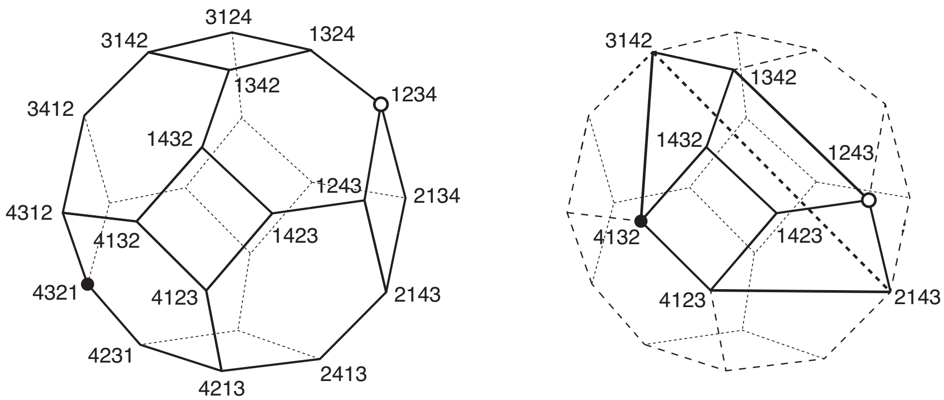

FIGURE 3. Some moment polytopes from the $\mathfrak{sl}_4(\mathbb{R})$ f-KT hierachy. The left figure is the permutohedron $\mathsf{P}_{e,w_0} = \mathrm{Perm}_4$, and the right one is the Bruhat interval polytope $\mathsf{P}_{v,w}$ with $w = s_2s_3s_2s_1$ and $v = s_3$.

7. THE FULL SYMMETRIC TODA HIERARCHY

In this section we discuss the full symmetric Toda hierarchy for a symmetric Lax matrix $\mathcal{L}$ in connection with the f-KT hierarchy for $L \in \epsilon + \mathfrak{b}^-$. We first review work of Bloch-Gekhtman [BG98] which provides a map from the f-KT hierarchy to the symmetric Toda hierarchy. Although this map is not always defined, in Theorem 7.9 we will give a sufficient condition, in terms of points $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$ of the tnn flag variety, under which the map is defined. This allows us to map the corresponding flows in the f-KT hierarchy to flows in the symmetric Toda hierarchy, and to compute the moment map images for such flows in the symmetric Toda hierarchy.

In the process we will explain the commutative diagram,

$$\begin{array}{ccc} L & \xrightarrow{Ad_{\beta^{-1}}} & \mathcal{L} \\ {\scriptstyle Ad_u}\big\downarrow & & \big\downarrow{\scriptstyle Ad_q} \\ C_\Lambda & \xrightarrow{Ad_{E^{-1}}} & \Lambda \end{array}$$

where E is the Vandermonde matrix (4.1), and $u \in U^-$, $\beta \in B^+$ and $q \in \mathrm{O}_n(\mathbb{R})$ are obtained uniquely from $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$.

7.1. **The full symmetric Toda hierarchy and the f-KT hierarchy.** We now explain the construction of Bloch-Gekhtman [BG98] for mapping flows $L = L(\mathbf{t})$ for the f-KT hierarchy to flows $\mathcal{L} = \mathcal{L}(\mathbf{t})$ for the symmetric Toda hierarchy. Note that this construction is not defined at those $\mathbf{t}$ where $L(\mathbf{t})$ is singular.

First recall from (5.1) that for each $L \in \mathcal{F}_\Lambda$, there is a unique element $u \in U^-$ such that

$$L = u^{-1} C_\Lambda u,$$

where C_Λ is the companion matrix. Since $C_\Lambda = E\Lambda E^{-1}$, we have

$$L = \mathrm{Ad}_\gamma \Lambda = \gamma \Lambda \gamma^{-1}, \qquad \text{where} \quad \gamma := u^{-1}E.$$

Lemma 7.1. Let $\dot\gamma$ denote the derivative of $\gamma = \gamma(\mathbf{t})$ with respect to t_m. Then we have

$$\dot\gamma\gamma^{-1} = (L^m)_{\geq 0} - L^m = -(L^m)_{<0}.$$

Proof. Note that

$$\dot{L} = \dot{\gamma}\gamma^{-1}L - L\dot{\gamma}\gamma^{-1} = [\dot{\gamma}\gamma^{-1}, L],$$

where we have used the fact that $\frac{\partial}{\partial t_m}\gamma^{-1} = -\gamma^{-1}\dot{\gamma}\gamma^{-1}$. From $\gamma = u^{-1}E$, we also have

$$\dot{\gamma}\gamma^{-1} = -u^{-1}\dot{u} \in \mathfrak{u}^-.$$

Let $F = (L^m)_{\geq 0} - \dot{\gamma}\gamma^{-1}$ and $K = L^m - F$. To prove the lemma, it suffices to show that $F = L^m$, or equivalently, $K = 0$. The fact that $\dot{L} = [\dot{\gamma}\gamma^{-1}, L]$ and $\dot{L} = [(L^m)_{\geq 0}, L]$ (the Lax equation) implies that $[F, L] = 0$. Then Lemma 4.2 implies that $F = \sum_{i=0}^{n-1} a_i L^i$ for some constants a_i.

Now we show $K = L^m - F = 0$. Using the definition of F and $\dot{\gamma}\gamma^{-1} \in \mathfrak{u}^-$, we have $(F)_{\geq 0} = (L^m)_{\geq 0}$, and hence $(K)_{\geq 0} = 0$. Therefore all the eigenvalues of K are 0. Also note that using the Cayley-Hamilton lemma, one can write

$$K = \sum_{i=0}^{n-1} b_i L^i$$

for unique b_i's. Then we have $0 = b\,E$ with $b = (b_0, b_1, \dots, b_{n-1})$. This implies that $b = 0$, which shows $K = 0$. Therefore we have $F = L^m$, as desired. □

Following Bloch-Gekhtman [BG98], we associate a symmetric matrix $\mathcal{L}$ to L:

$$\mathcal{L} = \psi(L) := \mathrm{Ad}_{\beta^{-1}} L = \beta^{-1} L \beta, \qquad \text{with} \quad \beta \in B^+. \tag{7.1}$$

The matrix β is given by Cholesky matrix factorization, i.e. for a fixed $\gamma = u^{-1}E$,

$$\gamma\gamma^T = \beta\beta^T. \tag{7.2}$$

This implies that

$$\mathcal{L}^T = \beta^T\gamma^{-T}\Lambda\gamma^T\beta^{-T} = \beta^{-1}\gamma\Lambda\gamma^{-1}\beta = \mathcal{L},$$

where $x^{-T} = (x^{-1})^T = (x^T)^{-1}$.

Remark 7.2. Since $\gamma\gamma^T = u^{-1}EE^Tu^{-T}$ whose principal minors are all positive, the matrix $\gamma\gamma^T$ is positive definite. This guarantees that the Cholesky factorization is unique.

The following result shows that the map (7.1) transforms the f-KT hierarchy into the symmetric Toda hierarchy.

Theorem 7.3. [BG98, Theorem 3.1] If $L = L(\mathbf{t})$ satisfies the f-KT hierarchy, the symmetric matrix $\mathcal{L} = \mathcal{L}(\mathbf{t})$ in (7.1) satisfies the symmetric Toda hierarchy, i.e.

$$\frac{\partial \mathcal{L}}{\partial t_m} = [\pi_{\mathfrak{so}}(\mathcal{L}^m), \mathcal{L}], \qquad \text{with} \quad \pi_{\mathfrak{so}}(\mathcal{L}^m) := (\mathcal{L}^m)_{>0} - (\mathcal{L}^m)_{<0}.$$

Proof. Let $\dot{\mathcal{L}}$ denote the t_m-derivative of $\mathcal{L}$. Using (7.1) it is straightforward to check that

$$\dot{\mathcal{L}} = [B, \mathcal{L}], \qquad \text{where} \qquad B = -\beta^{-1}\dot{\beta} + \beta^{-1}\dot{\gamma}\gamma^{-1}\beta.$$

Using $\beta\beta^T = \gamma\gamma^T$ and its t_m-derivative, one can show that B is a skew-symmetric matrix, i.e. $B^T = -B$, denoted by $B \in \mathfrak{so}(n)$. Also using $L^m = (L^m)_{<0} + (L^m)_{\geq 0}$ and Lemma 7.1, we have the decomposition

$$\begin{aligned} B &= -\beta^{-1}\dot{\beta} + \beta^{-1}\dot{\gamma}\gamma^{-1}\beta = -\beta^{-1}\dot{\beta} + \beta^{-1}\left[(L^m)_{\geq 0} - L^m\right]\beta \\ &= -\beta^{-1}\dot{\beta} + \beta^{-1}(L^m)_{\geq 0}\beta - \mathcal{L}^m \\ &= \left(-\beta^{-1}\dot{\beta} + \beta^{-1}(L^m)_{\geq 0}\beta - (\mathcal{L}^m)_{\geq 0}\right) - (\mathcal{L}^m)_{<0}. \end{aligned}$$

Note that the first term $(-\beta^{-1}\dot{\beta} + \dots)$ in the last line belongs to $\mathfrak{b}^+$. Then we use the fact that $B \in \mathfrak{so}(n)$ to conclude that $B = \pi_{\mathfrak{so}}(\mathcal{L}^m)$. □

Remark 7.4. It is known that the full symmetric Toda flow in Theorem 7.3 is complete (i.e. regular for all $\mathbf{t}$), see e.g. [DLNT86]. However, the full-Kostant Toda flow with general initial data is not complete – it can have a singularity. Therefore, as noted in [BG98], whenever the solution $L(\mathbf{t})$ becomes singular, the map from $L(\mathbf{t})$ to $\mathcal{L}(\mathbf{t})$ does not exist. In Theorem 7.9 below we show that if the initial matrix L^0 comes from a point of the tnn flag variety, as in Section 5.2, then the map from $L(\mathbf{t})$ to $\mathcal{L}(\mathbf{t})$ exists for all $\mathbf{t} \in \mathbb{R}^{n-1}$.

7.2. **The initial matrix $\mathcal{L}^0$ from $G^{>0}_{\mathbf{v}_+,\mathbf{w}}$.** Recall from Section 5.2 that from each point $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$ we can construct an initial matrix L^0 for the full Kostant-Toda lattice. By setting $\mathcal{L}^0 = \beta_0^{-1} L^0 \beta_0$, where β_0 represents β at $t = 0$, we can also use $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$ to construct the initial data $\mathcal{L}^0 = \mathcal{L}(0)$ for the full symmetric Toda hierarchy.

Proposition 7.5. Fix $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$ and define L^0 as in Section 5.2, i.e. $L^0 = u_0^{-1} C_\Lambda u_0 = u_0^{-1} E \Lambda E^{-1} u_0$, where u_0 is determined by the equation $Eg = u_0 b_0$. Let $\mathcal{L}^0 = \psi(L^0)$ be the symmetric matrix associated to L^0 by (7.1). Then

$$\mathcal{L}^0 = q_0^T \Lambda q_0,$$

where $q_0 \in \mathrm{SO}_n(\mathbb{R})$ is determined by the QR-factorization $g = q_0 r_0$, where $r_0 \in B^+$ is determined up to rescaling its columns by ± 1.

Proof. We have that $\mathcal{L}^0 = \psi(L^0) = \beta_0^{-1} L^0 \beta_0 = \beta_0^{-1} u_0^{-1} E \Lambda E^{-1} u_0 \beta_0$, where by (7.2) with $\gamma_0 = u_0^{-1} E$, we have

$$\gamma_0 \gamma_0^T = u_0^{-1} E (u_0^{-1} E)^T = \beta_0 \beta_0^T.$$

But then since $Eg = u_0 b_0$, we have $\mathcal{L}^0 = \beta_0^{-1} b_0 g^{-1} \Lambda g b_0^{-1} \beta_0$ and $(b_0 g^{-1})(b_0 g^{-1})^T = \beta_0 \beta_0^T$, which implies that

$$(\beta_0^{-1} b_0 g^{-1})(\beta_0^{-1} b_0 g^{-1})^T = I. \tag{7.3}$$

This shows that $q_0 := (\beta_0^{-1} b_0 g^{-1})^{-1} \in \mathrm{O}_n(\mathbb{R})$, and hence we have

$$\mathcal{L}^0 = q_0^T \Lambda q_0.$$

If we set $r_0 := \beta_0^{-1} b_0 \in B^+$ then we have

$$g = q_0 r_0 \qquad \text{for} \qquad q_0 \in \mathrm{O}_n(\mathbb{R}), \quad r_0 \in B^+,$$

which is the QR-factorization of g. (Note that the QR-factorization is unique up to the signs of the diagonal entries of r_0.) Since we are working in $G = \mathrm{SL}_n(\mathbb{R})$, q_0 in fact lies in $\mathrm{SO}_n(\mathbb{R})$. ∎

7.3. **The QR-factorization and the τ-functions of the full symmetric Toda lattice.** Recall that we have the following isomorphism for the real flag variety:

$$\mathrm{SL}_n(\mathbb{R})/B^+ \cong \mathrm{SO}_n(\mathbb{R})/T_{\mathfrak{so}}, \qquad \text{where} \quad T_{\mathfrak{so}} := \mathrm{diag}(\pm 1, \ldots, \pm 1).$$

When we associate an initial point L^0 for the full Kostant-Toda lattice to a point gB^+ of the flag variety we are using the first point of view on the flag variety. Proposition 7.5 shows that our association of an initial point $\mathcal{L}^0$ for the symmetric Toda lattice is quite natural from the second point of view.

Proposition 7.6 below shows that one can use the QR-factorization to solve the full symmetric Toda lattice (see also [Sym82]).

Proposition 7.6. The solution $\mathcal{L}(t)$ of the full symmetric Toda lattice is given by

$$\mathcal{L}(t) = q(t)^{-1} \mathcal{L}^0 q(t) = r(t) \mathcal{L}^0 r(t)^{-1},$$

where $q(t) \in \mathrm{SO}_n(\mathbb{R})$ and $r(t) \in B^+$ obtained by the QR-factorization of the matrix $\exp(t\mathcal{L}^0)$, i.e.

$$\exp(t\mathcal{L}^0) = q(t) r(t). \tag{7.4}$$

Proof. Differentiating (7.4), we have

$$\mathcal{L}^0 \exp(t\mathcal{L}^0) = \mathcal{L}^0 qr = qr\mathcal{L}^0 = \dot{q}r + q\dot{r},$$

where $q = q(t)$, $r = r(t)$, and $\dot{q}$, $\dot{r}$ means the derivatives of $q(t)$ and $r(t)$ with respect to t. Then we have

$$q^{-1}\mathcal{L}^0 q = r\mathcal{L}^0 r^{-1} = q^{-1}\dot{q} + \dot{r}r^{-1}.$$

Following the arguments in Proposition 4.5, we can also show

$$\mathcal{L} = q^{-1}\mathcal{L}^0 q = r\mathcal{L}^0 r^{-1}.$$

Here we have used the following projections of $\mathcal{L}$,

$$\begin{aligned} q^{-1}\dot{q} &= -\pi_{\mathfrak{so}}(\mathcal{L}) = -(\mathcal{L})_{>0} + (\mathcal{L})_{<0} \\ \dot{r}r^{-1} &= \mathcal{L} + \pi_{\mathfrak{so}}(\mathcal{L}) = \mathrm{diag}(\mathcal{L}) + 2(\mathcal{L})_{>0}. \end{aligned} \tag{7.5}$$

Those equations are needed when we compute the derivative of $\mathcal{L}$ as in Proposition 4.5. □

As in the case of the f-KT hierarchy, one can also easily obtain the solution of the full symmetric Toda hierarchy by extending the QR-factorization (7.4) with multi-times $\mathbf{t} = (t_1, \ldots, t_{n-1})$.

Proposition 7.7. Consider the QR-factorization

$$\exp(\Theta_{\mathcal{L}^0}(\mathbf{t})) = q(\mathbf{t})r(\mathbf{t}) \qquad \text{with} \quad q(\mathbf{t}) \in \mathrm{SO}_n(\mathbb{R}) \quad \text{and} \quad r(\mathbf{t}) \in B^+. \tag{7.6}$$

The solution $\mathcal{L}(\mathbf{t})$ of the full symmetric Toda hierarchy is then given by

$$\mathcal{L}(\mathbf{t}) = q(\mathbf{t})^{-1}\mathcal{L}^0 q(\mathbf{t}) = r(\mathbf{t})\mathcal{L}^0 r(\mathbf{t})^{-1}.$$

We also have an analogue of Proposition 5.1 for the full symmetric Toda hierarchy.

Proposition 7.8. Consider the full symmetric Toda hierarchy, with the initial data

$$\mathcal{L}^0 = q_0^T \Lambda q_0, \qquad \text{with} \quad q_0 \in \mathrm{SO}_n(\mathbb{R}),$$

where q_0 is obtained by the QR-factorization of $g \in G^{>0}_{\mathbf{v}^+,\mathbf{w}}$, i.e. $g = q_0 r_0$ with $r_0 \in B^+$. Using the QR-factorization (7.6), we have $\mathcal{L}(\mathbf{t}) = q(\mathbf{t})^{-1}\mathcal{L}^0 q(\mathbf{t})$. Then the symmetric Toda flow maps to the flag variety by the *diagonal embedding* d_Λ as in the following diagram.

$$\begin{array}{ccc} \mathcal{L}^{\mathbf{0}} & \xrightarrow{d_\Lambda} & q_0\, B^+ \\ {\scriptstyle Ad_{q(\mathbf{t})^{-1}}}\big\downarrow & & \big\downarrow \\ \mathcal{L}(\mathbf{t}) & \xrightarrow{d_\Lambda} & \left\{ \begin{array}{l} q_0\, q(\mathbf{t})\, B^+ \\ = q_0 \exp(\Theta_{\mathcal{L}^{\mathbf{0}}}(\mathbf{t}))\, B^+ \\ = \exp(\Theta_\Lambda(\mathbf{t}))\, q_0\, B^+ \end{array} \right. \end{array} \tag{7.7}$$

Here the diagonal embedding from symmetric matrices $\mathcal{L}$ to G/B^+ takes $q^T\Lambda q$ to qB^+, and the full symmetric Toda flow corresponds to the $\exp(\Theta_\Lambda(\mathbf{t}))$-torus action on the point $q_0 B^+$.

The main results of this section are Theorem 7.9 on the asymptotic behavior of the full symmetric Toda lattice, and Theorem 7.13 on the moment polytope for the full symmetric Toda hierarchy.

Theorem 7.9. Let $g \in G^{>0}_{\mathbf{v}^+,\mathbf{w}}$. Define u_0 by $Eg = u_0 b_0$ with $u_0 \in U^-$ and $b_0 \in B^+$, and define q_0 by $g = q_0 r_0$ with $q_0 \in \mathrm{SO}_n(\mathbb{R})$ and $r_0 \in B^+$. Let L^0 be the Hessenberg matrix $L^0 = u_0^{-1} C_\Lambda u_0$, and let $\mathcal{L}^0$ be the symmetric matrix $\mathcal{L}^0 = q_0^T \Lambda q_0$. Then $L(t)$ is regular for all $t = t_1$, and the solution $\mathcal{L}(t)$ of the full symmetric Toda equation with the initial matrix $\mathcal{L}^0$ has the same asymptotic behavior as the solution $L(t)$ of the full Kostant-Toda equation with the initial matrix L^0. More specifically, if

$$L(t) \quad \longrightarrow \quad \epsilon + \mathrm{diag}(\lambda_{z^\pm(1)}, \ldots, \lambda_{z^\pm(n)}) \qquad \text{as } t \to \pm\infty$$

for some $z^{\pm} \in \mathfrak{S}_n$, then the corresponding solution $\mathcal{L}(t)$ satisfies

$$\mathcal{L}(t) \quad \longrightarrow \quad \mathrm{diag}(\lambda_{z^{\pm}(1)}, \ldots, \lambda_{z^{\pm}(n)}) \qquad \text{as } t \to \pm\infty.$$

Therefore we have that

$$\mathcal{L}(t) \;\longrightarrow\; \begin{cases} \mathrm{diag}(\lambda_{v(1)}, \lambda_{v(2)}, \ldots, \lambda_{v(n)}) & \text{as} \quad t \to -\infty \\ \mathrm{diag}(\lambda_{w(1)}, \lambda_{w(2)}, \ldots, \lambda_{w(n)}) & \text{as} \quad t \to \infty \end{cases}$$

Proof. Proposition 5.9 implies that $L(t)$ is regular for all t. Now we use Theorem 7.3 and Proposition 7.5, which imply that $\mathcal{L}(t) = \beta(t)^{-1}L(t)\beta(t)$ for $\beta(t) \in B^+$. Note that the matrix $L^\infty := \epsilon + \mathrm{diag}(\lambda_{z(1)}, \ldots, \lambda_{z(n)})$ is in $\mathfrak{b}^+$ and hence conjugating it by an element of B^+ results in an element of $\mathfrak{b}^+$. The only symmetric matrices which lie in $\mathfrak{b}^+$ are diagonal matrices, so the corresponding limit point of $\mathcal{L}(t)$ is diagonal. Finally, the only diagonal matrix which is conjugate by B^+ to L^∞ is $\mathrm{diag}(\lambda_{z(1)}, \ldots, \lambda_{z(n)})$. The second part of the theorem now follows from Theorem 5.13. ◻

Remark 7.10. Note that the equation $\dot{r} = (\mathrm{diag}(\mathcal{L}) + 2(\mathcal{L})_{>0})r$ from (7.5) gives the following equation for the diagonal elements of the matrix r:

$$\frac{dr_{k,k}}{dt} = \alpha_{k,k} r_{k,k},$$

where $\alpha_{k,k}$ is the k-th element of the diagonal part of $\mathcal{L}$. That is, we have

$$\alpha_{k,k} = \frac{d}{dt} \ln r_{k,k}.$$

Also, from the QR-factorization $\exp(t\mathcal{L}^0) = qr$, one can write

$$\exp(2t\mathcal{L}^0) = r^T q^T q r = r^T r,$$

where we have used $\exp(t\mathcal{L}^0)^T = \exp(t\mathcal{L}^0) = r^T q^T = qr$ with $q^T q = I$. Now we define the τ-functions for the full symmetric Toda lattice as the principal minors of $\exp(2t\mathcal{L}^0)$, i.e.

$$\tau_k^{sym}(t) = [\exp(2t\mathcal{L}^0)]_k = [r^T r]_k = \prod_{i=1}^{k} (r_{i,i})^2.$$

This then gives the formula for the diagonal elements $\alpha_{k,k}$ of the symmetric Lax matrix $\mathcal{L}$,

$$\alpha_{k,k} = \frac{1}{2}\frac{d}{dt}(\ln r_{k,k}^2) = \frac{1}{2}\frac{d}{dt} \ln \frac{\tau_k^{sym}}{\tau_{k-1}^{sym}}.$$

This formula can be used to give an alternative proof of Theorem 7.9.

Remark 7.11. The first part of Theorem 7.9 recovers recent results of Chernyakov-Sharygin-Sorin, see [CSS12, Theorem 3.1 and Corollary 3.3].

The following lemma will be helpful for proving Theorem 7.13 below.

Lemma 7.12. Let g and q_0 be as in Proposition 7.5. Let A_k and Q_k be the $n \times k$ submatrices of g and q_0 consisting of the first k column vectors of g and q_0, respectively. Then the matroids $\mathcal{M}(A_k)$ and $\mathcal{M}(Q_k)$ are the same.

Proof. This can be shown by calculating the minors Δ_I with $I = \{i_1 < \ldots < i_k\}$:

$$\begin{aligned} \Delta_I(A_k) &= \langle e_{i_1} \wedge \cdots \wedge e_{i_k},\ g \cdot e_1 \wedge \cdots \wedge e_k \rangle = \langle e_{i_1} \wedge \cdots \wedge e_{i_k},\ q_0\, r_0 \cdot e_1 \wedge \cdots \wedge e_k \rangle \\ &= (r_0^{11} \cdots r_0^{kk}) \langle e_{i_1} \wedge \cdots \wedge e_{i_k},\ q_0 \cdot e_1 \wedge \cdots \wedge e_k \rangle = (r_0^{11} \cdots r_0^{kk}) \Delta_I(Q_k), \end{aligned}$$

where the r_0^{ii}'s are the diagonal elements of the matrix r_0, which are all nonzero. ◻

Theorem 7.13. Use the same hypotheses as in Theorem 7.9. Then the closure of the image of the moment map for the full symmetric Toda hierarchy has the same Bruhat interval polytope $\mathsf{P}_{v,w}$ as does the full Kostant-Toda hierarchy.

Proof. First note from (7.7) that the full symmetric Toda hierarchy gives the torus action $e^{\Theta_\Lambda(\mathbf{t})}$ on the flag variety. Then following the arguments on the moment map for the f-KT hierarchy, one can see that the moment map for the full symmetric Toda hierarchy is given by

$$\widetilde{\varphi}(\mathbf{t};g)=\sum_{k=1}^{n-1}\widetilde{\varphi}_k(\mathbf{t};g)\qquad\text{with}\quad \widetilde{\varphi}_k(\mathbf{t};g)=\sum_{I\in\mathcal{M}(Q_k)}\widetilde{\alpha}_I^k(\mathbf{t};g)\,\mathsf{L}(I),$$

$$\text{and}\qquad \widetilde{\alpha}_I^k(\mathbf{t};g)=\frac{\left(\Delta_I(Q_k e^{\Theta_\Lambda(\mathbf{t})})\right)^2}{\sum_{J\in\mathcal{M}(Q_k)}\left(\Delta_J(Q_k e^{\Theta_\Lambda(\mathbf{t})})\right)^2}.$$

Note here that $\widetilde{\varphi}_k(\mathbf{t};g)=\mu_k(Q_k e^{\Theta_\Lambda(\mathbf{t})})$. Then Lemma 7.12 implies that $\widetilde{\alpha}_I^k(\mathbf{t};g)=\alpha_I^k(\mathbf{t};g)$ for all $I\in\mathcal{M}(Q_k)=\mathcal{M}(A_k)$ and $k=1,\dots,n-1$. That is, the moment polytope for the full symmetric Toda hierarchy is exactly the same as that of the f-KT hierarchy. The present theorem is then reduced to Theorem 6.10. □

Remark 7.14. It is natural to wonder what happens if instead of using solutions coming from points gB^+ of the tnn flag variety, we use solutions coming from arbitrary points gB^+ of the real flag variety. In this more general case, we no longer have the result of Proposition 5.9 which guarantees that the f-KT flows are regular. However, we may still have a unique QR-factorization of g with positive diagonal entries in r_0 ($g=q_0r_0$), and the flow of the full symmetric Toda lattice with the initial data $\mathcal{L}^0=q_0^T\Lambda q_0$ *is* regular. This can be seen from the positivity of the τ-functions defined by

$$\tau_k^{sym}(t)=\left[\exp(2t\mathcal{L}^0)\right]_k=\left[q_0^T e^{2t\Lambda}q_0\right]_k=\sum_{I\in\mathcal{M}(Q_k)}\Delta_I(Q_k)^2e^{2\theta_I(t)},$$

where Q_k is the submatrix consisting of the first k columns of q_0, and $\theta_I(t)=\sum_{j=1}^{k}\lambda_{i_j}t$ with $I=\{i_1,\dots,i_k\}$. Since the f-KT flows are in general not regular but the full symmetric Toda flows *are* regular, there is no canonical way to map flows of the f-KT lattice to the full symmetric Toda lattice.

Appendix A. Bruhat interval polytopes

In this section we study some basic combinatorial properties of the Bruhat interval polytopes that arose in Section 6. A more extensive study will be taken up in [TW14]. Recall that for $u\le v\in\mathfrak{S}_n$, the Bruhat interval polytope $\mathsf{P}_{u,v}$ is defined as the convex hull of the permutation vectors z such that $u\le z\le v$. We show that each Bruhat interval polytope is a Minkowski sum of some matroid polytopes, and is also a *generalized permutohedron*, as defined by Postnikov [Pos09].

In order to make this section self-contained, we will recall the definitions of the various polytopes we need. To be more consistent with the combinatorial literature, we will define our polytopes in $\mathbb{R}^n$ (rather than in the weight space); however, this does not affect any of our statements about the polytopes. We start by defining a *matroid* using the *basis axioms*.

Definition A.1. Let $\mathcal{M}$ be a nonempty collection of k-element subsets of $[n]$ such that: if I and J are distinct members of $\mathcal{M}$ and $i\in I\setminus J$, then there exists an element $j\in J\setminus I$ such that $I\setminus\{i\}\cup\{j\}\in\mathcal{M}$. Then $\mathcal{M}$ is called the *set of bases of a matroid* of *rank k* on the *ground set* $[n]$; or simply a *matroid.*

Definition A.2. Given the set of bases $\mathcal{M}\subset\binom{[n]}{k}$ of a matroid, the *matroid polytope* $\Gamma_\mathcal{M}$ of $\mathcal{M}$ is the convex hull of the indicator vectors of the bases of $\mathcal{M}$:

$$\Gamma_\mathcal{M}:=\mathrm{Conv}\{e_I\mid I\in\mathcal{M}\}\subset\mathbb{R}^n,$$

where $e_I := \sum_{i\in I} e_i$, and $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

Note that "a matroid polytope" refers to the polytope of a specific matroid in its specific position in $\mathbb{R}^n$.

Recall that any element $A \in Gr_{k,n}$ gives rise to a matroid $\mathcal{M}(A)$ of rank k on the ground set $[n]$: specifically, the bases of $\mathcal{M}(A)$ are precisely the k-element subsets I such that $\Delta_I(A) \neq 0$.

Definition A.3. The *usual permutohedron* Perm_n in $\mathbb{R}^n$ is the convex hull of the $n!$ points obtained by permuting the coordinates of the vector $(1, 2, \dots, n)$.

There is a beautiful generalization of permutohedra due to Postnikov [Pos09].

Definition A.4. A *generalized permutohedron* is a polytope which is obtained by moving the vertices of the usual permutohedron in such a way that directions of edges are preserved, but some edges (and higher dimensional faces) may degenerate.

The main topic of this section is Bruhat interval polytopes.

Definition A.5. Let v and $w \in \mathfrak{S}_n$ such that $v \leq w$ in (strong) Bruhat order. We identify each permutation $z \in \mathfrak{S}_n$ with the corresponding vector $(z(1), \dots, z(n)) \in \mathbb{R}^n$. Then the *Bruhat interval polytope* $\mathsf{P}_{v,w}$ is defined as the convex hull of all vectors $(z(1), \dots, z(n))$ for z such that $v \leq z \leq w$.

See Figure 3 for some examples of Bruhat interval polytopes.

Theorem A.6. Let $v, w \in \mathfrak{S}_n$ such that $v \leq w$. Then the Bruhat interval polytope $\mathsf{P}_{v,w}$ is the Minkowski sum of $n-1$ matroid polytopes $\mathsf{P}_1, \dots, \mathsf{P}_{n-1}$. Specifically, if we choose any reduced expression $\mathbf{w}$ for w, and the corresponding positive expression $\mathbf{v}_+$ for v in $\mathbf{w}$, and choose any $g \in G^{>0}_{\mathbf{v}_+,\mathbf{w}}$ then P_k is the matroid polytope associated to the matroid $\mathcal{M}(\pi_k(g))$. See (2.2) and (3.1) for the definitions of $G^{>0}_{\mathbf{v}_+,\mathbf{w}}$ and π_k.

Proof. This result is equivalent to Corollary 6.11. □

Remark A.7. In fact $\mathsf{P}_{v,w}$ is the Minkowski sum of $n-1$ *positroid polytopes*, which were recently studied in [ARW13].

Corollary A.8. Every Bruhat interval polytope is a generalized permutohedron.

Proof. This follows from the fact that matroid polytopes are generalized permutohedra (see [ABD10, Section 2]), and the Minkowski sum of two generalized permutohedra is again a generalized permutohedron (see [ABD10, Lemma 2.2]). □

Our next result is about edges of Bruhat interval polytopes.

Before stating this result, we first recall the elegant characterization of edges of matroid polytopes, due to Gelfand, Goresky, MacPherson, and Serganova [GGMS87].

Theorem A.9 ([GGMS87]). Let $\mathcal{M}$ be a collection of subsets of $[n]$ and let $\Gamma_{\mathcal{M}} := \mathrm{Conv}\{e_I \mid I \in \mathcal{M}\} \subset \mathbb{R}^n$. Then $\mathcal{M}$ is the collection of bases of a matroid if and only if every edge of $\Gamma_{\mathcal{M}}$ is a parallel translate of $e_i - e_j$ for some $i, j \in [n]$.

Theorem A.10. Let $\mathsf{P}_{u,v}$ be a Bruhat interval polytope such that $u, v \in \mathfrak{S}_n$. Then every edge of $\mathsf{P}_{u,v}$ connects two vertices $y, z \in \mathfrak{S}_n$ such that $z = yt$ where t is a transposition (i, j) in $\mathfrak{S}_n$.

Proof. From Theorem A.6, we have that every Bruhat interval polytope $\mathsf{P}_{u,v}$ (for $u \leq v \in \mathfrak{S}_n$) is the Minkowski sum of $n-1$ matroid polytopes $\mathsf{P}_1, \dots, \mathsf{P}_{n-1}$. It follows that each edge E of $\mathsf{P}_{u,v}$ is the Minkowski sum of m parallel edges of those matroid polytopes P_i (for $m \geq 1$) and $n-m$ vertices (one from each of the other matroid polytopes). Therefore by Theorem A.9, the vertices of E must have the form $y = (y(1), y(2), \dots, y(n))$ and $z = (z(1), z(2), \dots, z(n))$ where $(z(1), z(2), \dots, z(n)) = (y(1), y(2), \dots, y(n)) + m(e_i - e_j)$ for some i, j. Therefore $z = yt$ where t is the transposition (i, j). □

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