NEWTON-OKOUNKOV BODIES, CLUSTER DUALITY AND MIRROR SYMMETRY FOR GRASSMANNIANS

K. RIETSCH AND L. WILLIAMS

Abstract. In this article we use cluster structures and mirror symmetry to explicitly describe a natural class of Newton-Okounkov bodies for Grassmannians. We consider the Grassmannian $\mathbb{X} = Gr_{n-k}(\mathbb{C}^n)$, as well as the mirror dual *Landau-Ginzburg model* $(\check{\mathbb{X}}^\circ, W_q : \check{\mathbb{X}}^\circ \to \mathbb{C})$, where $\check{\mathbb{X}}^\circ$ is the complement of a particular anti-canonical divisor in a Langlands dual Grassmannian $\check{\mathbb{X}} = Gr_k((\mathbb{C}^n)^*)$, and the superpotential W_q has a simple expression in terms of Plücker coordinates [MR13]. Grassmannians simultaneously have the structure of an $\mathcal{A}$-cluster variety and an $\mathcal{X}$-cluster variety [Sco06, Pos]; roughly speaking, a cluster variety is obtained by gluing together a collection of tori along birational maps [FZ02, FG06]. Given a plabic graph or, more generally, a cluster seed G, we consider two associated coordinate systems: a *network* or *$\mathcal{X}$-cluster chart* $\Phi_G : (\mathbb{C}^*)^{k(n-k)} \to \mathbb{X}^\circ$ and a *Plücker cluster* or *$\mathcal{A}$-cluster chart* $\Phi_G^\vee : (\mathbb{C}^*)^{k(n-k)} \to \check{\mathbb{X}}^\circ$. Here $\mathbb{X}^\circ$ and $\check{\mathbb{X}}^\circ$ are the open positroid varieties in $\mathbb{X}$ and $\check{\mathbb{X}}$, respectively. To each $\mathcal{X}$-cluster chart Φ_G and ample 'boundary divisor' D in $\mathbb{X} \setminus \mathbb{X}^\circ$, we associate a *Newton-Okounkov body* $\Delta_G(D)$ in $\mathbb{R}^{k(n-k)}$, which is defined as the convex hull of rational points; these points are obtained from the multi-degrees of leading terms of the Laurent polynomials $\Phi_G^*(f)$ for f on $\mathbb{X}$ with poles bounded by some multiple of D. On the other hand using the $\mathcal{A}$-cluster chart $\Phi_G^\vee$ on the mirror side, we obtain a set of rational polytopes – described in terms of inequalities – by writing the superpotential W_q as a Laurent polynomial in the $\mathcal{A}$-cluster coordinates, and then "tropicalising". Our main result is that the Newton-Okounkov bodies $\Delta_G(D)$ and the polytopes obtained by tropicalisation on the mirror side coincide. As an application, we construct degenerations of the Grassmannian to normal toric varieties corresponding to (dilates of) these Newton-Okounkov bodies. Additionally, when the cluster seed G corresponds to a plabic graph, we give an explicit formula in terms of Young diagrams, for the lattice points of the Newton-Okounkov bodies. This formula has an interpretation in terms of the quantum Schubert calculus of Grassmannians [FW04].

Contents

1. Introduction

1.1. Suppose that $\mathbb{X} = Gr_{n-k}(\mathbb{C}^n)$ is the Grassmannian of codimension k planes in $\mathbb{C}^n$, embedded in $\mathbb{P}(\bigwedge^{n-k}\mathbb{C}^n)$ via the Plücker embedding. Let $N := k(n-k)$ denote the dimension of $\mathbb{X}$. Grassmannians can be thought of as very close to toric varieties. In particular, the Grassmannian $\mathbb{X}$ has a distinguished anticanonical divisor $D_{\mathrm{ac}} = D_1 + \ldots + D_n$ made up of n hyperplanes, which generalises the usual torus-invariant anticanonical divisor of $\mathbb{CP}^{n-1}$. We denote the complement of the divisor D_{ac} by $\mathbb{X}^\circ$; this is a generalisation of the open torus-orbit in a toric variety.

We now view the Grassmannian $\mathbb{X}$ as the compactification of $\mathbb{X}^\circ$ by the boundary divisors $D_1, \ldots, D_n$. We consider ample divisors of the form $D = r_1 D_1 + \ldots + r_n D_n$ in $\mathbb{X}$, and their associated finite-dimensional subspaces

$$L_D := H^0(\mathbb{X}, \mathcal{O}(D)) \subset \mathbb{C}(\mathbb{X}).$$

Explicitly, L_D is the space of rational functions on $\mathbb{X}$ that are regular on $\mathbb{X}^\circ$ and for which the order of pole along D_i is bounded by r_i. By the Borel-Weil theorem, L_D may be identified with the irreducible representation $V_{r\omega_{n-k}}$ of $GL_n(\mathbb{C})$ where $r = \sum r_i$ and ω_{n-k} is the fundamental weight associated to $\mathbb{X} = Gr_{n-k}(\mathbb{C}^n)$.

In the toric setting one would associate to an ample divisor such as D its moment polytope $P(D)$, see [Ful93]. This is a lattice polytope in $\mathfrak{t}_c^*$, the dual of the Lie algebra of the compact torus T_c acting on the toric variety. It has the key property that its lattice points are in bijection with a basis of L_D, and the lattice points of the dilation $rP(D)$ are in bijection with a basis of L_{rD}.

There is a vast generalisation of this construction initiated by Okounkov, which applies in our setting of $\mathbb{X} = Gr_{n-k}(\mathbb{C}^n)$, and which can be used to associate to an ample divisor such as $D = \sum r_i D_i$ in $\mathbb{X}$ a convex body $\Delta(D)$ in $\mathbb{R}^N$, see [Oko96, Oko03, LM09, KK08, KK12a]. This so-called *Newton-Okounkov body* $\Delta(D)$ again encodes the dimension of each L_{rD} via the set of lattice points in the r-th dilation. In recent years Newton-Okounkov bodies have attracted a lot of attention; they have applications to toric degenerations and connections to integrable systems [And13, HK15]. However in general, Newton-Okounkov bodies are quite difficult to compute: they are not necessarily rational polytopes, or even polytopes [KLM12].

The main goal of this paper is to use mirror symmetry to describe the Newton-Okounkov bodies of divisors D as above, for a particular class of naturally occurring valuations. We show that they are rational polytopes, by giving formulas for the inequalities cutting them out. We also give explicit formulas for their lattice points. We now describe our results in more detail.

1.2. We consider certain open embedded tori inside $\mathbb{X}^\circ$ called *network tori.* These tori $\mathbb{T}_G$ were introduced by Postnikov [Pos], with their Plücker coordinates described succinctly by Talaska [Tal08]. They are associated to planar bicolored (plabic) graphs G, which have associated dual quivers $Q(G)$; the faces of G (equivalently, the vertices of $Q(G)$) are naturally labeled by a collection $\mathcal{P}_G$ of Young diagrams. The network tori form part of a cluster Poisson variety structure (also known as '$\mathcal{X}$-cluster structure'), and we also consider more general $\mathcal{X}$-cluster tori associated to quivers but not necessarily coming from plabic graphs; we continue to denote the tori, quivers, and vertices of the quivers by $\mathbb{T}_G$, $Q(G)$, and $\mathcal{P}_G$. As part of the data such a torus has specific $\mathcal{X}$-cluster coordinates $\mathcal{X}\mathrm{Coord}_{\mathbb{X}}(G)$ which are indexed by $\mathcal{P}_G$. The data of the quiver together with the torus coordinates is called an *$\mathcal{X}$-cluster seed* and denoted $\Sigma_G^{\mathcal{X}}$. As we show in Section 6, for a general $\mathcal{X}$-cluster seed $\Sigma_G^{\mathcal{X}}$ we also have an open embedding

$$\Phi_G : (\mathbb{C}^*)^{\mathcal{P}_G} \xrightarrow{\sim} \mathbb{T}_G \subset \mathbb{X}^\circ.$$

Using this embedding and a choice of ordering on $\mathcal{P}_G$, we define a lowest-order-term valuation

$$\mathrm{val}_G : \mathbb{C}(\mathbb{X}) \smallsetminus \{0\} \to \mathbb{Z}^{\mathcal{P}_G}.$$

The Newton-Okounkov body for a divisor D with this choice of valuation is defined to be

$$\Delta_G(D) := \overline{\mathrm{ConvexHull}\left(\bigcup_{r=1}^{\infty} \frac{1}{r}\mathrm{val}_G(L_{rD})\right)}.$$

Our goal is to describe $\Delta_G(D)$ for a general $\mathcal{X}$-cluster seed using mirror symmetry for $\mathbb{X}$.

1.3. We recall the mirror Landau-Ginzburg model $(\check{\mathbb{X}}^\circ, W_q)$ for the Grassmannian $\mathbb{X}$ introduced in [MR04]. Here $\check{\mathbb{X}}^\circ$ is the analogue of $\mathbb{X}^\circ = \mathbb{X} \setminus D_{\mathrm{ac}}$, but inside a Langlands dual Grassmannian $\check{\mathbb{X}}$, and $W_q : \check{\mathbb{X}}^\circ \to \mathbb{C}$ is a regular function called the *superpotential*. The superpotential is given by an explicit formula in terms of Plücker coordinates as a sum of n terms (and it depends on a single parameter q).

For example, if $\mathbb{X} = Gr_3(\mathbb{C}^7)$ then the superpotential on $\check{\mathbb{X}}^\circ$ is given by the expression

$$W = \frac{p_{\ydiagram{4,1}}}{p_{\ydiagram{4}}} + \frac{p_{\ydiagram{4,4,1}}}{p_{\ydiagram{4,4}}} + q\frac{p_{\ydiagram{3,3}}}{p_{\ydiagram{4,4,4}}} + \frac{p_{\ydiagram{4,3,3}}}{p_{\ydiagram{3,3,3}}} + \frac{p_{\ydiagram{3,2,2}}}{p_{\ydiagram{2,2,2}}} + \frac{p_{\ydiagram{2,1,1}}}{p_{\ydiagram{1,1,1}}} + \frac{p_{\ydiagram{1}}}{p_\varnothing},$$

where the p_λ are Plücker coordinates for $\check{\mathbb{X}} = Gr_4(\mathbb{C}^7)$; see Section 2 for an explanation of the notation.

As another example, if $\mathbb{X} = Gr_2(\mathbb{C}^5)$, then the superpotential on $\check{\mathbb{X}}^\circ$ is

$$W = \frac{p_{\ydiagram{3,1}}}{p_{\ydiagram{3}}} + q\frac{p_{\ydiagram{2}}}{p_{\ydiagram{3,3}}} + \frac{p_{\ydiagram{3,2}}}{p_{\ydiagram{2,2}}} + \frac{p_{\ydiagram{2,1}}}{p_{\ydiagram{1,1}}} + \frac{p_{\ydiagram{1}}}{p_\varnothing}. \tag{1.1}$$

The n summands of the superpotential individually give rise to functions which in this case are

$$W_1 = \frac{p_{\ydiagram{3,1}}}{p_{\ydiagram{3}}}, \quad W_2 = \frac{p_{\ydiagram{2}}}{p_{\ydiagram{3,3}}}, \quad W_3 = \frac{p_{\ydiagram{3,2}}}{p_{\ydiagram{2,2}}}, \quad W_4 = \frac{p_{\ydiagram{2,1}}}{p_{\ydiagram{1,1}}}, \quad W_5 = \frac{p_{\ydiagram{1}}}{p_\varnothing}. \tag{1.2}$$

We will often use the normalisation $p_\varnothing = 1$ so that the Plücker coordinates are actual coordinates on $\check{\mathbb{X}}^\circ$.

1.4. Besides the network tori $\mathbb{T}_G$, there is a different collection of open tori $\mathbb{T}_G^\vee$ in $\check{\mathbb{X}}^\circ$ indexed by plabic graphs G, each one corresponding to a maximal algebraically independent set of Plücker coordinates [Pos, Sco06]. We call these collections of Plücker coordinates the *Plücker clusters* of $\check{\mathbb{X}}$. By [Sco06] they are part of an $\mathcal{A}$-cluster structure on $\mathbb{C}[\check{\mathbb{X}}^\circ]$ in the sense of Fomin and Zelevinsky [FZ02]. As before we also consider more general $\mathcal{A}$-cluster tori associated to quivers not necessarily coming from plabic graphs; we continue to denote them by $\mathbb{T}_G^\vee$. As part of the data such a torus has specific $\mathcal{A}$-cluster coordinates $\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G)$ indexed by the vertices $\mathcal{P}_G$ of the quiver, which are Plücker coordinates when the quiver comes from a plabic graph. The data of the quiver together with the torus coordinates is called an $\mathcal{A}$-*cluster seed* and denoted $\check{\Sigma}_G^{\mathcal{A}}$. We think of the $\mathcal{A}$-cluster coordinates as encoding an open embedding

$$\Phi_G^\vee : (\mathbb{C}^*)^{\mathcal{P}_G} \xrightarrow{\sim} \mathbb{T}_G^\vee \subset \check{\mathbb{X}}^\circ.$$

Given an $\mathcal{A}$-cluster torus $\mathbb{T}_G^\vee$, we may restrict W and each W_i to the torus $\mathbb{T}_G^\vee$ and write it in the coordinates of the $\mathcal{A}$-cluster. From the $\mathcal{A}$-cluster seed and the superpotential together we thus obtain Laurent polynomials

$$\mathbf{W}_i^G = W_i|_{\mathbb{T}_G^\vee}, \quad i = 1, \dots, n, \quad \text{and} \quad \mathbf{W}^G = \sum_i q^{\delta_{i,n-k}} \mathbf{W}_i^G.$$

To these Laurent polynomials, together with a choice of integers $r_1, \dots, r_n$, we may associate a (possibly empty or unbounded) intersection of half-spaces $\Gamma_G(r_1, \dots, r_n)$ by a tropicalisation construction, see Section 9.2. We describe this construction by giving an example.

Let $\mathbb{X} = Gr_2(\mathbb{C}^5)$, with superpotential given by (1.1). If we write W and the W_i from (1.2) in terms of the Plücker cluster indexed by $\mathcal{P}_G = \{\ydiagram{1}, \ydiagram{1,1}, \ydiagram{2}, \ydiagram{2,2}, \ydiagram{3}, \ydiagram{3,3}\}$, we get the Laurent polynomial

$$\mathbf{W}^G = p_{\ydiagram{1}} + \frac{p_{\ydiagram{1,1}}}{p_{\ydiagram{1}}} + \frac{p_{\ydiagram{2,2}}}{p_{\ydiagram{1}}\,p_{\ydiagram{2}}} + \frac{p_{\ydiagram{3,3}}}{p_{\ydiagram{2}}\,p_{\ydiagram{3}}} + q\frac{p_{\ydiagram{2}}}{p_{\ydiagram{3,3}}} + \frac{p_{\ydiagram{3}}}{p_{\ydiagram{2}}} + \frac{p_{\ydiagram{3,3}}\,p_{\ydiagram{1}}}{p_{\ydiagram{2}}\,p_{\ydiagram{2,2}}} + \frac{p_{\ydiagram{2}}}{p_{\ydiagram{1}}} + \frac{p_{\ydiagram{2,2}}}{p_{\ydiagram{1}}\,p_{\ydiagram{1,1}}}, \tag{1.3}$$

as well as

(1.4)

$$\mathbf{W}_1^G = \frac{p_{\ydiagram{1,1}}}{p_{\ydiagram{1}}} + \frac{p_{\ydiagram{2,2}}}{p_{\ydiagram{1}}\,p_{\ydiagram{2}}} + \frac{p_{\ydiagram{3,3}}}{p_{\ydiagram{2}}\,p_{\ydiagram{3}}}, \qquad \mathbf{W}_2^G = \frac{p_{\ydiagram{2}}}{p_{\ydiagram{3,3}}}, \qquad \mathbf{W}_3^G = \frac{p_{\ydiagram{3}}}{p_{\ydiagram{2}}} + \frac{p_{\ydiagram{3,3}}\,p_{\ydiagram{1}}}{p_{\ydiagram{2}}\,p_{\ydiagram{2,2}}}, \qquad \mathbf{W}_4^G = \frac{p_{\ydiagram{2}}}{p_{\ydiagram{1}}} + \frac{p_{\ydiagram{2,2}}}{p_{\ydiagram{1}}\,p_{\ydiagram{1,1}}}, \qquad \mathbf{W}_5^G = p_{\ydiagram{1}}.$$

Each Laurent polynomial $\mathbf{W}_i^G$ gives rise to a piecewise-linear function $\mathrm{Trop}(\mathbf{W}_i^G) : \mathbb{R}^{\mathcal{P}_G} \to \mathbb{R}$ obtained by replacing multiplication by addition, division by subtraction, and addition by min. For any choice of

$r_1, \dots, r_5 \in \mathbb{Z}$ we then define $\Gamma_G(r_1, \dots, r_5) \in \mathbb{R}^{\mathcal{P}_G}$ by the following explicit inequalities in terms of variables $d = (d_{(1)}, d_{(1,1)}, d_{(2)}, d_{(2,2)}, d_{(3)}, d_{(3,3)}) \in \mathbb{R}^{\mathcal{P}_G}$:

$$\begin{aligned}
\mathrm{Trop}(\mathbf{W}^G_1)(d) + r_1 &= \min\left(d_{(1,1)} - d_{(1)}, d_{(2,2)} - d_{(1)} - d_{(2)}, d_{(2,2)} - d_{(2)} - d_{(3)}\right) + r_1 \geq 0,\\
\mathrm{Trop}(\mathbf{W}^G_2)(d) + r_2 &= d_{(2)} - d_{(3,3)} + r_2 \geq 0,\\
\mathrm{Trop}(\mathbf{W}^G_3)(d) + r_3 &= \min\left(d_{(3)} - d_{(2)}, d_{(3,3)} + d_{(1)} - d_{(2)} - d_{(3,3)}\right) + r_3 \geq 0,\\
\mathrm{Trop}(\mathbf{W}^G_4)(d) + r_4 &= \min\left(d_{(2)} - d_{(1)}, d_{(2,2)} - d_{(1)} - d_{(1,1)}\right) + r_4 \geq 0,\\
\mathrm{Trop}(\mathbf{W}^G_5)(d) + r_5 &= d_{(1)} + r_5 \geq 0.
\end{aligned}$$

There is an important special case where $r = r_{n-k} \geq 0$ and $r_i = 0$ for all other i. (In the running example $n = 5$ and $k = 3$, so $r = r_2$.) In this case the polytope defined by the construction is also denoted Γ^r_G. The polytope Γ^r_G can be expressed directly in terms of the superpotential $\mathbf{W}^G = W|_{\mathbb{T}^\vee_G \times \mathbb{C}^*}$ as

$$\Gamma^r_G = \{d \in \mathbb{R}^{\mathcal{P}_G} \mid \mathrm{Trop}(\mathbf{W}^G)(d, r) \geq 0\}, \tag{1.5}$$

see Definition 9.7 for the notation. When $r = 1$, we refer to this polytope as the *superpotential polytope* $\Gamma^1_G = \Gamma_G$ for the seed $\check{\Sigma}^{\mathcal{A}}_G$.

1.5. We now put the two sides together to state the first main theorem. Recall the original Grassmannian $\mathbb{X} = Gr_{n-k}(\mathbb{C}^n)$ with its anti-canonical divisor $D_{\mathrm{ac}} = D_1 + \dots + D_n$, its $\mathcal{X}$-cluster seeds, and the definition of the Newton-Okounkov body.

Theorem 1.1. *Suppose D is an ample divisor in $\mathbb{X}$ of the form $D = r_1 D_1 + \dots + r_n D_n$ and $\Sigma^{\mathcal{X}}_G$ is an $\mathcal{X}$-cluster seed in $\mathbb{X}^\circ$. The associated Newton-Okounkov body $\Delta_G(D)$ is a rational polytope and we have*

$$\Delta_G(D) = \Gamma_G(r_1, \dots, r_n),$$

where $\Gamma_G(r_1, \dots, r_n)$ is the polytope constructed from the superpotential $W : \check{\mathbb{X}}^\circ \times \mathbb{C}^*_q \to \mathbb{C}$ and the $\mathcal{A}$-cluster seed $\check{\Sigma}^{\mathcal{A}}_G$ of $\check{\mathbb{X}}^\circ$.

When $D = D_{n-k}$ we also denote $\Delta_G(D_{n-k})$ simply by Δ_G. The above result implies that

$$\Delta_G = \Gamma_G, \tag{1.6}$$

where Γ_G is the superpotential polytope from (1.5). This key special case is the content of Theorem 15.17. To prove Theorem 15.17, we show that for a distinguished choice of G (indexing the "rectangles" cluster), both Δ_G and Γ_G coincide with the Gelfand-Tsetlin polytope. We then "lift" Γ_G to generalised Puiseux series and show that when the seed G changes via a mutation, Γ_G is transformed via a piecewise linear "tropical mutation". We also show that when we mutate G, Δ_G is transformed via the same tropical mutation: our proof on this side uses deep properties of the *theta basis* of [GHKK14], including the Fock-Goncharov conjecture that elements of the theta basis are *pointed*, see Theorem 15.14. In the case where Γ_G is an integral polytope we prove that $\Gamma_G = \Delta_G$ without using [GHKK14], see Theorem 15.11.

If we choose a network torus coming from a plabic graph G, then the associated Laurent expansion $\mathbf{W}^G$ of W can be read off from G using a formula of Marsh and Scott [MS16a]. We thus obtain an explicit formula in terms of perfect matchings for the inequalities defining the Newton-Okounkov body, see Section 17.

It follows from our results that Δ_G is a rational polytope. In Section 16 we build on this fact to show that from each seed $\Sigma^{\mathcal{X}}_G$ we obtain a flat degeneration of $\mathbb{X}$ to the toric variety associated to the dual fan constructed from the polytope Δ_G. Note however that Δ_G is not in general integral; of the 34 polytopes Δ_G associated to plabic graphs for $Gr_3(\mathbb{C}^6)$, precisely two are non-integral, see Section 8. In each of those cases, there is a unique non-integral vertex which corresponds to the twist of a Plücker coordinate. Since the first version of this paper appeared on the arXiv, the polytopes arising from $Gr_3(\mathbb{C}^6)$ have been studied in [BFF+16].

In Section 18 we prove Theorem 1.1 in the general $D = \sum r_i D_i$ case by relating $\Delta_G(D)$ to $\Delta_G(D_{n-k})$ and $\Gamma_G(r_1, \dots, r_n)$ to Γ_G and deducing the general result from Theorem 15.17.

1.6. Our second main result concerns an explicit description of the lattice points of the Newton-Okounkov body $\Delta_G = \Delta_G(D_{n-k})$ when G is a plabic graph. Recall that the homogeneous coordinate ring of $\mathbb{X}$ is generated by Plücker coordinates which are naturally indexed by the set $\mathcal{P}_{k,n}$ of Young diagrams fitting inside an $(n-k) \times k$ rectangle. We denote these Plücker coordinates by P_λ with $\lambda \in \mathcal{P}_{k,n}$. Note that the upper-case P_λ (Plücker coordinate of $\mathbb{X}$) should not be confused with the lower-case p_λ (Plücker coordinate of $\check{\mathbb{X}}$). The largest of the Young diagrams in $\mathcal{P}_{k,n}$ is the entire $(n-k) \times k$ rectangle, and we denote its corresponding Plücker coordinate by $P_{\max}$. The set $\{P_\lambda / P_{\max} \mid \lambda \in \mathcal{P}_{k,n}\}$ is a natural basis for $H^0(\mathbb{X}, \mathcal{O}(D_{n-k}))$.

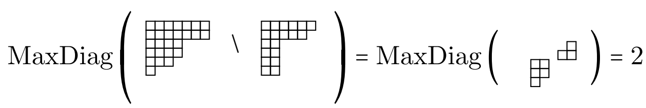
The following result says that the valuations $\mathrm{val}_G(P_\lambda/P_{\max})$ are precisely the $\binom{n}{k}$ lattice points of the Newton-Okounkov body Δ_G, and gives an explicit formula for them.

Theorem 1.2 (Corollary 15.18)**.** *Let G be any reduced plabic graph giving a network torus for $\mathbb{X}^\circ$. Then the Newton-Okounkov body Δ_G has $\binom{n}{k}$ lattice points $\{\mathrm{val}_G(P_\lambda/P_{\max}) \mid \lambda \in \mathcal{P}_{k,n}\} \subseteq \mathbb{Z}^{\mathcal{P}_G}$, with coordinates given by*

$$\mathrm{val}_G(P_\lambda/P_{\max})_\mu = \mathrm{MaxDiag}(\mu \smallsetminus \lambda)$$

for any partition $\mu \in \mathcal{P}_G$. Here $\mathrm{MaxDiag}(\mu \smallsetminus \lambda)$ denotes the maximal number of boxes in a slope -1 diagonal in the skew partition $\mu \smallsetminus \lambda$, see Definition 13.3.

Note that the right hand side of the formula depends neither on the plabic graph G nor on the Grassmannian, that is, on k or n. We illustrate the function MaxDiag with an example:



Also note that if $\mu \subseteq \lambda$ then necessarily $\mathrm{MaxDiag}(\mu \smallsetminus \lambda) = 0$, so the theorem implies that the μ-coordinate of $\mathrm{val}_G(P_\lambda/P_{\max})$ vanishes. Indeed, if $\lambda = \max$ then the formula says that all coordinates of the valuation vanish, which recovers the fact that the constant function 1 has valuation 0.

Interestingly, the function $\mathrm{MaxDiag}(\mu \smallsetminus \lambda)$ in Theorem 1.2 has an interpretation in quantum cohomology: by a result of Fulton and Woodward [FW04], it is equal to the smallest degree d such that q^d appears in the Schubert expansion of the product of two Schubert classes $\sigma_\mu \sigma_{\lambda^c}$ in the quantum cohomology ring $QH^*(\mathbb{X})$. See also [Yon03] and [Pos05]. We also prove a parallel result in Section 19 which says that if we consider the *highest-order-term* valuation val^G instead of the lowest-order-term valuation val_G, then $\mathrm{val}^G(P_\lambda/P_{\max})_\mu$ is equal to the largest degree d such that q^d appears in the Schubert expansion of $\sigma_\mu \sigma_{\lambda^c}$.

While our proof of Theorem 1.2 does not rely on Theorem 1.1, both proofs use the general philosophy of mirror symmetry. We think of the valuation $\mathrm{val}_G(P_\lambda/P_{\max})$ as an element of the character lattice of the $\mathcal{X}$-cluster network torus $\mathbb{T}_G$. Then we reinterpret this *character lattice* as the *cocharacter lattice* of the dual torus $\mathbb{T}_G^\vee$. We consider the dual torus to be naturally an $\mathcal{A}$-cluster torus in a Langlands dual Grassmannian $\check{\mathbb{X}}$, using the cluster algebra structure of [FZ02, Sco06]. Then we show that $\mathrm{val}_G(P_\lambda/P_{\max})$ represents a *tropical point* of $\check{\mathbb{X}}$ with regard to this cluster structure. The formula in Theorem 1.2 is obtained by the explicit construction of an element of $\check{\mathbb{X}}(\mathbb{R}_{>0}((t)))$ which represents this tropical point.

1.7. We note that tropicalisation in the Langlands dual world is well-known to play a fundamental role in the parametrization of basis elements of representations of a reductive algebraic group $\mathcal{G}$; this goes back to Lusztig and his work on the canonical basis [Lus90, Lus10]. The particular construction of the polytope Γ_G we use here is related to the construction of Berenstein and Kazhdan in their theory of geometric crystals [BK07]. The cluster charts we use are specific to Grassmannians, but we note that there is an isomorphism, [MR04, Theorem 4.9], between the superpotential $W_q : \mathbb{X}^\circ \to \mathbb{C}$ and the function used in [BK07] in the maximal parabolic setting. The function from [BK07] also agrees with the Lie-theoretic superpotential associated to $\mathbb{X} = \mathcal{G}/\mathcal{P}$ in [Rie08].

The connection between the lattice points of the tropicalised superpotential polytopes and the theta basis of the dual cluster algebra, which enters into our first main theorem, appears as an instance of the cluster duality conjectures between cluster $\mathcal{X}$-varieties and cluster $\mathcal{A}$-varieties developed by Fock and

Goncharov [FG09, FG06]. For Theorem 1.1 we make use of the deep properties of the theta basis of Gross, Hacking, Keel and Kontsevich [GHKK14] for a cluster $\mathcal{X}$-variety, see Section 15.2. The duality theory of cluster algebras has also been explored and applied in other works such as [GS15, GS16, Mag15].

For a Grassmannian $Gr_2(\mathbb{C}^n)$, the plabic graphs are in bijection with triangulations of an n-gon, and in this case polytopes isomorphic to ours were obtained earlier by Nohara and Ueda. These polytopes were shown in [NU14] to be integral (unlike in the general case), and were used to construct toric degenerations of the Grassmannian $Gr_2(\mathbb{C}^n)$, see also [BFF$^+$16].

1.8. This project originated out of the observation that Gelfand-Tsetlin polytopes appear to be naturally associated, but by very different constructions, both to the Grassmannian $\mathbb{X}$ and to its mirror, using a transcendence basis as input data. It also arose out of the wish to better understand the superpotential for Grassmannians from [MR13]. As far as we know this is the first time these ideas from mirror symmetry have been brought to bear on the problem of constructing Newton-Okounkov bodies. In the other direction, we expect that explicit descriptions of Newton-Okounkov bodies should allow one to find formulas for superpotentials.

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2. Notation for Grassmannians

2.1. The Grassmannian $\mathbb{X}$. Let $\mathbb{X}$ be the Grassmannian of $(n-k)$-planes in $\mathbb{C}^n$. We will denote its dimension by $N = k(n-k)$. An element of $\mathbb{X}$ can be represented as the column-span of a full-rank $n\times(n-k)$ matrix modulo right multiplication by nonsingular $(n-k)\times(n-k)$ matrices. Let $\binom{[n]}{n-k}$ be the set of all $(n-k)$-element subsets of $[n] := \{1,\dots,n\}$. For $J\in\binom{[n]}{n-k}$, let $P_J(A)$ denote the maximal minor of an $n\times(n-k)$ matrix A located in the row set J. The map $A\mapsto(P_J(A))$, where J ranges over $\binom{[n]}{n-k}$, induces the *Plücker embedding* $\mathbb{X}\hookrightarrow\mathbb{P}^{\binom{n}{n-k}-1}$, and the P_J are called *Plücker coordinates*.

We also think of $\mathbb{X}$ as a homogeneous space for the group $GL_n(\mathbb{C})$, acting on the left. We fix the standard pinning of $GL_n(\mathbb{C})$ consisting of upper and lower-triangular Borel subgroups B_+, B_-, maximal torus T in the intersection, and simple root subgroups $x_i(t)$ and $y_i(t)$ given by exponentiating the standard upper and lower-triangular Chevalley generators e_i, f_i with $i = 1,\dots,n-1$. We denote the Lie algebra of T by $\mathfrak{h}$ and we have fundamental weights $\omega_i\in\mathfrak{h}^*$ as well as simple roots $\alpha_i\in\mathfrak{h}^*$.

For $\mathbb{X} = Gr_{n-k}(\mathbb{C}^n)$ there is a natural identification between $H^2(\mathbb{X},\mathbb{C})$ and the subspace of $\mathfrak{h}^*$ spanned by ω_{n-k}, under which ω_{n-k} is identified with the hyperplane class of $\mathbb{X}$ in the Plücker embedding.

2.2. The mirror dual Grassmannian $\check{\mathbb{X}}$. Let $(\mathbb{C}^n)^*$ denote a vector space of row vectors. We then let $\check{\mathbb{X}} = Gr_k((\mathbb{C}^n)^*)$ be the 'mirror dual' Grassmannian of k-planes in the vector space $(\mathbb{C}^n)^*$. An element of $\check{\mathbb{X}}$ can be represented as the row-span of a full-rank $k\times n$ matrix M. This new Grassmannian is considered to be a homogeneous space via a *right* action by the Langlands dual group $GL_n^\vee(\mathbb{C})$ (which is isomorphic to $GL_n(\mathbb{C})$, but we distinguish the two groups nevertheless). For this group we use the same notations as introduced in the preceding paragraph for GL_n, but with an added superscript $^\vee$. In the Langlands dual setting the $r\omega_{n-k}$ that corresponded to a line bundle on $\mathbb{X}$ can now be considered (as $(r\omega_{n-k}^\vee)^\vee$) to represent a one-parameter subgroup of $T^\vee$, or element of $T^\vee(\mathbb{C}((t)))$, if t is the parameter. Note that the Plücker coordinates of $\check{\mathbb{X}}$ are naturally parameterized by $\binom{[n]}{k}$; for every k-subset I in $[n]$ the Plücker coordinate p_I is associated to the $k \times k$ minor of M with column set given by I.

2.3. Young diagrams. It is convenient to index Plücker coordinates of both $\mathbb{X}$ and $\check{\mathbb{X}}$ using Young diagrams. Recall that $\mathcal{P}_{k,n}$ denotes the set of Young diagrams fitting in an $(n-k)\times k$ rectangle. There is a natural bijection between $\mathcal{P}_{k,n}$ and $\binom{[n]}{n-k}$, defined as follows. Let μ be an element of $\mathcal{P}_{k,n}$, justified so that its top-left corner coincides with the top-left corner of the $(n-k)\times k$ rectangle. The south-east border of μ is then cut out by a path from the northeast to southwest corner of the rectangle, which consists of k west steps and $(n-k)$ south steps. After labeling the n steps by the numbers $\{1,\dots,n\}$, we map μ to the labels of the south steps. This gives a bijection from $\mathcal{P}_{k,n}$ to $\binom{[n]}{n-k}$. If we use the labels of the west steps instead, we get a bijection from $\mathcal{P}_{k,n}$ to $\binom{[n]}{k}$. Therefore the elements of $\mathcal{P}_{k,n}$ index the Plücker coordinates P_μ on $\mathbb{X}$ and simultaneously the Plücker coordinates on $\check{\mathbb{X}}$, which we denote by p_μ.

For $0 \le i \le n-1$, set $J_i := [i+1, i+k]$, interpreted cyclically as a subset of $[1,n]$. We let μ_i denote the Young diagram with west steps given by J_i. Then when $i \le n-k$, we have that μ_i is the rectangular $i\times k$ Young diagram, and when $i \ge n-k$, it is the rectangular $(n-k)\times(n-i)$ Young diagram. Note that μ_{n-k} is the whole $(n-k)\times k$ rectangle, so we also write $\max := \mu_{n-k}$.

2.4. The open positroid strata $\mathbb{X}^\circ$ and $\check{\mathbb{X}}^\circ$. We use the special Young diagrams μ_i from Section 2.3 to define a distinguished anticanonical divisor $D_{\mathrm{ac}} = \bigcup_{i=1}^n D_i$ where $D_i = \{P_{\mu_i} = 0\}$ in $\mathbb{X}$, and similarly an anticanonical divisor $\check{D}_{\mathrm{ac}} = \bigcup_{i=1}^n \{p_{\mu_i} = 0\}$ in $\check{\mathbb{X}}$.

Definition 2.1. We define $\mathbb{X}^\circ$ to be the complement of the divisor $D_{\mathrm{ac}} = \bigcup_{i=1}^n \{P_{\mu_i} = 0\}$,

$$\mathbb{X}^\circ := \mathbb{X} \setminus D_{\mathrm{ac}} = \{x \in \mathbb{X} \mid P_{\mu_i}(x) \neq 0 \ \forall i \in [n]\}.$$

And we define $\check{\mathbb{X}}^\circ$ to be the complement of the divisor $\check{D}_{\mathrm{ac}} = \bigcup_{i=1}^n \{p_{\mu_i} = 0\}$,

$$\check{\mathbb{X}}^\circ := \check{\mathbb{X}} \setminus \check{D}_{\mathrm{ac}} = \{x \in \check{\mathbb{X}} \mid p_{\mu_i}(x) \neq 0 \ \forall i \in [n]\}.$$

These varieties come up in [GSSV12] and [KLS13].

3. Plabic graphs for Grassmannians

In this section we review Postnikov's notion of *plabic graphs* [Pos], which we will then use to define network charts and cluster charts for the Grassmannian.

Definition 3.1. A *plabic (or planar bicolored) graph* is an undirected graph G drawn inside a disk (considered modulo homotopy) with n *boundary vertices* on the boundary of the disk, labeled $b_1,\dots,b_n$ in clockwise order, as well as some colored *internal vertices*. These internal vertices are strictly inside the disk and are colored in black and white. We will always assume that G is bipartite, and that each boundary vertex b_i is adjacent to one white vertex and no other vertices.

See Figure 1 for an example of a plabic graph.

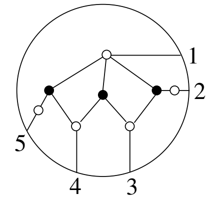


FIGURE 1. A plabic graph

There is a natural set of local transformations (moves) of plabic graphs, which we now describe. Note that we will always assume that a plabic graph G has no isolated components (i.e. every connected component contains at least one boundary vertex). We will also assume that G is *leafless*, i.e. if G has an internal vertex of degree 1, then that vertex must be adjacent to a boundary vertex.

(M1) SQUARE MOVE (Urban renewal). If a plabic graph has a square formed by four trivalent vertices whose colors alternate, then we can switch the colors of these four vertices (and add some degree 2 vertices to preserve the bipartiteness of the graph).

(M2) CONTRACTING/EXPANDING A VERTEX. Any degree 2 internal vertex not adjacent to the boundary can be deleted, and the two adjacent vertices merged. This operation can also be reversed. Note that this operation can always be used to change an arbitrary square face of G into a square face whose four vertices are all trivalent.

(M3) MIDDLE VERTEX INSERTION/REMOVAL. We can always remove or add degree 2 vertices at will, subject to the condition that the graph remains bipartite.

See Figure 2 for depictions of these three moves.

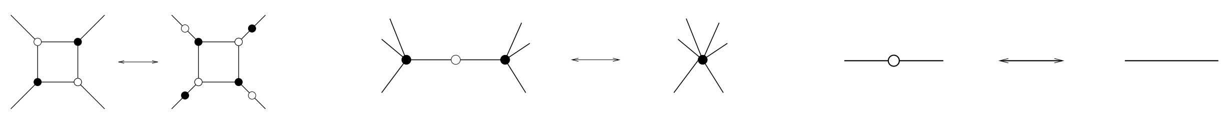


FIGURE 2. A square move, an edge contraction/expansion, and a vertex insertion/removal.

(R1) PARALLEL EDGE REDUCTION. If a plabic graph contains two trivalent vertices of different colors connected by a pair of parallel edges, then we can remove these vertices and edges, and glue the remaining pair of edges together.

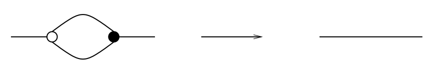


FIGURE 3. Parallel edge reduction

Definition 3.2. Two plabic graphs are called *move-equivalent* if they can be obtained from each other by moves (M1)-(M3). The *move-equivalence class* of a given plabic graph G is the set of all plabic graphs which are move-equivalent to G. A leafless plabic graph without isolated components is called *reduced* if there is no graph in its move-equivalence class to which we can apply (R1).

Definition 3.3. Let G be a reduced plabic graph with boundary vertices $b_1, \dots, b_n$. The *trip* T_i from b_i is the path obtained by starting from b_i and traveling along edges of G according to the rule that each time we reach an internal black vertex we turn (maximally) right, and each time we reach an internal white vertex we turn (maximally) left. This trip ends at some boundary vertex $b_{\pi(i)}$. In this way we associate a *trip permutation* $\pi_G = (\pi(1), \dots, \pi(n))$ to each reduced plabic graph G, and we say that G has *type* π_G.

As an example, the trip permutation associated to the reduced plabic graph in Figure 1 is $(3, 4, 5, 1, 2)$.

Remark 3.4. Let $\pi_{k,n} = (n-k+1, n-k+2, \dots, n, 1, 2, \dots, n-k)$. In this paper we will be particularly concerned with reduced plabic graphs whose trip permutation is $\pi_{k,n}$. Note that the trip permutation of a plabic graph is preserved by the local moves (M1)-(M3), but not by (R1). For reduced plabic graphs the converse holds, namely it follows from [Pos, Theorem 13.4] that any two reduced plabic graphs with trip permutation $\pi_{k,n}$ are move-equivalent.

Next we use the trips to label each face of a reduced plabic graph by a partition.

Definition 3.5. Let G be a reduced plabic graph of type $\pi_{k,n}$. Note that each trip T_i partitions the disk containing G into two parts: the part on the left of T_i, and the part on the right. Place an i in each face of G which is to the left of T_i. After doing this for all $1 \le i \le n$, each face will contain an $(n-k)$-element subset of $\{1, 2, \dots, n\}$. Finally we realise that $(n-k)$-element subset as the south steps of a corresponding Young diagram in $\mathcal{P}_{k,n}$. We let $\widetilde{\mathcal{P}}_G$ denote the set of Young diagrams inside $\mathcal{P}_{k,n}$ associated in this way to G. Note that $\widetilde{\mathcal{P}}_G$ always contains $\varnothing$ as the partition labeling a boundary region. We therefore set $\mathcal{P}_G := \widetilde{\mathcal{P}}_G \setminus \{\varnothing\}$. Each reduced plabic graph G of type $\pi_{k,n}$ will have precisely $N+1$ faces, where $N = k(n-k)$ [Pos].

The left of Figure 4 shows the labeling of each face of our running example by a Young diagram in $\mathcal{P}_{k,n}$ (here $k = 3$ and $n = 5$).

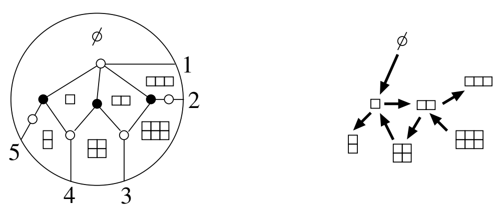


FIGURE 4. A plabic graph G with trip permutation $\pi_{3,5}$, with faces labeled by Young diagrams in $\mathcal{P}_{3,5}$, and the corresponding quiver $Q(G)$. Here $\mathcal{P}_G = \{(3,3),(2,2),(1,1),(3),(2),(1)\}$.

We next describe quivers and quiver mutation, and how they relate to moves on plabic graphs. Quiver mutation was first defined by Fomin and Zelevinsky [FZ02] in order to define cluster algebras.

Definition 3.6 (Quiver)**.** A *quiver* Q is a directed graph; we will assume that Q has no loops or 2-cycles. If there are i arrows from vertex λ to μ, then we will set $b_{\lambda\mu} = i$ and $b_{\mu\lambda} = -i$. Each vertex is designated either *mutable* or *frozen*. The skew-symmetric matrix $B = (b_{\lambda\mu})$ is called the *exchange matrix* of Q.

Definition 3.7 (Quiver Mutation)**.** Let λ be a mutable vertex of quiver Q. The quiver mutation Mut_λ transforms Q into a new quiver $Q' = \mathrm{Mut}_\lambda(Q)$ via a sequence of three steps:

(1) For each oriented two path $\mu \to \lambda \to \nu$, add a new arrow $\mu \to \nu$ (unless μ and ν are both frozen, in which case do nothing).
(2) Reverse the direction of all arrows incident to the vertex λ.
(3) Repeatedly remove oriented 2-cycles until unable to do so.

We say that two quivers Q and Q' are *mutation equivalent* if Q can be transformed into a quiver isomorphic to Q' by a sequence of mutations.

Definition 3.8. Let G be a reduced plabic graph. We associate a quiver $Q(G)$ as follows. The vertices of $Q(G)$ are labeled by the faces of G. We say that a vertex of $Q(G)$ is *frozen* if the corresponding face is incident to the boundary of the disk, and is *mutable* otherwise. For each edge e in G which separates two faces, at least one of which is mutable, we introduce an arrow connecting the faces; this arrow is oriented so that it "sees the white endpoint of e to the left and the black endpoint to the right" as it crosses over e. We then remove oriented 2-cycles from the resulting quiver, one by one, to get $Q(G)$. See Figure 4.

The following lemma is straightforward, and is implicit in [Sco06].

Lemma 3.9. *If G and G' are related via a square move at a face, then $Q(G)$ and $Q(G')$ are related via mutation at the corresponding vertex.*

Because of Lemma 3.9, we will subsequently refer to "mutating" at a nonboundary face of G, meaning that we mutate at the corresponding vertex of quiver $Q(G)$. Note that in general the quiver $Q(G)$ admits mutations at vertices which do not correspond to moves of plabic graphs. For example, G might have a hexagonal face, all of whose vertices are trivalent; in that case, $Q(G)$ admits a mutation at the corresponding vertex, but there is no move of plabic graphs which corresponds to this mutation.

In Section 4 and Section 5, we will explain how to associate to each plabic graph G a *network chart* and a *cluster chart* in $\mathbb{X}^\circ$, and similarly in $\check{\mathbb{X}}^\circ$.

4. Cluster charts from plabic graphs

In this section we fix a reduced plabic graph G of type $\pi_{k,n}$ and use it to construct a cluster chart for each of the open positroid varieties $\mathbb{X}^\circ$ and $\check{\mathbb{X}}^\circ$ from Definition 2.1. Our exposition will for the most part focus on $\check{\mathbb{X}}^\circ$. References for this construction are [Sco06, Pos], see also [MR13, Section 7].

Recall from Definition 3.5 that we have a labeling of each face of G by some Young diagram in $\widetilde{\mathcal{P}}_G \subset \mathcal{P}_{k,n}$. We now interpret each Young diagram in $\mathcal{P}_{k,n}$ as a k-element subset of $\{1,2,\dots,n\}$ via its west steps, see Section 2. It follows from [Sco06] that the collection

$$\widetilde{\mathcal{A}\mathrm{Coord}}_{\check{\mathbb{X}}}(G) := \{p_\mu \mid \mu \in \widetilde{\mathcal{P}}_G\} \tag{4.1}$$

of Plücker coordinates indexed by the faces of G is a *cluster* for the *cluster algebra* associated to the homogeneous coordinate ring of $\check{\mathbb{X}}$. In particular, these Plücker coordinates are called *cluster variables* and are algebraically independent; moreover, *any* Plücker coordinate for $\check{\mathbb{X}}$ can be written as a positive Laurent polynomial in the variables from $\widetilde{\mathcal{A}\mathrm{Coord}}_{\check{\mathbb{X}}}(G)$.

Among the elements of $\widetilde{\mathcal{A}\mathrm{Coord}}_{\check{\mathbb{X}}}(G)$ there are always n Plücker coordinates $\{p_{\mu_i} \mid 0 \le i \le n-1\}$, called *frozen variables*. They are present in each $\widetilde{\mathcal{A}\mathrm{Coord}}_{\check{\mathbb{X}}}(G)$ because each reduced plabic graph of type $\pi_{k,n}$ has n boundary regions which are labeled by the Young diagrams μ_i defined in Section 2.3.

Let

$$\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G) := \left\{ \frac{p_\mu}{p_\varnothing} \mid p_\mu \in \widetilde{\mathcal{A}\mathrm{Coord}}\mathbb{X}(G) \setminus \{p_\varnothing\} \right\} \subset \mathbb{C}(\mathbb{X}).$$

If we choose the normalization of Plücker coordinates on $\mathbb{X}^\circ$ such that $p_\varnothing = p_{\{1,\dots,k\}} = 1$, we get a map

$$\Phi^\vee_G = \Phi^\vee_{G,\mathcal{A}} : (\mathbb{C}^*)^{\mathcal{P}_G} \to \check{\mathbb{X}}^\circ \subset \check{\mathbb{X}} \tag{4.2}$$

which we call a *cluster chart*, which satisfies $p_\nu(\Phi^\vee_G((t_\mu)_\mu)) = t_\nu$ for $\nu \in \mathcal{P}_G$. Here $\mathcal{P}_G$ is as in Definition 3.5. When it is clear that we are setting $p_\varnothing = 1$ we may write

$$\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G) := \{p_\mu \mid \mu \in \mathcal{P}_G\}. \tag{4.3}$$

Definition 4.1 (Cluster torus $\mathbb{T}^\vee_G$)**.** Define the open dense torus $\mathbb{T}^\vee_G$ in $\check{\mathbb{X}}^\circ$ as the image of the cluster chart $\Phi^\vee_G$,

$$\mathbb{T}^\vee_G := \Phi^\vee_G((\mathbb{C}^*)^{\mathcal{P}_G}) = \{x \in \check{\mathbb{X}} \mid p_\mu(x) \neq 0 \text{ for all } \mu \in \mathcal{P}_G\}.$$

We call $\mathbb{T}^\vee_G$ the *cluster torus* associated to G.

Remark 4.2. The $p_\mu \in \mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G)$ restrict to coordinates on the open torus $\mathbb{T}^\vee_G$ in $\check{\mathbb{X}}$. Therefore we can think of $\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G)$ as a transcendence basis of $\mathbb{C}(\check{\mathbb{X}})$. Moreover by iterating the Plücker relations, we can express any Plücker coordinate as a rational function in the elements of $\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G)$ with coefficients which are all nonnegative, so $\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G)$ is a positive transcendence basis.

Example 4.3. We continue our example from Figure 4. The Plücker coordinates labeling the faces of G are $\widetilde{\mathcal{A}\mathrm{Coord}}_{\check{\mathbb{X}}}(G) = \{p_{\{1,2,3\}}, p_{\{1,2,4\}}, p_{\{1,3,4\}}, p_{\{2,3,4\}}, p_{\{1,2,5\}}, p_{\{1,4,5\}}, p_{\{3,4,5\}}\}$.

We next describe cluster $\mathcal{A}$-mutation, and how it relates to the clusters associated to plabic graphs G.

Definition 4.4. Let Q be a quiver with vertices V and associated exchange matrix B. We associate a *cluster variable* a_μ to each vertex $\mu \in V$. If λ is a mutable vertex of Q, then we define a new set of variables $\mathrm{Mut}^{\mathcal{A}}_\lambda(\{a_\mu\}) := \{a'_\mu\}$ where $a'_\mu = a_\mu$ if $\mu \neq \lambda$, and otherwise, a'_λ is determined by the equation

$$a_\lambda a'_\lambda = \prod_{b_{\mu\lambda}>0} a_\mu^{b_{\mu\lambda}} + \prod_{b_{\mu\lambda}<0} a_\mu^{-b_{\mu\lambda}}. \tag{4.4}$$

We say that $(\mathrm{Mut}_\lambda(Q), \{a'_\mu\})$ is obtained from $(Q, \{a_\mu\})$ by $\mathcal{A}$-*seed mutation* in direction λ, and we refer to the ordered pairs $(\mathrm{Mut}_\lambda(Q), \{a'_\mu\})$ and $(Q, \{a_\mu\})$ as *labeled* $\mathcal{A}$-*seeds*. We say that two labeled $\mathcal{A}$-seeds are $\mathcal{A}$-mutation equivalent if one can be obtained from the other by a sequence of $\mathcal{A}$-seed mutations.

Using the terminology of Definition 4.4, each reduced plabic graph G gives rise to a labeled $\mathcal{A}$-seed $(Q(G), \widetilde{\mathcal{A}\mathrm{Coord}}_{\check{\mathbb{X}}}(G))$. Lemma 4.5 below, which is easy to check, shows that our labeling of faces of each plabic graph by a Plücker coordinate is compatible with the $\mathcal{A}$-mutation. More specifically, performing a square move on a plabic graph corresponds to a three-term Plücker relation. Therefore whenever two plabic graphs are connected by moves, the corresponding $\mathcal{A}$-seeds are $\mathcal{A}$-mutation equivalent.

Lemma 4.5. *Let G be a reduced plabic graph with cluster variables $\widetilde{\mathcal{A}\mathrm{Coord}}_{\check{\mathbb{X}}}(G) := \{p_\mu \mid \mu \in \widetilde{\mathcal{P}}_G\}$, and let ν_1 be a square face of G formed by four trivalent vertices, see Figure 5. Let G' be obtained from G by performing a square move at face ν_1, and $\widetilde{\mathcal{A}\mathrm{Coord}}_{\check{\mathbb{X}}}(G')$ be the corresponding cluster variables. Then $\widetilde{\mathcal{A}\mathrm{Coord}}_{\check{\mathbb{X}}}(G') = \mathrm{Mut}^{\mathcal{A}}_{\nu_1}(\{p_\mu\})$. In particular, the Plücker coordinates labeling the faces of G and G' satisfy the* three-term Plücker relation

$$p_{\nu_1} p_{\nu_1'} = p_{\nu_2} p_{\nu_4} + p_{\nu_3} p_{\nu_5}.$$

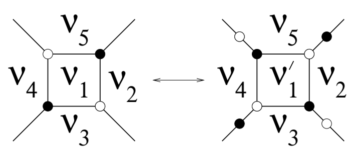


FIGURE 5

Remark 4.6. By Lemma 4.5 and Remark 3.4, all $\mathcal{A}$-seeds coming from plabic graphs G of type $\pi_{k,n}$ are $\mathcal{A}$-mutation equivalent. Recall that we can mutate at interior faces of G which are not squares; however, this will lead to quivers that no longer correspond to plabic graphs. Nevertheless, we can consider an arbitrary labeled $\mathcal{A}$-seed $(Q, \{a_\mu\})$ which is $\mathcal{A}$-mutation equivalent to an $\mathcal{A}$-seed coming from a reduced plabic graph of type $\pi_{k,n}$; we say that $(Q, \{a_\mu\})$ also has type $\pi_{k,n}$. In this case we still have a cluster chart for $\mathbb{X}^\circ$ which is obtained from the cluster chart $\Phi^\vee_G$ of Equation (4.2) by composing the $\mathcal{A}$-seed mutations of Equation (4.4), and it will have a corresponding cluster torus. Abusing notation, we will continue to index such $\mathcal{A}$-seeds, cluster charts, and cluster tori by G (rather than $(Q, \{a_\mu\})$), but will take care to indicate when we are working with an arbitrary $\mathcal{A}$-seed rather than one coming from a plabic graph.

Remark 4.7 (The case of $\mathbb{X}^\circ$)**.** The plabic graph G which determines a seed of an $\mathcal{A}$-cluster structure on $\check{\mathbb{X}}^\circ$ also determines a seed of an $\mathcal{A}$-cluster structure on $\mathbb{X}^\circ$. Namely we set

$$\mathcal{A}\mathrm{Coord}_{\mathbb{X}}(G) = \left\{ \frac{P_\mu}{P_{\max}} \mid \mu \in \widetilde{\mathcal{P}}_G \setminus \{\max\} \right\}.$$

The associated torus chart is denoted by $\Phi^{\mathcal{A}}_G$. Again quiver mutation in general gives rise to many more seeds than these. But these seeds still correspond to torus charts in $\mathbb{X}^\circ$ and we use the same notation $\Phi^{\mathcal{A}}_G$ also for these more general charts.

5. Network charts from plabic graphs

In this section we will explain how to use a reduced plabic graph G of type $\pi_{k,n}$ to construct a network chart for $\mathbb{X}^\circ$, the open positroid variety in $\mathbb{X} = Gr_{n-k}(\mathbb{C}^n)$. Network charts were originally introduced [Pos, Tal08] as a way to parameterize the *positive part* of the Grassmannian. There is a notion of mutation for network charts, which was described in the Grassmannian setting by Postnikov [Pos, Section 12]. More generally, the notion of mutation can be defined for arbitrary quivers; it is called *mutation of y-patterns* in [FZ07, (2.3)] and cluster $\mathcal{X}$-mutation by Fock and Goncharov [FG09, Equation 13]. In this article we will not restrict ourselves to network charts from plabic graphs, but will consider more general network charts associated to quivers Q mutation equivalent to $Q(G)$, see Section 6.

Definition 5.1. The *totally positive part* $\mathbb{X}(\mathbb{R}_{>0})$ of the Grassmannian $\mathbb{X}$ is the subset of the real Grassmannian $Gr_{n-k}(\mathbb{R}^n)$ consisting of the elements for which all Plücker coordinates are in $\mathbb{R}_{>0}$.

This definition is equivalent to Lusztig's original definition [Lus94] of the totally positive part of a generalized partial flag variety G/P applied in the Grassmannian case. (One proof of the equivalence of definitions comes from [TW13], which related the Marsh-Rietsch parametrizations of cells [MR04] of Lusztig's totally non-negative Grassmannian to the parametrizations of cells coming from network charts.)

Network charts are defined using *perfect orientations* and *flows* in plabic graphs.

Definition 5.2. A *perfect orientation* $\mathcal{O}$ of a plabic graph G is a choice of orientation of each of its edges such that each black internal vertex u is incident to exactly one edge directed away from u; and each white internal vertex v is incident to exactly one edge directed towards v. A plabic graph is called *perfectly orientable* if it admits a perfect orientation. The *source set* $I_{\mathcal{O}} \subset [n]$ of a perfect orientation $\mathcal{O}$ is the set of i for which b_i is a source of $\mathcal{O}$ (considered as a directed graph). Similarly, if $j \in \overline{I}_{\mathcal{O}} := [n] - I_{\mathcal{O}}$, then b_j is a sink of $\mathcal{O}$. If G has type $\pi_{k,n}$, then each perfect orientation of G will have a source set of size $n-k$ [Pos].

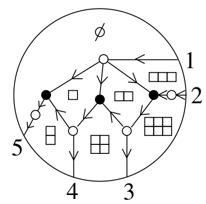


FIGURE 6. A perfect orientation $\mathcal{O}$ of a plabic graph. The source set is $I_{\mathcal{O}} = \{1, 2\}$.

The following lemma appeared in [PSW07].[1]

Lemma 5.3 ([PSW07, Lemma 3.2 and its proof])**.** *Each reduced plabic graph G has an acyclic perfect orientation $\mathcal{O}$. Moreover, we may choose $\mathcal{O}$ so that the set of boundary sources I is the index set for the lexicographically minimal non-vanishing Plücker coordinate on the corresponding cell. (In particular, if G is of type $\pi_{k,n}$, then we can choose $\mathcal{O}$ so that $I = \{1, \dots, n-k\}$.) Then given another reduced plabic graph G' which is move-equivalent to G, we can transform $\mathcal{O}$ into a perfect orientation $\mathcal{O}'$ for G', such that $\mathcal{O}'$ is also acyclic with boundary sources I, using oriented versions of the moves (M1), (M2), (M3). Up to rotational symmetry, we will only need to use the oriented version of the move (M1) shown in Figure 7.*

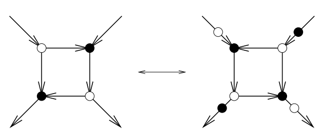


FIGURE 7. Oriented square move

Remark 5.4. By Lemma 5.3, a reduced plabic graph G of type $\pi_{k,n}$ always has an acyclic perfect orientation $\mathcal{O}$ with source set $I_{\mathcal{O}} = \{1, \dots, n-k\}$, as in Figure 6. Moreover it follows from [PSW09, Lemma 4.5] that this is the unique perfect orientation with source set $\{1, \dots, n-k\}$. From now on we will always choose our perfect orientation to be acyclic with source set $\{1, \dots, n-k\}$; we prefer this choice because then the variable $x_{\varnothing}$ never appear in the expressions for flow polynomials, and we always have $P_{\max} = 1$.

Recall from Definition 3.5 that we label each face of G by a Young diagram in $\widetilde{\mathcal{P}}_G \subset \mathcal{P}_{k,n}$. Let

$$\mathcal{X}\widetilde{\mathrm{Coord}}_{\mathbb{X}}(G) := \{x_\mu \mid \mu \in \widetilde{\mathcal{P}}_G\} \tag{5.1}$$

be a set of parameters which are indexed by the Young diagrams μ labeling faces of G. Since one of the faces of G is labeled by the empty partition, $\varnothing$, we also set

$$\mathcal{X}\mathrm{Coord}_{\mathbb{X}}(G) := \{x_\mu \mid \mu \in \mathcal{P}_G\} = \mathcal{X}\widetilde{\mathrm{Coord}}_{\mathbb{X}}(G) \setminus \{x_{\varnothing}\}. \tag{5.2}$$

[1]The published version of [PSW07], namely [PSW09], did not include the lemma, because it turned out to be unnecessary.

A *flow* F from $I_{\mathcal{O}}$ to a set J of boundary vertices with $|J| = |I_{\mathcal{O}}|$ is a collection of paths in $\mathcal{O}$, all pairwise vertex-disjoint, such that the sources of these paths are $I_{\mathcal{O}} - (I_{\mathcal{O}} \cap J)$ and the destinations are $J - (I_{\mathcal{O}} \cap J)$.

Note that each path w in $\mathcal{O}$ partitions the faces of G into those which are on the left and those which are on the right of the walk. We define the *weight* $\mathrm{wt}(w)$ of each such path to be the product of parameters x_μ, where μ ranges over all face labels to the left of the path. And we define the *weight* $\mathrm{wt}(F)$ of a flow F to be the product of the weights of all paths in the flow.

Fix a perfect orientation $\mathcal{O}$ of a reduced plabic graph G of type $\pi_{k,n}$. Given $J \in \binom{[n]}{n-k}$, we define the *flow polynomial*

$$P_J^G = \sum_F \mathrm{wt}(F), \tag{5.3}$$

where F ranges over all flows from $I_{\mathcal{O}}$ to J.

Example 5.5. We continue with our running example from Figure 6. There are two flows F from $I_{\mathcal{O}}$ to $\{2,4\}$, and $P^G_{\{2,4\}} = x_{(3)}x_{(2,2)}x_{(3,3)} + x_{(2)}x_{(3)}x_{(2,2)}x_{(3,3)}$. There is one flow from $I_{\mathcal{O}}$ to $\{3,4\}$, and $P^G_{\{3,4\}} = x_{(2)}x_{(3)}x_{(2,2)}x_{(3,3)}^2$.

We now describe the network chart for $\mathbb{X}^\circ$ associated to a plabic graph G. The result concerning the totally positive Grassmannian below comes from [Pos, Section 6], while the extension to $\mathbb{X}^\circ$ comes from [TW13] (see also [MS16b]).

Theorem 5.6 ([Pos, Section 6])**.** *Let G be a reduced plabic graph of type $\pi_{k,n}$, and choose an aylic perfect orientation $\mathcal{O}$ with source set $I_{\mathcal{O}} = \{1, \dots, n-k\}$. Let A be the $(n-k) \times n$ matrix with rows indexed by $I_{\mathcal{O}}$ whose (i,j)-entry equals*

$$(-1)^{|\{i' \in [n-k] : i < i' < j\}|} \sum_{p: i \to j} \mathrm{wt}(w),$$

where the sum is over all paths w in $\mathcal{O}$ from i to j. Then the map Φ_G sending $(x_\mu)_{\mu \in \mathcal{P}_G} \in (\mathbb{C}^*)^{\mathcal{P}_G}$ to the element of $\mathbb{X}$ represented by A is an injective map onto a dense open subset of $\mathbb{X}^\circ$. The restriction of Φ_G to $(\mathbb{R}_{>0})^{\mathcal{P}_G}$ gives a parametrization of the totally positive Grassmannian $\mathbb{X}(\mathbb{R}_{>0})$. We call the map Φ_G a network chart *for $\mathbb{X}^\circ$.*

Example 5.7. For example, the graph and orientation in Figure 6 gives for the matrix A

$$\Phi_G((x_\mu)_{\mu \in \mathcal{P}_G}) = \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{bmatrix} \overset{1}{1} & \overset{2}{0} & \overset{3}{-x_{(3)}x_{(3,3)}} & \overset{4}{-x_{(3)}x_{(3,3)}x_{(2,2)}(1+x_{(2)})} & \overset{5}{-x_{(3)}x_{(3,3)}x_{(2,2)}x_{(1,1)}(1+x_{(2)}+x_{(2)}x_{(1)})} \\ 0 & 1 & x_{(3,3)} & x_{(3,3)}x_{(2,2)} & x_{(3,3)}x_{(2,2)}x_{(1,1)} \end{bmatrix}.$$

The following result gives a formula for the Plücker coordinates of points in the image of Φ_G. In our setting, the result is essentially the Lindstrom-Gessel-Viennot Lemma. More general versions of Theorem 5.8, which work for arbitrary perfect orientations of a reduced plabic graph, can be found in [Pos] and [Tal08].

Theorem 5.8. *Let G be as in Theorem 5.6 and let $J \in \binom{[n]}{n-k}$. Then Plücker coordinate P_λ of $\Phi_G((x_\mu)_{\mu \in \mathcal{P}_G})$, i.e. the minor with column set J of the matrix A, is equal to the flow polynomial P_J^G from* (5.3).

Definition 5.9 (Network torus $\mathbb{T}_G$)**.** Define the open dense torus $\mathbb{T}_G$ in $\mathbb{X}^\circ$ to be the image of the network chart Φ_G, namely $\mathbb{T}_G := \Phi_G((\mathbb{C}^*)^{\mathcal{P}_G})$. We call $\mathbb{T}_G$ the *network torus* associated to G.

Definition 5.10 (Positive transcendence bases)**.** We say that a transcendence basis $\mathcal{T}$ for the field of rational functions on a Grassmannian is *positive* if each Plücker coordinate is a rational function in the elements of $\mathcal{T}$ with coefficients which are all nonnegative.

Example 5.11. Since the image of Φ_G lands in $\mathbb{X}^\circ$ [TW13, MS16b], we can view the parameters $\mathcal{X}\mathrm{Coord}_{\mathbb{X}}(G)$ as rational functions on $\mathbb{X}$ which restrict to coordinates on the open torus $\mathbb{T}_G$. Therefore we can think of $\mathcal{X}\mathrm{Coord}_{\mathbb{X}}(G)$ as a transcendence basis of $\mathbb{C}(\mathbb{X})$. Moreover it is clearly positive.

Example 5.12. We continue with our running example from Figure 6 and Example 5.7. The formulas for the Plücker coordinates of $\Phi_G((x_\mu)_{\mu\in\mathcal{P}_G})$ are:

$$P_{\{1,2\}} = 1, \qquad P_{\{1,3\}} = x_{(3,3)},$$
$$P_{\{1,4\}} = x_{(2,2)}x_{(3,3)}, \qquad P_{\{1,5\}} = x_{(1,1)}x_{(2,2)}x_{(3,3)},$$
$$P_{\{2,3\}} = x_{(3)}x_{(3,3)}, \qquad P_{\{2,4\}} = x_{(3)}x_{(2,2)}x_{(3,3)}(1+x_{(2)}),$$
$$P_{\{2,5\}} = x_{(3)}x_{(1,1)}x_{(2,2)}x_{(3,3)}(1+x_{(2)}+x_{(1)}x_{(2)}), \qquad P_{\{3,4\}} = x_{(2)}x_{(3)}x_{(2,2)}x_{(3,3)}^2,$$
$$P_{\{3,5\}} = x_{(2)}x_{(3)}x_{(1,1)}x_{(2,2)}x_{(3,3)}^2(1+x_{(1)}), \qquad P_{\{4,5\}} = x_{(1)}x_{(2)}x_{(3)}x_{(1,1)}x_{(2,2)}^2x_{(3,3)}^2.$$

One may obtain these Plücker coordinates either directly from the matrix in Example 5.7 or by computing flow polynomials from Figure 6. Note that $x_\varnothing$ does not appear in the flow polynomials since the region labeled by $\varnothing$ is to the right of every path from $I_\mathcal{O}$ to $[n]\setminus I_\mathcal{O}$. One may invert the map Φ_G and express the x_μ as rational functions in the Plücker coordinates, thus describing $\mathcal{X}\mathrm{Coord}_\mathbb{X}(G)$ as a subset of $\mathbb{C}(\mathbb{X})$.

Definition 5.13 (Strongly minimal, strongly maximal, and pointed)**.** We say that a Laurent monomial $\prod_\mu x_\mu^{a_\mu}$ appearing in a Laurent polynomial P is *strongly minimal* (respectively, *strongly maximal*) in P if for every other Laurent monomial $\prod_\mu x_\mu^{b_\mu}$ occurring in P, we have $a_\mu \le b_\mu$ (respectively, $a_\mu \ge b_\mu$) for all μ.

If P has a strongly minimal Laurent monomial with coefficient 1, then we say P is *pointed*. Consider a plabic graph G and perfect orientation with source set $\{1,\dots,n-k\}$. Recall that the flow polynomial P_J is a sum over flows from $\{1,\dots,n-k\}$ to J. We call a flow from $\{1,\dots,n-k\}$ to J *strongly minimal* (respectively, *strongly maximal*) if it has a strongly minimal (respectively, strongly maximal) weight monomial in P_J.

Remark 5.14. In Example 5.12, each flow polynomial $P_{\{i,j\}}$ has a strongly minimal and a strongly maximal term. This is true in general; see Corollary 11.4.

We next describe cluster $\mathcal{X}$-mutation, and how it relates to network parameters.

Definition 5.15. Let Q be a quiver with vertices V, associated exchange matrix B (see Definition 3.6), and with a parameter x_μ associated to each vertex $\mu\in V$. If λ is a mutable vertex of Q, then we define a new set of parameters $\mathrm{Mut}_\lambda^\mathcal{X}(\{x_\mu\}) := \{x'_\mu\}$ where

$$x'_\mu = \begin{cases} \frac{1}{x_\lambda} & \text{if } \mu=\lambda,\\ x_\mu(1+x_\lambda)^{b_{\lambda\mu}} & \text{if there are } b_{\lambda\mu} \text{ arrows from } \lambda \text{ to } \mu \text{ in } Q,\\ \frac{x_\mu}{(1+x_\lambda^{-1})^{b_{\mu\lambda}}} & \text{if there are } b_{\mu\lambda} \text{ arrows from } \mu \text{ to } \lambda \text{ in } Q,\\ x_\mu & \text{otherwise.} \end{cases} \tag{5.4}$$

We say that $(\mathrm{Mut}_\lambda(Q),\{x'_\mu\})$ is obtained from $(Q,\{x_\mu\})$ by *$\mathcal{X}$-seed mutation* in direction λ, and we refer to the ordered pairs $(\mathrm{Mut}_\lambda(Q),\{x'_\mu\})$ and $(Q,\{x_\mu\})$ as *labeled $\mathcal{X}$-seeds*. We say that two labeled $\mathcal{X}$-seeds are $\mathcal{X}$-mutation equivalent if one can be obtained from the other by a sequence of $\mathcal{X}$-seed mutations.

One can easily verify that $\mathrm{Mut}_\lambda^\mathcal{X}$ is an involution. If f is a rational expression in the parameters $\{x_\mu\}$, we use $\mathrm{Mut}_\lambda^\mathcal{X}(f)$ to denote the new expression for f obtained by rewriting it in terms of the $\{x'_\mu\}$.

Using the terminology of Definition 5.15, each reduced plabic graph G gives rise to a labeled $\mathcal{X}$-seed $(Q(G),\widetilde{\mathcal{X}\mathrm{Coord}}_\mathbb{X}(G))$. The following lemma, which is easy to check, shows that our flow polynomial expressions for Plücker coordinates are compatible with the $\mathcal{X}$-mutation. In other words, whenever two plabic graphs are connected by moves, the corresponding $\mathcal{X}$-seeds are $\mathcal{X}$-mutation equivalent.

Lemma 5.16. *Let G be a reduced plabic graph with network parameters $\widetilde{\mathcal{X}\mathrm{Coord}}_\mathbb{X}(G) := \{x_\mu \mid \mu\in\widetilde{\mathcal{P}}_G\}$ and associated quiver $Q(G)$, and let λ be a square face of G formed by four trivalent vertices. Let G' be obtained from G by performing a square move at λ. Then for each $J\in\binom{[n]}{n-k}$, the mutation $\mathrm{Mut}_\lambda^G(P_J^G(\{x_\mu\}))$ of the flow polynomial P_J^G is equal to the flow polynomial $P_J^{G'}(\{x'_\mu\})$ expressed in the network parameters of G'.*

Remark 5.17. By Lemma 5.16 and Remark 3.4, all $\mathcal{X}$-seeds coming from plabic graphs G of type $\pi_{k,n}$ are $\mathcal{X}$-mutation equivalent. Recall that we can mutate at faces of G which are not squares (in other words, at arbitrary nonfrozen vertices of $Q(G)$, not just those with precisely two incoming and two outgoing arrows); however, this will lead to quivers that no longer correspond to plabic graphs. Nevertheless, we can consider an arbitrary labeled $\mathcal{X}$-seed $(Q,\{x_\mu\})$ which is $\mathcal{X}$-mutation equivalent to an $\mathcal{X}$-seed coming from a reduced plabic graph of type $\pi_{k,n}$; we say that $(Q,\{x_\mu\})$ also has type $\pi_{k,n}$. In this case we still have a (generalised) network chart, also called $\mathcal{X}$-cluster chart, which is obtained from the network chart Φ_G of Theorem 5.6 by composing the $\mathcal{X}$-seed mutations of Equation (5.4); and there is a corresponding network or $\mathcal{X}$-cluster torus, see also Section 6. Abusing notation, we will continue to index such $\mathcal{X}$-seeds, network charts, and network tori by G (rather than $(Q,\{x_\mu\})$, but will take care to indicate when we are working with an arbitrary $\mathcal{X}$-seed rather than one coming from a plabic graph.

Recall from Remark 5.4 that our conventions guarantee that $x_\varnothing$ never appears in the expressions for flow polynomials (which are Plücker coordinates). Since $\varnothing$ labels a frozen vertex of our quiver $Q(G)$, when we perform arbitrary mutations on $Q(G)$ (possibly leaving the setting of plabic graphs), our expressions for Plücker coordinates will continue to be independent of the parameter associated to the frozen vertex $\varnothing$.

6. The twist map and general $\mathcal{X}$-cluster tori

In this section we define the *twist map* on $\mathbb{X}^\circ$, and, following Marsh-Scott [MS16a] and Muller-Speyer [MS16b], we explain how it connects network and cluster parameterizations coming from the same plabic graph G. We then use the twist map to deduce that the regular function on a network torus $\mathbb{T}_G$ coming from a Plücker coordinate stays regular after an arbitrary sequence of $\mathcal{X}$-cluster mutations. Thus we will see that the $\mathcal{X}$-cluster tori embed into $\mathbb{X}^\circ$ where they glue together.

The *twist map* is an automorphism of $\mathbb{X}^\circ$ which allows one to relate cluster charts and network charts. It was first defined in the context of double Bruhat cells by Berenstein, Fomin, and Zelevinsky [BFZ96], and subsequently defined for $\mathbb{X}^\circ$ by Marsh and Scott [MS16a]. Shortly thereafter it was defined for all positroid varieties (including $\mathbb{X}^\circ$) by Muller and Speyer [MS16b], using a slightly different convention. We follow the conventions and terminology of [MS16b] in this paper.

Definition 6.1. Let A denote an $(n-k)\times n$ matrix representing an element of $\mathbb{X}^\circ$. Let A_i denote the ith column of A, with indices taken cyclically; that is, $A_{i+n}=A_i$. Let $\langle -,-\rangle$ denote the standard Euclidean inner product on $\mathbb{C}^{n-k}$.

The *left twist* of A is the $(n-k)\times n$ matrix such that, for all i, the ith column $\overleftarrow{\tau}(A)_i$ satisfies

$$\langle \overleftarrow{\tau}(A)_i \mid A_i\rangle = 1, \text{ and}$$
$$\langle \overleftarrow{\tau}(A)_i \mid A_j\rangle = 0 \text{ if } A_j \text{ is not in the span of } \{A_{j+1},A_{j+2},\dots,A_{i-1},A_i\}.$$

Theorem 6.2 ([MS16b, Theorem 6.7 and Corollary 6.8])**.** *The map $\overleftarrow{\tau}$ is a regular automorphisms of $\mathbb{X}^\circ$.*

The inverse of $\overleftarrow{\tau}$, though we will not need it here, is called the right twist. The following theorem is a version of [MS16b, Theorem 7.1]. (It is also closely related to [MS16a, Theorem 1.1].) However, in [MS16b], the network tori were parameterized in terms of variables associated to edges rather than faces of G, so the notation looks different.

Theorem 6.3 ([MS16b, Theorem 7.1])**.** *There is an isomorphism of tori $\overleftarrow{\partial}=\overleftarrow{\partial}_G$ such that the following diagram commutes.*

$$\begin{array}{ccc} (\mathbb{C}^*)^{\tilde{\mathcal{P}}_G\setminus\varnothing} & \overset{\overleftarrow{\partial}}{\leftarrow} & (\mathbb{C}^*)^{\tilde{\mathcal{P}}_G\setminus\max} \\ \Phi_G^{\mathcal{X}}\downarrow & & \downarrow\Phi_G^{\mathcal{A}} \\ \mathbb{X}^\circ & \overset{\overleftarrow{\tau}}{\leftarrow} & \mathbb{X}^\circ \end{array}.$$

The left twist is closely related to the exchange matrix.

Proposition 6.4 ([MS16b, Corollary 5.11][Mul16])**.** *Let G be a reduced plabic graph of type $\pi_{k,n}$, and $B=B(G)$ the associated exchange matrix. Then there exists an adjusted matrix $\tilde{B}=B+M$, where*

$M \in \mathbb{Z}^{\mathcal{P}_G \times \mathcal{P}_G}$ has the property that $M_{\mu,\nu} = 0$ unless both μ and ν index frozen variables, such that the left twist is given by

$$(\overleftarrow{\partial})^*(x_\mu) = \prod_{\nu \in \mathcal{P}_G} P_\nu^{\tilde{B}_{\mu,\nu}},$$

in terms of the $\mathcal{X}$- and $\mathcal{A}$-cluster charts associated to G. In particular the pullback of the network parameter x_μ, when μ is mutable, is encoded in the original exchange matrix.

For mutable μ this proposition is simply [MS16b, Corollary 5.11], restated using the exchange matrix. The adjustment required for frozen μ (choice of M) is technical and was left out from the paper [MS16b] on those grounds, [Mul16].

Let $\mathcal{X}$ and $\mathcal{A}$ denote the spaces obtained by gluing together all of the $\mathcal{X}$-cluster tori, respectively, the $\mathcal{A}$-cluster tori, for varying seeds, using the rational maps given by mutation. From the work of Scott [Sco06] we know that we have an embedding $\mathcal{A} \hookrightarrow \mathbb{X}^\circ$. Our goal is to prove the analogous result for $\mathcal{X}$.

Let $\mathcal{X}^{\mathrm{net}}$ be the union of the network tori (associated to plabic graphs) glued together via the mutation maps. Recall that the network parametrisations define an embedding $\mathcal{X}^{\mathrm{net}} \hookrightarrow \mathbb{X}^\circ$.

Proposition 6.5. *The map $\mathcal{X}^{\mathrm{net}} \hookrightarrow \mathbb{X}^\circ$ extends to an embedding $\mathcal{X} \hookrightarrow \mathbb{X}^\circ$.*

Remark 6.6. This proposition can be interpreted in the following concrete way. Recall that any Plücker coordinate on $\mathbb{X}^\circ$ is expressed as a polynomial P_λ^G in terms of network coordinates if G is a plabic graph. If we apply a sequence of mutations (which are birational maps) to express P_λ^G in terms of $\mathcal{X}\mathrm{Coord}_{\mathbb{X}}(G')$ for a general $\mathcal{X}$-cluster, then the resulting expression is always a Laurent polynomial.

We recall a result about twists, generalising a construction from [GSV03] and [FG09], which applies in our setting as follows.

Proposition 6.7 ([Wil13, Proposition 4.7])**.** *Fix a seed G with exchange matrix $B = B(G)$. Suppose $M \in \mathbb{Z}^{\mathcal{P}_G \times \mathcal{P}_G}$ satisfies that $M_{\mu,\nu} = 0$ unless both μ and ν index frozen variables. Let $\tilde{B} = B + M$. Let us denote by $\{X_\mu\}$ the $\mathcal{X}$-cluster variables associated to G, and by $\{A_\mu\}$ the $\mathcal{A}$-cluster variables associated to G. Consider the map p_M^G from the $\mathcal{X}$-cluster torus $\mathbb{T}_G^{\mathcal{X}}$ to the $\mathcal{A}$-cluster torus $\mathbb{T}_G^{\mathcal{A}}$ associated to G defined by the formula*

$$(p_M^G)^*(X_\mu) = \prod_{\nu \in \mathcal{P}_G} A_\nu^{\tilde{B}_{\mu,\nu}}.$$

This map is compatible with mutation and extends to a regular map $p_M : \mathcal{A} \to \mathcal{X}$. In particular, whenever G and G' are adjacent seeds related by mutation at ν, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{T}_G^{\mathcal{A}} & \xrightarrow{\mathrm{Mut}_\nu^{\mathcal{A}}} & \mathbb{T}_{G'}^{\mathcal{A}} \\ \downarrow p_M^G & & \downarrow p_M^{G'} \\ \mathbb{T}_G^{\mathcal{X}} & \xrightarrow{\mathrm{Mut}_\nu^{\mathcal{X}}} & \mathbb{T}_{G'}^{\mathcal{X}} \end{array},$$

where $p_M^{G'}$ is defined in terms of the matrix $\mathrm{Mut}_\nu(B) + M = \mathrm{Mut}_\nu(B + M)$.

We are now in a position to prove Proposition 6.5

Proof of Proposition 6.5. The map $\mathcal{X}^{\mathrm{net}} \hookrightarrow \mathbb{X}^\circ$ can be extended to a rational map $\mathcal{X} \to \mathbb{X}^\circ$ using mutation. By the combination of Proposition 6.7 and Proposition 6.4 we have the commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{p_M} & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathbb{X}^\circ & \xrightarrow{\overleftarrow{\tau}} & \mathbb{X}^\circ \end{array},$$

where the left hand vertical map is the embedding of [Sco06], while the right hand vertical map is so far only known to be rational. By Proposition 6.4, we have that on a cluster torus associated to a plabic graph G, the map p_M^G is given by $\overleftarrow{\partial} = [\tilde{B}(G)_{\mu,\nu}]$, and by Theorem 6.3 it is invertible. Since mutation preserves the rank of a matrix [BFZ05, Lemma 3.2], the global map $p_M : \mathcal{A} \to \mathcal{X}$ is also invertible. Now the diagram implies that the vertical map on the right must be an embedding, just like the map on the left. □

7. The Newton-Okounkov body $\Delta_G(D)$

In this section we define the *Newton-Okounkov body* $\Delta_G(D)$ associated to an ample divisor in $\mathbb{X}$ of the form $D = r_1 D_1 + \cdots + r_n D_n$, see Section 2.4, along with a choice of transcendence basis $\mathcal{X}\mathrm{Coord}_{\mathbb{X}}(G)$ of $\mathbb{C}(\mathbb{X})$, see Definition 5.9. The theory of Newton-Okounkov bodies was developed by Kaveh and Khovanskii, and Lazarsfeld and Mustata, see [KK12a, KK12b, LM09], building on Okounkov's original construction [Oko96, Oko98, Oko03] which was inspired also by a formula for moment polytopes due to Brion [Bri87]. Our exposition below mainly follows [KK12a]. A key property of a Newton-Okounkov body associated to a divisor D is that its Euclidean volume encodes the volume of D, i.e. the asymptotics of $\dim(H^0(\mathbb{X}, \mathcal{O}(rD)))$ as $r \to \infty$. In our setting we will see that the lattice points of $\Delta_G(rD)$ count the dimension of the space of sections $H^0(\mathbb{X}, \mathcal{O}(rD))$ also for all finite r.

Fix a reduced plabic graph G or a labelled $\mathcal{X}$-seed $\Sigma_G^{\mathcal{X}}$ of type $\pi_{k,n}$. To define the Newton-Okounkov body $\Delta_G(D)$ we first construct a valuation val_G on $\mathbb{C}(\mathbb{X})$ from the transcendence basis $\mathcal{X}\mathrm{Coord}_{\mathbb{X}}(G)$.

Definition 7.1 (The valuation val_G)**.** Given a general $\mathcal{X}$-seed $\Sigma_G^{\mathcal{X}}$ of type $\pi_{k,n}$, we fix a total order $<$ on the parameters $x_\mu \in \mathcal{X}\mathrm{Coord}_{\mathbb{X}}(G)$. This order extends to a term order on monomials in the parameters $\mathcal{X}\mathrm{Coord}_{\mathbb{X}}(G)$ which is lexicographic with respect to $<$. For example if $x_\mu < x_\nu$ then $x_\mu^{a_1} x_\nu^{a_2} < x_\mu^{b_1} x_\nu^{b_2}$ if either $a_1 < b_1$, or if $a_1 = b_1$ and $a_2 < b_2$. We define a valuation

$$\mathrm{val}_G : \mathbb{C}(\mathbb{X}) \smallsetminus \{0\} \to \mathbb{Z}^{\mathcal{P}_G} \tag{7.1}$$

as follows. Let f be a polynomial in the Plücker coordinates for $\mathbb{X}$. We use Theorem 5.8, Definition 5.9, and Proposition 6.5 to write f uniquely as a Laurent polynomial in $\mathcal{X}\mathrm{Coord}_{\mathbb{X}}(G)$. We then choose the lexicographically minimal term $\prod_{\mu \in \mathcal{P}_G} x_\mu^{a_\mu}$ and define $\mathrm{val}_G(f)$ to be the associated exponent vector $(a_\mu)_\mu \in \mathbb{Z}^{\mathcal{P}_G}$. In general for $(f/g) \in \mathbb{C}(\mathbb{X}) \smallsetminus \{0\}$ (here f, g are polynomials in the Plücker coordinates), the valuation is defined by $\mathrm{val}_G(f/g) = \mathrm{val}_G(f) - \mathrm{val}_G(g)$. Note that we will only be applying val_G to functions whose $\mathcal{X}$-cluster expansions are Laurent however.

Definition 7.2 (The Newton-Okounkov body $\Delta_G(D)$)**.** Let $D \subset \mathbb{X}$ be a divisor in the complement of $\check{\mathbb{X}}^\circ$, that is we have $D = \sum r_i D_i$, compare Section 2.4. Denote by L_{rD}, the subspace of $\mathbb{C}(\mathbb{X})$ given by

$$L_{rD} := H^0(\mathbb{X}, \mathcal{O}(rD)).$$

By abuse of notation we write $\mathrm{val}_G(L)$ for $\mathrm{val}_G(L \smallsetminus \{0\})$. We define the *Newton-Okounkov body* associated to val_G and the divisor D by

$$\Delta_G(D) = \overline{\mathrm{ConvexHull}\left(\bigcup_r \frac{1}{r} \mathrm{val}_G(L_{rD})\right)}. \tag{7.2}$$

If we choose $D = D_{n-k}$, we will refer to $\Delta_G(D)$ simply as Δ_G.

Definition 7.3. For any subset $\mathcal{S}$ of $\mathbb{R}^{\mathcal{P}_G}$ we denote its subset of lattice points by $\mathrm{Lattice}(\mathcal{S}) := \mathcal{S} \cap \mathbb{Z}^{\mathcal{P}_G}$.

Remark 7.4 (Toy example)**.** Suppose $\Delta \subset \mathbb{R}^m$ is a convex m-dimensional polytope. Associated to Δ consider the set $\mathrm{Lattice}(r\Delta)$ of lattice points in the dilation $r\Delta$. Then we observe that

$$\Delta = \overline{\mathrm{ConvexHull}\left(\bigcup_r \frac{1}{r} \mathrm{Lattice}(r\Delta)\right)}. \tag{7.3}$$

In particular if for a polytope $\Delta \in \mathbb{R}^{\mathcal{P}_G}$ the lattice points $\mathrm{Lattice}(r\Delta)$ coincide $\mathrm{val}_G(L_{rD})$ from Definition 7.2, then it immediately follows that Δ is the Newton-Okounkov body $\Delta_G(D)$.

Remark 7.5 (The special case of D_{n-k})**.** We will often choose our divisor D in $\mathbb{X}$ to be $D_{n-k} = \{P_{\max} = 0\}$. We note that explicitly $H^0(\mathbb{X}, \mathcal{O}(rD_{n-k}))$ is the linear subspace of $\mathbb{C}(\mathbb{X})$ described as follows

$$H^0(\mathbb{X}, \mathcal{O}(rD_{n-k})) = L_r := \left\langle \frac{M}{(P_{\max})^r} \mid M \in \mathcal{M}_r \right\rangle, \tag{7.4}$$

where $\mathcal{M}_r$ is the set of all degree r monomials in the Plücker coordinates. Recall that $H^0(\mathbb{X}, \mathcal{O}(rD_{n-k}))$ is naturally an irreducible representation of $GL_n(\mathbb{C})$, namely it is isomorphic to $V^*_{r\omega_{n-k}}$. The identity

(7.4) says that $\mathbb{X}$ is *projectively normal* and follows from representation theory, see [GW11]. Namely, restriction of sections gives a nonzero equivariant map of $GL_n(\mathbb{C})$-representations, $H^0(\mathbb{P}(\bigwedge^{n-k}\mathbb{C}^n), \mathcal{O}(r)) \to H^0(\mathbb{X}, \mathcal{O}(rD_{n-k}))$, which must be surjective since its target is irreducible.

For simplicity of notation we will usually write $\mathrm{val}_G(M)$ for $\mathrm{val}_G(M/P_{\max}^r)$. Thus we write $\mathrm{val}_G(P_\lambda)$ instead of $\mathrm{val}_G(P_\lambda/P_{\max})$ and talk about the valuation of a Plücker coordinate.

Starting from the divisor D_{n-k} we introduce a set of lattice polytopes Conv_G^r.

Definition 7.6 (The polytope Conv_G^r)**.** For each reduced plabic graph G of type $\pi_{k,n}$ and related valuation val_G we define lattice polytopes Conv_G^r in $\mathbb{R}^{\mathcal{P}_G}$ by

$$\mathrm{Conv}_G^r := \mathrm{ConvexHull}\left(\mathrm{val}_G(L_r)\right),$$

for L_r as in (7.4). When $r = 1$, we also write $\mathrm{Conv}_G := \mathrm{Conv}_G^1$.

The lattice polytope Conv_G (resp. Conv_G^r) is what val_G associates to the divisor D (resp. rD) directly, without taking account of the asymptotic behaviour of the powers of $\mathcal{O}(D)$. Since we will fix $D = D_{n-k}$ when considering the polytopes Conv_G^r, we don't indicate the dependence on D in the notation Conv_G^r.

Remark 7.7. Note that we used a total order $<$ on the parameters in order to define val_G, and different choices give slightly differing valuation maps. However Δ_G and the polytopes Conv_G^r, will turn out not to depend on our choice of total order, and that choice will not enter into our proofs.

Remark 7.8 (Valuations associated to flags)**.** The valuations used in Okounkov's original construction come from flags of subvarieties $X \supset X_1 \supset \cdots X_{N-1} \supset X_N = \{pt\}$, see also [LM09, Section 1.1]. Our valuations val_G definitely do not all come from flags. For example in the case of the rectangles cluster, if the ordering on the x_μ is not compatible with inclusion of Young diagrams, then our valuation cannot come from a flag. In general, our definition can be interpreted as choosing, via a network chart, a birational isomorphism of $\mathbb{X}$ with $\mathbb{C}^N$, and then taking a standard flag of linear subspaces in $\mathbb{C}^N$.

We immediately point out some fundamental properties of the sets $\mathrm{val}_G(L_r)$ defining our polytopes Conv_G^r. The first property is a version of the key lemma from [Oko96]. It says, in the terminology of [KK12a], that the valuation val_G has *one-dimensional leaves*.

Lemma 7.9 (Version of [Oko96, Lemma from Section 2.2])**.** *Consider* $\mathbb{C}(\mathbb{X})$ *with the valuation* val_G *from Definition 7.1. For any finite-dimensional linear subspace* L *of* $\mathbb{C}(\mathbb{X})$, *the cardinality of the image* $\mathrm{val}_G(L)$ *equals the dimension of* L. *In particular, the cardinality of the set* $\mathrm{val}_G(L_r)$ *equals the dimension of the vector space* L_r *from* (7.4), *namely it is the dimension of the representation* $V_{r\omega_{n-k}}$ *of* $GL_n(\mathbb{C})$.

The proof uses the valuation and the total order on $\mathbb{Z}^{\mathcal{P}_G}$ to define in the natural way a filtration

$$L = (L)_{\geq \mathbf{a}_1} \supset (L)_{\geq \mathbf{a}_2} \supset (L)_{\geq \mathbf{a}_3} \supset \cdots \supset (L)_{\geq \mathbf{a}_m} \supset \{0\},$$

of L indexed by $\mathrm{val}_G(L) = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$, where $L_{\geq a} = \{f \in L \mid \mathrm{val}_G(f) \geq \mathbf{a}\}$ and similarly with $\geq$ replaced by $>$. The result follows by observing that successive quotients $(L)_{\geq \mathbf{a}}/(L)_{>\mathbf{a}}$ are isomorphic to $\mathbb{C}$ by the isomorphism which takes the coefficient of the leading term.

Example 7.10. We now take $r = 1$ and compute the polytope Conv_G associated to Example 5.12. Computing the valuation of each Plücker coordinate we get the result shown in Table 1. Therefore Conv_G is the convex hull of the set of points $\{(0,0,0,0,0,0), (1,0,0,0,0,0), (1,1,0,0,0,0), (1,1,1,0,0,0), (1,0,0,1,0,0), (1,1,0,1,0,0), (1,1,1,1,0,0), (2,1,0,1,1,0), (2,1,1,1,1,0), (2,2,1,1,1,1)\}$.

It will follow from results in Section 15.1 that in this example, $\mathrm{Conv}_G = \Delta_G$.

7.1. The rectangles cluster. We define a particular reduced plabic graph $G_{k,n}^{\mathrm{rec}}$ with trip permutation $\pi_{k,n}$. This is a reduced plabic graph whose internal faces are arranged into an $(n-k) \times k$ grid pattern, as shown in Figure 8. (It is easy to check that the plabic graph $G_{k,n}^{\mathrm{rec}}$ is reduced, using e.g. [KW14, Theorem 10.5].) When one uses Definition 3.5 to label faces by Young diagrams, one obtains the labeling of faces by rectangles which is shown in the figure. The generalization of this figure for arbitrary k and n is straightforward. Note that the plabic graph from Figure 4 is $G_{3,5}^{\mathrm{rec}}$. Moreover, the plabic graph $G_{k,n}^{\mathrm{rec}}$ has a nice perfect orientation $\mathcal{O}^{\mathrm{rec}}$, which is shown in Figure 16. The source set is $\{1, 2, \dots, n-k\}$.

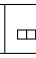
Plücker						
$P_{1,2}$	0	0	0	0	0	0
$P_{1,3}$	1	0	0	0	0	0
$P_{1,4}$	1	1	0	0	0	0
$P_{1,5}$	1	1	1	0	0	0
$P_{2,3}$	1	0	0	1	0	0
$P_{2,4}$	1	1	0	1	0	0
$P_{2,5}$	1	1	1	1	0	0
$P_{3,4}$	2	1	0	1	1	0
$P_{3,5}$	2	1	1	1	1	0
$P_{4,5}$	2	2	1	1	1	1

TABLE 1. The valuations $\mathrm{val}_G(P_J)$ of the Plücker coordinates

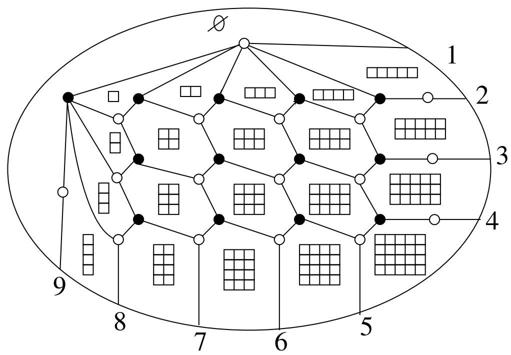
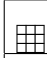
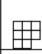
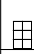
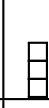


FIGURE 8. The plabic graph $G^{\mathrm{rec}}_{5,9}$ with trip permutation $\pi_{5,9}$, with faces labeled by $\mathcal{P}_{5,9}$.

8. A non-integral example of Δ_G for $Gr_3(\mathbb{C}^6)$

We say that two plabic graphs are *equivalent modulo (M2) and (M3)* if they can be related by any sequence of moves of the form (M2) and (M3) as defined in Section 3. For $Gr_3(\mathbb{C}^6)$, there are precisely 34 equivalence classes of plabic graphs of type $\pi_{3,6}$ modulo (M2) and (M3). Milena Hering pointed out to us an example of such a plabic graph G^1 such that Δ_{G^1} is non-integral. We then did a computer check with Polymake and found that among the 34 equivalence classes, only two give rise to non-integral Newton-Okounkov polytopes: the graph G^1 as well as the closely related graph G^2 shown in Figure 9. The other 32 equivalence classes give rise to integral Newton-Okounkov polytopes. Note that we computed the Newton-Okounkov polytopes Δ_G by using the inequality description of Γ_G, and Theorem 15.17, to be proved later in this paper, which says that $\Delta_G = \Gamma_G$.

The polytope Δ_{G^1} has a single non-integral vertex with coordinates as follows.

								
$\frac{3}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

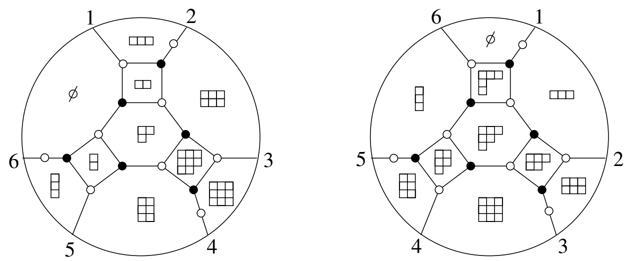


FIGURE 9. The plabic graphs G^1 and G^2 such that Δ_{G^1} and Δ_{G^2} are not integral.

Note that this non-integral vertex represents half the valuation of the flow polynomial for the element $f = (P_{124}P_{356} - P_{123}P_{456})/P_{\max}^2 \in L_2$. This element (and plabic graph) appear in [MS16b, Section A.3], where the authors observe that up to column rescaling, f is the twist of the Plücker coordinate P_{246}. (Their conventions for labeling faces of plabic graphs are slightly different from ours.)

For $Gr_3(\mathbb{C}^7)$, there are precisely 259 equivalence classes of plabic graphs of type $\pi_{3,7}$ modulo (M2) and (M3). Of the corresponding Newton-Okounkov polytopes, precisely 216 are integral and 43 are non-integral.

9. The superpotential and its associated polytopes

9.1. The superpotential W. Following [MR13], we define the superpotential mirror dual to $\mathbb{X}$. We refer to [MR13, Section 6] for more detail. Recall definitions from Sections 2 and 4.

Definition 9.1. Let $\mu_i^{\square}$ be the Young diagram associated to the k-element subset of horizontal steps $J_i^+ := [i+1, i+k-1] \cup \{i+k+1\}$, where the index i is always interpreted modulo n. Then for $i \neq n-k$, the Young diagram $\mu_i^{\square}$ turns out to be the unique diagram in $\mathcal{P}_{k,n}$ obtained by adding a single box to μ_i. And for $i = n-k$, the Young diagram $\mu_{n-k}^{\square}$ associated to J_{n-k}^+ is the rectangular $(n-k-1)\times(k-1)$ Young diagram obtained from μ_{n-k} by removing a rim hook.

We define the *superpotential* dual to the Grassmannian $\mathbb{X}$ to be the regular function $W : \check{\mathbb{X}}^\circ \times \mathbb{C}^* \to \mathbb{C}$ given by

$$W = \sum_{i=1}^{n} q^{\delta^i_{n-k}} \frac{p_{\mu_i^{\square}}}{p_{\mu_i}}, \tag{9.1}$$

where q is the coordinate on the $\mathbb{C}^*$ factor. We also write $\mathbb{C}_q^*$ for $\mathbb{C}^*$ with coordinate q.

For $i = 1, \dots, n$ we also define $W_i \in \mathbb{C}[\check{\mathbb{X}}^\circ]$ by

$$W_i := \frac{p_{\mu_i^{\square}}}{p_{\mu_i}} = \frac{p_{J_i^+}}{p_{J_i}}, \tag{9.2}$$

so that $W = \sum_{i=1}^n q^{\delta^i_{n-k}} W_i$. We may also write $W_q(x) := W(x,q)$, and refer to $W_q : \check{\mathbb{X}}^\circ \to \mathbb{C}^*$ as the superpotential, when there is no risk of confusion.

Example 9.2. For $k = 3$ and $n = 5$ we have $\mathbb{X} = Gr_2(\mathbb{C}^5)$ and $\check{\mathbb{X}} = Gr_3((\mathbb{C}^5)^*)$. The anticanonical divisor $\check{D}_{\mathrm{ac}}$ is given by

$$\check{D}_{\mathrm{ac}} = \{p_{(3)} = 0\} \cup \{p_{(3,3)} = 0\} \cup \{p_{(2,2)} = 0\} \cup \{p_{(1,1)} = 0\} \cup \{p_{\varnothing} = 0\},$$

compare Section 2.4, and

$$W = \frac{p_{(3,1)}}{p_{(3)}} + q\frac{p_{(2)}}{p_{(3,3)}} + \frac{p_{(3,2)}}{p_{(2,2)}} + \frac{p_{(2,1)}}{p_{(1,1)}} + \frac{p_{(1)}}{p_{\varnothing}}.$$

Definition 9.3 (Universally positive)**.** We say that a Laurent polynomial is *positive* if all of its coefficients are in $\mathbb{R}_{>0}$. An element $h \in \mathbb{C}[\check{\mathbb{X}}^\circ]$ is called *universally positive* (for the $\mathcal{A}$-cluster structure) if for every $\mathcal{A}$-cluster seed $\check{\Sigma}_G^{\mathcal{A}}$ the expansion $\mathbf{h}^G$ of h in $\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G)$ is a positive Laurent polynomial. Similarly $f \in \mathbb{C}[\check{\mathbb{X}}^\circ \times \mathbb{C}_q^*]$ is called universally positive if its expansion $\mathbf{f}^G$ in the variables $\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G) \cup \{q\}$ is given by a positive Laurent polynomial for every seed $\check{\Sigma}_G^{\mathcal{A}}$.

Remark 9.4. Recall from Section 4 the $\mathcal{A}$-cluster algebra structure on the homogeneous coordinate ring of the Grassmannian. In the formula (9.2) for W_i, the numerator is a Plücker coordinate (and hence a cluster variable), and the denominator is a frozen variable. Therefore by the positivity of the Laurent phenomenon [LS15, GHKK14], W_i is an example of a universally positive element of $\mathbb{C}[\check{\mathbb{X}}^\circ]$. Similarly, the superpotential W comes from the cluster algebra with q adjoined and is universally positive in the extended sense. Proposition 9.5 below gives the cluster expansion of W_q in terms of the rectangles cluster.

Proposition 9.5 ([MR13])**.** *If we let $i \times j$ denote the Young diagram which is a rectangle with i rows and j columns, then on the subset of $\check{\mathbb{X}}^\circ$ where all $p_{i\times j} \neq 0$, the superpotential W_q equals*

$$(9.3)\qquad W_q = \frac{p_{1\times 1}}{p_\varnothing} + \sum_{i=2}^{n-k}\sum_{j=1}^{k} \frac{p_{i\times j}\ p_{(i-2)\times(j-1)}}{p_{(i-1)\times(j-1)}\ p_{(i-1)\times j}} + q\frac{p_{(n-k-1)\times(k-1)}}{p_{(n-k)\times k}} + \sum_{i=1}^{n-k}\sum_{j=2}^{k} \frac{p_{i\times j}\ p_{(i-1)\times(j-2)}}{p_{(i-1)\times(j-1)}\ p_{i\times(j-1)}}.$$

Here of course if i or j equals 0, then $p_{i\times j} = p_\varnothing$.

The Laurent polynomial (9.3) can be encoded in a diagram (shown in Figure 10 for $k = 3$ and $n = 5$), see [MR13]. Namely it is the Laurent polynomial obtained by summing over all the arrows the Laurent

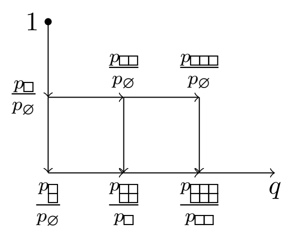


FIGURE 10. The diagram defining the superpotential for $k = 3$ and $n = 5$.

monomials obtained by dividing the expression at the head by the expression at the tail of the arrow. So in this example, we have

$$(9.4)\qquad W_q = p_{1\times1} + \frac{p_{2\times1}}{p_{1\times1}} + \frac{p_{2\times2}\ p_\varnothing}{p_{1\times1}\ p_{1\times2}} + \frac{p_{2\times3}\ p_\varnothing}{p_{1\times2}\ p_{1\times3}} + \frac{p_{1\times2}}{p_{1\times1}} + \frac{p_{1\times3}}{p_{1\times2}} + \frac{p_{2\times2}\ p_\varnothing}{p_{1\times1}\ p_{2\times1}} + \frac{p_{2\times3}\ p_{1\times1}}{p_{1\times2}\ p_{2\times2}} + q\frac{p_{1\times2}}{p_{2\times3}},$$

where we have chosen the normalization of Plücker coordinates on $\check{\mathbb{X}}^\circ$ given by $p_\varnothing = 1$.

Remark 9.6. The quiver underlying the diagram above was introduced by [BCFKvS00] where it was encoding the EHX Laurent polynomial superpotential [EHX97] associated to a Grassmannian (in the vein of Givental's quiver for the full flag variety [Giv97]). It was related to the Peterson variety in [Rie06] before appearing in connection with the rectangles cluster in [MR13].

9.2. Polytopes via tropicalisation. In this section we define a polytope Γ_G^r in terms of inequalities, which are obtained by restricting the superpotential to the cluster torus $\mathbb{T}_G^\vee$ and applying a tropicalisation procedure, see [MS15] and references therein. We also define a polytope $\Gamma_G(r_1, \ldots, r_n)$, which generalizes Γ_G^r, and which will be discussed in Section 18.

Definition 9.7 (naive Tropicalisation)**.** To any Laurent polynomial $\mathbf{h}$ in variables $X_1, \ldots, X_m$ with coefficients in $\mathbb{R}_{>0}$ we associate a piecewise linear map $\mathrm{Trop}(\mathbf{h}) : \mathbb{R}^m \to \mathbb{R}$ called the *tropicalisation* of $\mathbf{h}$ as follows. We set $\mathrm{Trop}(X_i)(y_1, \ldots, y_m) = y_i$. If $\mathbf{h_1}$ and $\mathbf{h_2}$ are two positive Laurent polynomials, and $a_1, a_2 \in \mathbb{R}_{>0}$, then we impose the condition that

$$(9.5)\qquad \mathrm{Trop}(a_1\mathbf{h_1} + a_2\mathbf{h_2}) = \min(\mathrm{Trop}(\mathbf{h_1}), \mathrm{Trop}(\mathbf{h_2})), \text{ and } \mathrm{Trop}(\mathbf{h_1}\mathbf{h_2}) = \mathrm{Trop}(\mathbf{h_1}) + \mathrm{Trop}(\mathbf{h_2}).$$

This defines $\mathrm{Trop}(\mathbf{h})$ for all positive Laurent polynomials $\mathbf{h}$, by induction.

Remark 9.8. Informally, $\mathrm{Trop}(\mathbf{h})$ is obtained by replacing multiplication by addition, and addition by min. For example if $\mathbf{h} = X_1^{-1}X_3^2 + 5X_2 + X_1X_2^{-3}X_3$ then $\mathrm{Trop}(\mathbf{h})(y_1, y_2, y_3) = \min(2y_3 - y_1, y_2, y_1 - 3y_2 + y_3)$.

Now let G be a reduced plabic graph of type $\pi_{k,n}$ with associated set of cluster variables $\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G)$, see (4.3). Suppose $\mathbf{h} : \mathbb{T}_G^{\vee} \times \mathbb{C}^* \to \mathbb{C}$ is a positive Laurent polynomial in the variables $\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G) \cup \{q\}$ with coefficients in $\mathbb{R}_{>0}$. In this case the tropicalisation is a (piecewise linear) map

$$\mathrm{Trop}(\mathbf{h}) : \mathbb{R}^{\mathcal{P}_G} \times \mathbb{R} \to \mathbb{R},$$

in variables that we denote $((v_\mu)_{\mu \in \mathcal{P}_G}, r)$. Similarly, if $\mathbf{h} : \mathbb{T}_G^{\vee} \to \mathbb{C}$, then $\mathrm{Trop}(\mathbf{h}) : \mathbb{R}^{\mathcal{P}_G} \to \mathbb{R}$.

Definition 9.9. Suppose $f \in \mathbb{C}[\check{\mathbb{X}}^\circ]$ is universally positive with $\mathcal{A}$-cluster expansion $\mathbf{f}^G$. Then we define $\mathrm{Trop}_G(f)$ to be the tropicalisation $\mathrm{Trop}(\mathbf{f}^G) : \mathbb{R}^{\mathcal{P}_G} \to \mathbb{R}$. Similarly, if $f \in \mathbb{C}[\check{\mathbb{X}}^\circ \times \mathbb{C}_q^*]$ is universally positive, so that $\mathbf{f}^G$ is a positive Laurent polynomial in the variables $\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G) \cup \{q\}$, then we use the same notation, $\mathrm{Trop}_G(f)$, to mean the map $\mathrm{Trop}(\mathbf{f}^G) : \mathbb{R}^{\mathcal{P}_G} \times \mathbb{R} \to \mathbb{R}$.

By Remark 9.4, the superpotential W is universally positive, so that $\mathrm{Trop}_G(W) : \mathbb{R}^{\mathcal{P}_G} \times \mathbb{R} \to \mathbb{R}$ is well-defined for any seed $\check{\Sigma}_G^{\mathcal{A}}$. We now use $\mathrm{Trop}_G(W)$ to define a polytope.

Definition 9.10. For $r \in \mathbb{R}$ we define the *superpotential polytope*

$$\Gamma_G^r = \{v \in \mathbb{R}^{\mathcal{P}_G} \mid \mathrm{Trop}_G(W)(v, r) \geq 0\}.$$

When $r = 1$, we will also write $\Gamma_G := \Gamma_G^1$.

Remark 9.11. Note that the right hand side is a convex subset of $\mathbb{R}^{\mathcal{P}_G}$ given by inequalities determined by the Laurent polynomial $W^G = W|_{\mathbb{T}_G^{\vee} \times \mathbb{C}^*}$. It will follow from Lemma 15.2 and Corollary 10.16 that Γ_G^r is in fact bounded and hence a convex polytope for $r \geq 0$. In this case it also follows directly from the definitions that $\Gamma_G^r = r\Gamma_G$. Hence we will primarily restrict our attention to Γ_G. If $r < 0$ we will have $\Gamma_G^r = \varnothing$ as follows from Proposition 18.6.

Example 9.12. Let G be the plabic graph from Figure 4. The superpotential W is written out in terms of $\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G) \cup \{q\}$ in (9.4). We obtain the following inequalities which define the polytpe Γ_G^r.

$$\begin{aligned}
&0 \leq v_{\ydiagram{1}} && 0 \leq v_{\ydiagram{1,1}} - v_{\ydiagram{1}} \\
&0 \leq v_{\ydiagram{2,2}} - v_{\ydiagram{1}} - v_{\ydiagram{2}} && 0 \leq v_{\ydiagram{3,3}} - v_{\ydiagram{2}} - v_{\ydiagram{3}} \\
&0 \leq v_{\ydiagram{2}} - v_{\ydiagram{1}} && 0 \leq v_{\ydiagram{3}} - v_{\ydiagram{2}} \\
&0 \leq v_{\ydiagram{2,2}} - v_{\ydiagram{1}} - v_{\ydiagram{1,1}} && 0 \leq v_{\ydiagram{3,3}} + v_{\ydiagram{1}} - v_{\ydiagram{2}} - v_{\ydiagram{2,2}} \\
&0 \leq r + v_{\ydiagram{2}} - v_{\ydiagram{3,3}} &&
\end{aligned}$$

One can check that in this case, Γ_G is precisely the polytope Conv_G from Example 7.10.

We also have a natural generalisation of the superpotential polytope defined as follows. Recall the summands $W_i \in \mathbb{C}[\check{\mathbb{X}}^\circ]$ of the superpotential from (9.2). Each W_i is itself universally positive and gives rise to a piecewise linear function $\mathrm{Trop}_G(W_i) : \mathbb{R}^{\mathcal{P}_G} \to \mathbb{R}$ for any $\mathcal{A}$-cluster seed $\check{\Sigma}_G^{\mathcal{A}}$.

Definition 9.13. Choose $r_1, \ldots, r_n \in \mathbb{R}$. We define the *generalized superpotential polytope* by

$$\Gamma_G(r_1, \ldots, r_n) = \bigcap_i \{v \in \mathbb{R}^{\mathcal{P}_G} \mid \mathrm{Trop}_G(W_i)(v) + r_i \geq 0\}. \tag{9.6}$$

In particular if $r_{n-k} = r$ and $r_i = 0$ for $i \neq n-k$, then $\Gamma_G(r_1, \ldots, r_n) = \Gamma_G^r$.

10. Tropicalisation, total positivity, and mutation

10.1. Total positivity and generalised Puiseux series. The $\mathcal{A}$-cluster structure on the Grassmannian $\check{\mathbb{X}}$, which is a *positive atlas* in the terminology of [FG06], gives rise to a 'tropicalised version' of $\check{\mathbb{X}}$. This, inspired by [Lus94], is defined in [FG06] as the analogue of the totally positive part with $\mathbb{R}_{>0}$ replaced by the tropical semifield $(\mathbb{R}, \min, +)$. We construct the tropicalisation of $\check{\mathbb{X}}$ and our polytopes in terms of total positivity over generalised Puiseux series, extending the original construction of [Lus94]. Our initial goal will be to describe how the polytopes $\Gamma_G(r_1, \ldots, r_n)$ behave under mutation of G.

Definition 10.1 (Generalised Puiseux series)**.** Following [Mar10], let $\mathbf{K}$ be the field of generalised Puiseux series in one variable with set of exponents taken from

$$\mathrm{MonSeq} = \{A \subset \mathbb{R} \mid \mathrm{Cardinality}(A \cap \mathbb{R}_{\le x}) < \infty \text{ for arbitrarily large } x \in \mathbb{R}\}.$$

Note that a set $A \in \mathrm{MonSeq}$ can be thought of as a strictly monotone increasing sequence of numbers which is either finite or countable tending to infinity. We write $(\alpha_m) \in \mathrm{MonSeq}$ if $(\alpha_m)_{m\in\mathbb{Z}_{\ge 0}}$ is such a strictly monotone increasing sequence, and we have

$$\mathbf{K} = \left\{ c(t) = \sum_{(\alpha_m)\in\mathrm{MonSeq}} c_{\alpha_m} t^{\alpha_m} \mid c_{\alpha_m} \in \mathbb{C} \right\}. \tag{10.1}$$

Note that $\mathbf{K}$ is complete and algebraically closed, see [Mar10]. We denote by $\mathbf{K}_{>0}$ the subsemifield of $\mathbf{K}$ defined by

$$\mathbf{K}_{>0} = \left\{ c(t) \in \mathbf{K} \mid c(t) = \sum_{(\alpha_m)\in\mathrm{MonSeq}} c_{\alpha_m} t^{\alpha_m},\ c_{\alpha_0} \in \mathbb{R}_{>0} \right\}. \tag{10.2}$$

We have an $\mathbb{R}$-valued valuation, $\mathrm{Val}_{\mathbf{K}} : \mathbf{K} \smallsetminus \{0\} \to \mathbb{R}$, given by $\mathrm{Val}_{\mathbf{K}}(c(t)) = \alpha_0$ if $c(t) = \sum c_{\alpha_m} t^{\alpha_m}$ where the lowest order term is assumed to have non-zero coefficient, $c_{\alpha_0} \neq 0$.

We also use the notation $\mathbf{L} := \mathbb{R}((t))$ for the field of real Laurent series in one variable. Note that $\mathbf{L} \subset \mathbf{K}$. We let $\mathbf{L}_{>0} = \mathbf{L} \cap \mathbf{K}_{>0}$, and denote by $\mathrm{Val}_{\mathbf{L}}$ the lowest-order-term valuation of $\mathbf{L}$.

Lusztig [Lus94] applied his theory of total positivity for an algebraic groups $\mathcal{G}$ not just to defining a notion of $\mathbb{R}_{>0}$-valued points, 'the totally positive part', inside $\mathcal{G}(\mathbb{R})$, but also to introducing $\mathbf{L}_{>0}$-valued points $\mathcal{G}(\mathbf{L})$. Moreover, he used this theory to describe his parametrisation of the canonical basis, see [Lus94, Section 10]. In our setting, there is a notion of totally positive part $\check{\mathbb{X}}(\mathbf{L}_{>0})$ in $\check{\mathbb{X}}(\mathbf{L})$ which plays a similar role, and which we employ in this section to give an interpretation to the lattice points of the generalised superpotential polytopes. Moreover we give an analogous interpretation of all of the points of our polytopes by applying the same construction with $\mathbf{L}_{>0}$ replaced by $\mathbf{K}_{>0}$.

Recall that we have fixed $p_\varnothing = 1$ on $\check{\mathbb{X}}^\circ$. We make the following definition.

Definition 10.2 (Positive parts of $\check{\mathbb{X}}$)**.** Recall from Definition 5.1 that the totally positive part of the Grassmannian $\check{\mathbb{X}}$ can be defined as the subset of the real Grassmannian where the Plücker coordinates p_λ are positive [Pos]. Now let $\mathbf{F}$ be an infinite field and $\mathbf{F}_{>0}$ a subset in $\mathbf{F} \smallsetminus \{0\}$ which is closed under addition, multiplication and inverse. For example $\mathbf{F} = \mathbb{R}$ with with the positive real numbers, or $\mathbf{F} = \mathbf{L}, \mathbf{K}$ with $\mathbf{F}_{>0}$ as in Definition 10.1. We define

$$\check{\mathbb{X}}(\mathbf{F}_{>0}) = \check{\mathbb{X}}^\circ(\mathbf{F}_{>0}) := \{x \in \check{\mathbb{X}}^\circ(\mathbf{F}) \mid p_\lambda(x) \in \mathbf{F}_{>0}, \quad \lambda \in \mathcal{P}_{k,n}\}.$$

Note that for any $x \in \check{\mathbb{X}}(\mathbf{K}_{>0})$, all of the Plücker coordinates $p_\lambda(x)$ are automatically nonzero, and that we have inclusions $\check{\mathbb{X}}(\mathbb{R}_{>0}) \subset \check{\mathbb{X}}(\mathbf{L}_{>0}) \subset \check{\mathbb{X}}(\mathbf{K}_{>0})$.

We record that we have the standard parametrisations of the totally positive part also in this situation.

Lemma 10.3. *Suppose $\Phi_G^\vee$ is an $\mathcal{A}$-cluster chart (see (4.2)). Suppose $\mathbf{F}$ and $\mathbf{F}_{>0}$ are as in Definition 10.2. We can consider $\Phi_G^\vee$ over the field $\mathbf{F}$. In this case we have that*

$$\check{\mathbb{X}}(\mathbf{F}_{>0}) = \Phi_G^\vee((\mathbf{F}_{>0})^{\mathcal{P}_G}), \tag{10.3}$$

and the map $\Phi_G^\vee : (\mathbf{F}_{>0})^{\mathcal{P}_G} \to \check{\mathbb{X}}(\mathbf{F}_{>0})$ is a bijection.

Proof. This follows in the usual way from the cluster algebra structure on the Grassmannian [Sco06], by virtue of which each cluster variable can be written as a subtraction-free rational function in any cluster. So in particular, if the elements of one cluster have values in $\mathbf{F}_{>0}$, then so do all cluster variables. □

Remark 10.4. The right hand side of the equation (10.3) is independent of G, by positivity of mutation. Note that the notion of the $\mathbf{F}_{>0}$-valued points extends to a general $\mathcal{A}$-cluster variety if we take (10.3) as the definition in place of Definition 10.2.

10.2. Tropicalisation of a positive Laurent polynomial. We record the following straightforward lemma which interprets the tropicalisation $\mathrm{Trop}(\mathbf{h})$ of a positive Laurent polynomial $\mathbf{h}$, see Definition 9.3 and Definition 9.7, in terms of the semifield $\mathbf{K}_{>0}$ and the valuation $\mathrm{Val}_{\mathbf{K}}$. See [Lus94, Proof of Proposition 9.4] and [SW05, Proposition 2.5] for related statements.

Lemma 10.5. *Let* $\mathbf{h} \in \mathbb{C}[X_1^{\pm 1}, \dots, X_m^{\pm 1}]$ *be a positive Laurent polynomial. We may evaluate* $\mathbf{h}$ *on* $(k_i)_{i=1}^m \in (\mathbf{K}_{>0})^m$*. On the other hand, associated to each* k_i *we have* $y_i := \mathrm{Val}_{\mathbf{K}}(k_i)$*, so that* $(y_i)_{i=1}^m \in \mathbb{R}^m$*. Then*

$$\mathrm{Trop}(\mathbf{h})(y_1, \dots, y_m) = \mathrm{Val}_{\mathbf{K}}(\mathbf{h}(k_1, \dots, k_m)).$$

In particular, $\mathrm{Val}_{\mathbf{K}}(\mathbf{h}(k_1, \dots, k_m))$ *depends only on the valuations* y_i *of the* k_i*.*

Proof. If $\mathbf{h} = X_i$ then both sides agree and equal to x_i. Clearly any product $\mathbf{h} = \mathbf{h}_1\mathbf{h}_2$ gives a $\mathbf{K}$-valuation equal to $\mathrm{Val}_{\mathbf{K}}(\mathbf{h}_1(k_1, \dots, k_m)) + \mathrm{Val}_{\mathbf{K}}(\mathbf{h}_2(k_1, \dots, k_m))$. Now let $\mathbf{h} = \mathbf{h}_1 + \mathbf{h}_2$. Because all of the coefficients of $\mathbf{h}_1, \mathbf{h}_2$ are positive and the leading terms of the k_i also have positive coefficients, there can be no cancellations when working out the valuation of the sum $(\mathbf{h}_1 + \mathbf{h}_2)(k_1, \dots, k_m)$. This implies that the latter valuation is given by $\min(\mathrm{Val}_{\mathbf{K}}(\mathbf{h}_1(k_1, \dots, k_m)), \mathrm{Val}_{\mathbf{K}}(\mathbf{h}_2(k_1, \dots, k_m)))$. Thus the right hand side has the same properties as define the left hand side, see Definition 9.7. □

10.3. Tropicalisation of $\check{\mathbb{X}}$ and Zones. We introduce a (positive) tropical version of our cluster variety $\check{\mathbb{X}}$ via an equivalence relation on elements of $\check{\mathbb{X}}(\mathbf{K}_{>0})$, analogous to Lusztig's construction of 'zones' in $U^+(\mathbf{L}_{>0})$ [Lus94]. This is also very close to the notion of positive tropical variety from [SW05, Section 2].

Definition 10.6 (Zones and tropical points)**.** Let us define an equivalence relation on $\check{\mathbb{X}}(\mathbf{K}_{>0})$ by

$$x \sim x' \quad :\Longleftrightarrow \quad \mathrm{Val}_{\mathbf{K}}(p_\lambda(x)) = \mathrm{Val}_{\mathbf{K}}(p_\lambda(x')) \ \text{ for all } \lambda \in \mathcal{P}_{k,n}.$$

We write $[x]$ for the equivalence class of $x \in \check{\mathbb{X}}(\mathbf{K}_{>0})$ and let $\mathrm{Trop}(\check{\mathbb{X}}) := \check{\mathbb{X}}(\mathbf{K}_{>0})/\sim$ denote the set of equivalence classes, also called *tropical points* of $\check{\mathbb{X}}$. If a tropical point has a representative $x \in \check{\mathbb{X}}(\mathbf{L}_{>0})$ then we call it a *zone* inspired by the terminology of Lusztig. The zones are precisely those tropical points $[x]$ for which all $\mathrm{Val}_{\mathbf{K}}(p_\lambda(x))$ lie in $\mathbb{Z}$.

Lemma 10.7. *For any seed* $\check{\Sigma}_G^{\mathcal{A}}$ *the following map is well-defined and gives a bijection,*

$$\pi_G : \mathrm{Trop}(\check{\mathbb{X}}) \to \mathbb{R}^{\mathcal{P}_G}, \qquad [x] \mapsto (\mathrm{Val}_{\mathbf{K}}(\varphi_\mu(x)))_\mu, \tag{10.4}$$

where the φ_μ *run over the set of cluster variables* $\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G)$*, and the indexing set of cluster variables is denoted* $\mathcal{P}_G$*.*

Definition 10.8 (Tropicalised $\mathcal{A}$-cluster mutation)**.** Suppose $\check{\Sigma}_G^{\mathcal{A}}$ and $\check{\Sigma}_{G'}^{\mathcal{A}}$ are general $\mathcal{A}$-cluster seeds of type $\pi_{k,n}$ which are related by a single mutation at a vertex ν_i. Let the cluster variables for $\check{\Sigma}_G^{\mathcal{A}}$ be indexed by $\mathcal{P}_G = \{\nu_1, \dots, \nu_N\}$. Recall that $\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G') = \mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G) \cup \{\varphi_{\nu_i'}\} \setminus \{\varphi_{\nu_i}\}$, and the $\mathcal{A}$-cluster mutation $\mathrm{Mut}_{\nu_i}^{\mathcal{A}}$ gives a positive Laurent polynomial expansion of the new variable $\varphi_{\nu_i'}$ in terms of $\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G)$, see (4.4). We tropicalise this change of coordinates between $\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G)$ and $\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G')$ and denote the resulting piecewise linear map by $\Psi_{G,G'}$. Explicitly, $\Psi_{G,G'} : \mathbb{R}^{\mathcal{P}_G} \to \mathbb{R}^{\mathcal{P}_{G'}}$ takes $(v_{\nu_1}, v_{\nu_2}, \dots, v_{\nu_N})$ to $(v_{\nu_1}, \dots, v_{\nu_{i-1}}, v_{\nu_i'}, v_{\nu_{i+1}}, \dots, v_{\nu_N})$, where

$$v_{\nu_i'} = \min\Big(\sum_{\nu_j \to \nu_i} v_{\nu_j}, \sum_{\nu_i \to \nu_j} v_{\nu_j}\Big) - v_{\nu_i}, \tag{10.5}$$

and the sums are over arrows in the quiver $Q(G)$ pointing towards ν_i or away from ν_i, respectively. We call $\Psi_{G,G'}$ a *tropicalised $\mathcal{A}$-cluster mutation.*

Remark 10.9. Note that if G and G' are plabic graphs related by the square move (M1) – we can suppose we are doing the square move at ν_1 in Figure 5 – then $\Psi_{G,G'}$ is simply given by

$$v_{\nu_1'} = \min(v_{\nu_2} + v_{\nu_4}, v_{\nu_3} + v_{\nu_5}) - v_{\nu_1}.$$

Lemma 10.10. *Suppose G and G' index arbitrary $\mathcal{A}$-seeds of type $\pi_{k,n}$ which are related by a single mutation at vertex ν_1, where the cluster variables are indexed by $(\nu_1, \dots, \nu_N)$. Then we have a commutative diagram*

$$\begin{array}{ccccc} & & \mathrm{Trop}(\check{\mathbb{X}}) & & \\ & \pi_G \swarrow & & \searrow \pi_{G'} & \\ \mathbb{R}^{\mathcal{P}_G} & & \overset{\Psi_{G,G'}}{\longrightarrow} & & \mathbb{R}^{\mathcal{P}_{G'}}, \end{array} \tag{10.6}$$

where the map along the bottom is the tropicalised $\mathcal{A}$-cluster mutation $\Psi_{G,G'}$ *from Definition 10.8.*

Proof of Lemmas 10.7 and 10.10. Recall that the cluster chart $\Phi^\vee_G$ from Lemma 10.3 gives a bijective parameterization of $\check{\mathbb{X}}(\mathbf{K}_{>0})$ where the inverse $(\Phi^\vee_G)^{-1} : \check{\mathbb{X}}(\mathbf{K}_{>0}) \longrightarrow (\mathbf{K}_{>0})^{\mathcal{P}_G}$ is precisely the map $x \mapsto (\varphi_\mu(x))_\mu$. We have the following composition of surjective maps

$$\mathrm{Comp}_G : \check{\mathbb{X}}(\mathbf{K}_{>0}) \overset{(\Phi^\vee_G)^{-1}}{\longrightarrow} (\mathbf{K}_{>0})^{\mathcal{P}_G} \overset{\mathrm{Val}_{\mathbf{K}}}{\longrightarrow} \mathbb{R}^{\mathcal{P}_G}.$$

We define an equivalence relation $\sim_G$ by letting $x \sim_G x'$ if and only if $\mathrm{Comp}_G(x) = \mathrm{Comp}_G(x')$. Clearly with this definition, Comp_G descends to a bijection $[\mathrm{Comp}_G] : \check{\mathbb{X}}(\mathbf{K}_{>0})/\sim_G \to \mathbb{R}^{\mathcal{P}_G}$. For Lemma 10.7 it suffices to show that the equivalence relation $\sim_G$ is independent of G and recovers the original equivalence relation $\sim$ from Definition 10.6. Then $[\mathrm{Comp}_G] = \pi_G$ and we are done.

If G and G' are related by a single mutation, see Definition 4.4, then we have a commutative diagram

$$\begin{array}{ccccc} & & \check{\mathbb{X}}(\mathbf{K}_{>0}) & & \\ \mathrm{Comp}_G & \swarrow & & \searrow & \mathrm{Comp}_{G'} \\ \mathbb{R}^{\mathcal{P}_G} & & \overset{\Psi_{G,G'}}{\longrightarrow} & & \mathbb{R}^{\mathcal{P}_{G'}}. \end{array} \tag{10.7}$$

where $\Psi_{G,G'}$ is the tropicalised cluster mutation. This follows by an application of Lemma 10.5. Since $\Psi_{G,G'}$ is a bijection (with inverse $\Psi_{G',G}$) it follows that $\mathrm{Comp}_G(x) = \mathrm{Comp}_G(x')$ if and only if $\mathrm{Comp}_{G'}(x) = \mathrm{Comp}_{G'}(x')$. Thus $\sim_G$ and $\sim_{G'}$ are the same equivalence relation. Therefore the equivalence relation $\sim_G$ is independent of G. If G is a plabic graph indexing a Plücker cluster then $x \sim x'$ implies that $x \sim_G x'$. On the other hand if $x \sim_G x'$ then also $x \sim_{G'} x'$ for any other G'. Therefore it follows that $\mathrm{Val}_{\mathbf{K}}(p_\lambda(x)) = \mathrm{Val}_{\mathbf{K}}(p_\lambda(x'))$ for all Plücker coordinates p_λ, since every Plücker coordinate appears in *some* seed $\check{\Sigma}^{\mathcal{A}}_{G'}$. As a consequence $x \sim_G x'$ implies $x \sim x'$ and Lemma 10.7 is proved.

Since all of the equivalence relations $\sim_G$ are equal to $\sim$, we can factor all of the vertical maps Comp_G through $\sim$ and then (10.7) turns into the commutative diagram of bijections which is precisely the one given in Lemma 10.10. ☐

Remark 10.11. The main observation of the above proof was that if we consider an arbitrary $\mathcal{A}$-seed $\check{\Sigma}^{\mathcal{A}}_G$ of type $\pi_{k,n}$, then for $x, x' \in \check{\mathbb{X}}(\mathbf{K}_{>0})$ we have

$$x \sim x' \iff \mathrm{Val}_{\mathbf{K}}(\varphi_\mu(x)) = \mathrm{Val}_{\mathbf{K}}(\varphi_\mu(x')) \text{ for all cluster variables } \varphi_\mu \text{ of } \check{\Sigma}^{\mathcal{A}}_G. \tag{10.8}$$

This says that equivalence of points in $\check{\mathbb{X}}(\mathbf{K}_{>0})$ can be checked using a single, arbitrarily chosen seed, and gives an alternative definition for the equivalence relation $\sim$. With this alternative definition (10.8) of $\sim$, the definition of 'tropical points' and 'zones' as equivalence classes generalises to an arbitrary $\mathcal{A}$-cluster algebra, compare Remark 10.4.

Remark 10.12. We note that the inverse of the tropicalised cluster mutation $\Psi_{G,G'}$ is always just given by $\Psi_{G',G}$. Since both maps $\Psi_{G,G'}$ and $\Psi_{G',G}$ map integral points to integral points we have that $\Psi_{G,G'}$ restricts to a bijection $\mathbb{Z}^{\mathcal{P}_G} \to \mathbb{Z}^{\mathcal{P}_{G'}}$ and the entire diagram (10.6) restricts to give the commutative diagram of bijections,

$$(10.9)\qquad \begin{array}{ccc} & \mathrm{Zones}(\check{\mathbb{X}}) & \\ \pi_G \swarrow & & \searrow \pi_{G'} \\ \mathbb{Z}^{\mathcal{P}_G} & \xrightarrow{\Psi_{G,G'}} & \mathbb{Z}^{\mathcal{P}_{G'}}. \end{array}$$

Therefore $\mathrm{Trop}(\check{\mathbb{X}})$ is endowed with an integral piecewise linear structure via the identifications π_G with the $\mathbb{R}^{\mathcal{P}_G}$ and the Lemmas 10.7 and 10.10.

10.4. Mutation of polytopes. In this section we give an interpretation of the superpotential polytopes Γ_G^r from Definition 9.10 and their generalisations $\Gamma_G(r_1,\dots,r_n)$ from Definition 9.13 in terms of $\mathrm{Trop}(\check{\mathbb{X}})$.

Definition 10.13. Suppose $h \in \mathbb{C}[\check{\mathbb{X}}^\circ]$ has the property that it is *universally positive* for the $\mathcal{A}$-cluster algebra structure of $\mathbb{C}[\check{\mathbb{X}}^\circ]$, as in Definition 9.3. Let $m \in \mathbb{R}$. In this case we define inside $\mathrm{Trop}(\check{\mathbb{X}})$ the set

$$\mathrm{PosSet}_{(m)}(h) := \{[x] \in \mathrm{Trop}(\check{\mathbb{X}}) \mid \mathrm{Val}_{\mathbf{K}}(h(x)) + m \geq 0\}.$$

For a given choice of seed $\check{\Sigma}_G^{\mathcal{A}}$ we also associate to h the subset of $\mathbb{R}^{\mathcal{P}_G}$,

$$\mathrm{PosSet}^G_{(m)}(h) := \{v \in \mathbb{R}^{\mathcal{P}_G} \mid \mathrm{Trop}(\mathbf{h}^G)(v) + m \geq 0\}.$$

Remark 10.14. Note that for $m = 0$ the set $\mathrm{PosSet}^G_{(0)}(h)$ is a (possibly trivial) polyhedral cone described as intersection of half-spaces. Introducing the $m \in \mathbb{R}$ amounts to shifting the half-spaces.

Lemma 10.15. *Given any seed $\check{\Sigma}_G^{\mathcal{A}}$, a universally positive $h \in \mathbb{C}[\check{\mathbb{X}}^\circ]$, and any $m \in \mathbb{R}$, the bijection $\pi_G : \mathrm{Trop}(\check{\mathbb{X}}) \to \mathbb{R}^{\mathcal{P}_G}$ from Lemma 10.7 restricts to give a bijection,*

$$\pi_G : \mathrm{PosSet}_{(m)}(h) \longrightarrow \mathrm{PosSet}^G_{(m)}(h),$$

between the sets from Definition 10.13, which we again denote π_G by abuse of notation. We have the following commutative diagram of bijections

$$(10.10)\qquad \begin{array}{ccc} & \mathrm{PosSet}_{(m)}(h) & \\ \pi_G \swarrow & & \searrow \pi_{G'} \\ \mathrm{PosSet}^G_{(m)}(h) & \xrightarrow{\Psi_{G,G'}} & \mathrm{PosSet}^{G'}_{(m)}(h). \end{array}$$

where the map $\Psi_{G,G'}$ along the bottom is the restriction of the tropicalised cluster mutation *from Lemma 10.10.*

Proof. The set $\mathrm{PosSet}^G_{(m)}(h)$ in $\mathbb{R}^{\mathcal{P}_G}$ is indeed the image of $\mathrm{PosSet}_{(m)}(h)$ under the bijection π_G from Lemma 10.7. This follows, since h is universally positive, from Lemma 10.5. The rest of the lemma is immediate from Lemma 10.10. □

Recall that the summands W_i of the superpotential are universally positive by Remark 9.4.

Corollary 10.16. *Let $r_1,\dots,r_n \in \mathbb{R}$ and choose $\check{\Sigma}_G^{\mathcal{A}}$ a general seed. The subset of* $\mathrm{Trop}(\check{\mathbb{X}})$ *defined by*

$$\Gamma(r_1,\dots,r_n) := \bigcap_{i=1}^n \mathrm{PosSet}_{(r_i)}(W_i)$$

is in bijection with the generalised superpotential polytope $\Gamma_G(r_1,\dots,r_n) = \bigcap_i \mathrm{PosSet}^G_{(r_i)}(W_i)$, by the restriction of the map π_G from Lemma 10.7. Moreover if $\check{\Sigma}_{G'}^{\mathcal{A}}$ is related to $\check{\Sigma}_G^{\mathcal{A}}$ by a cluster mutation $\mathrm{Mut}_\nu^{\mathcal{A}}$, then we have that the tropicalised cluster mutation *$\Psi_{G,G'}$ restricts to a bijection*

$$\Psi_{G,G'} : \Gamma_G(r_1,\dots,r_n) \to \Gamma_{G'}(r_1,\dots,r_n).$$

Proof. The equality $\Gamma_G(r_1,\dots,r_n) = \bigcap_i \mathrm{PosSet}^G_{(r_i)}(W_i)$ is just an equivalent restatement of Definition 9.13. The corollary is immediate from Lemma 10.15. □

Corollary 10.17. *The number of lattice points of $\Gamma_G(r_1,\dots,r_n)$ is independent of G.*

Proof. By Remark 10.12 and Corollary 10.16, if G' indexes a seed which is obtained by mutation from G, then the corresponding tropicalised cluster mutation $\Psi_{G,G'}$ restricts to a bijection from the lattice points of $\Gamma_G(r_1,\dots,r_n)$ to the lattice points of $\Gamma_{G'}(r_1,\dots,r_n)$. Since all seeds are connected by mutation, it follows that the number of lattice points of $\Gamma_G(r_1,\dots,r_n)$ is independent of the choice of seed. □

11. Combinatorics of perfect matchings

We now return to $\mathbb{X}$ and study the network expansions of the Plücker coordinates P_λ. Namely, in this section we use perfect matchings to show that for any Plücker coordinate the network expansions coming from plabic graphs always have a strongly minimal and a strongly maximal term, see Definition 5.13.

Let G be a bipartite plabic graph with boundary vertices labeled $1,2,\dots,n$. We assume that each boundary vertex is adjacent to one white vertex and no other vertices. A *matching* of G is a collection of edges of G which cover each internal vertex exactly once. For a matching M, we let $\partial M \subset [n]$ denote the subset of the boundary vertices covered by M. Given G, we say that J is *matchable* if there is at least one matching M of G with boundary $\partial M = J$.

There is a partial order on matchings, which makes the set of matchings with a fixed boundary into a *distributive lattice*. Let M be a matching of G and f an internal face of G such that M contains exactly half the edges in the boundary of f (the most possible). The *flip* (or *swivel*) of M at μ is the matching M', which contains the other half of the edges in the boundary of μ and is otherwise the same as M. Note that M' uses the same boundary edges as M does. We say that the flip of M at μ is a *flip up* from M to M', and we write $M \lessdot M'$, if, when we orient the edges in the boundary of μ clockwise, the matched edges in M go from white to black. Otherwise we say that the flip is a *flip down* from M to M', and we write $M' \lessdot M$. See the leftmost column of Figure 11. We let $\le$ denote the partial order on matchings generated by the cover relation $\lessdot$.

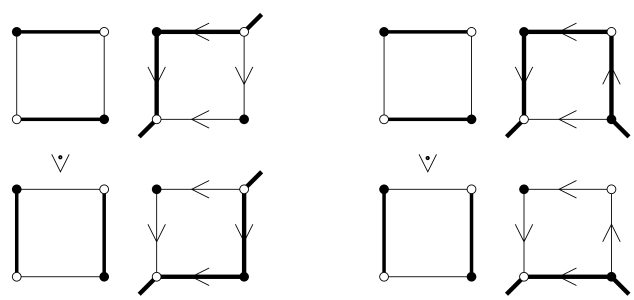


Figure 11. Flipping up a face, and some examples of the effect on corresponding flows.

The following result appears as [MS16b, Theorem B.1] and [MS16b, Corollaries B.3 and B.4], and is deduced from [Pro93, Theorem 2].

Theorem 11.1 ([MS16b, Theorem B.1] and [Pro93, Theorem 2])**.** *Let G be a reduced bipartite plabic graph, and let J be a matchable subset of $[n]$. Then the partial order $\le$ makes the set of matchings on G with boundary J into a finite distributive lattice, which we call* Match_J^G *(or* Match_λ^G*, if $\lambda \subseteq (n-k)\times k$ is the partition corresponding to J).*

In particular, the set Match_J^G *of matchings of G with boundary J has a unique minimal element M_J^{min} and a unique maximal element M_J^{max}, assuming the set is nonempty. Moreover, any two elements of* Match_J^G *are connected by a sequence of flips.*

Definition 11.2. Given G and J as in Theorem 11.1, we let $G(J)$ denote the subgraph of G consisting of the (closure of the) faces involved in a flip connecting elements of Match_J^G. And if $\lambda \in \mathcal{P}_{k,n}$ is the partition with vertical steps $J(\lambda)$, then we also use $G(\lambda)$ to denote $G(J(\lambda))$.

Note that the elements of Match_J^G can be identified with the perfect matchings of $G(J)$. Our next goal is to relate matchings of G to flows in a perfect orientation of G. The following lemma is easy to check; see Figure 12.

Lemma 11.3. *Let $\mathcal{O}$ be a perfect orientation of a plabic graph G, with source set $I_{\mathcal{O}}$. Let J be a set of boundary vertices with $|J| = |I_{\mathcal{O}}|$. There is a bijection between flows F from $I_{\mathcal{O}}$ to J, and matchings of G with boundary J. In particular, if G has type $\pi_{k,n}$, then $|J| = n-k$. The matching $M(F)$ associated to flow F is defined by*

$$M(F) = \{e \mid e \notin F \text{ and } e \text{ is directed towards its incident white vertex in } \mathcal{O}\} \cup \{e \mid e \in F \text{ and } e \text{ is directed away from its incident white vertex in } \mathcal{O}\}.$$

We write $F(M)$ for the flow corresponding to the matching M.

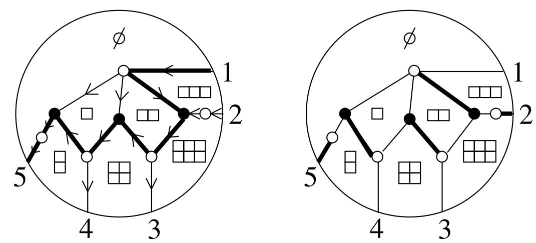


FIGURE 12. A flow F used in the flow polynomial P_{25}^G and the corresponding matching $M(F)$. Here F is the minimal flow for P_{25}^G and $M(F)$ is the minimal matching with boundary $\{2,5\}$.

We now use Theorem 11.1 to show that flow polynomials have strongly minimal and maximal terms. Recall the notations from Section 5.

Corollary 11.4. *Let G, $\mathcal{O}$, and J be as in Lemma 11.3. The flow polynomial $P_J^G = \sum \mathrm{wt}(F)$ has a strongly minimal term m_J^G such that m_J^G divides $\mathrm{wt}(F)$ for all flows F from $I_{\mathcal{O}}$ to J. And it has a strongly maximal term which is divisible by $\mathrm{wt}(F)$ for each flow F from $I_{\mathcal{O}}$ to J. If λ is the partition corresponding to J, we also write m_λ^G instead of m_J^G.*

Proof. A simple case by case analysis shows that if matching M' is obtained from $M(G)$ by flipping face μ up, i.e. $M(F) \lessdot M'$, then the flow $F' := F(M')$ satisfies $\mathrm{wt}(F') = \mathrm{wt}(F)x_\mu$. See Figure 11. The result now follows from Theorem 11.1, where the strongly minimal and maximal terms of P_J^G are the weights of the flows $F(M_J^{min})$ and $F(M_J^{max})$, respectively. □

12. MUTATION OF PLÜCKER COORDINATE VALUATIONS FOR $\mathbb{X}$

In this section we will again restrict our attention to plabic graphs (as opposed to $\mathcal{X}$-clusters), and will use the combinatorics of flow polynomials to describe explicitly how valuations of Plücker coordinates of $\mathbb{X}$ behave under mutation. This will be an important tool in proving Theorem 14.1, which describes all lattice points of Δ_G, when G is a reduced plabic graph of type $\pi_{k,n}$.

Theorem 12.1. *Suppose that G and G' are reduced plabic graphs of type $\pi_{k,n}$, which are related by a single move. If G and G' are related by one of the moves (M2) or (M3), then $\mathcal{P}_G = \mathcal{P}_{G'}$ and the polytopes $\mathrm{Conv}_G(D) \subset \mathbb{R}^{\mathcal{P}_G}$ and $\mathrm{Conv}_{G'}(D) \subset \mathbb{R}^{\mathcal{P}_{G'}}$ are identical. If G and G' are related by the square move (M1), then for any Plücker coordinate P_K of $\mathbb{X}$,*

$$\mathrm{val}_{G'}(P_K) = \Psi_{G,G'}(\mathrm{val}_G(P_K)),$$

for $\Psi_{G,G'} : \mathbb{R}^{\mathcal{P}_G} \to \mathbb{R}^{\mathcal{P}_{G'}}$ the tropicalized cluster mutation *from* (10.9), *and where we have written* $\mathrm{val}_G(P_K)$ *for* $\mathrm{val}_G(P_K/P_{\max})$.

Explicitly, suppose we obtain G from G' by a square move at the face labeled by ν_1 in Figure 5. Then any vertex $(V_{\nu_1}, V_{\nu_2}, \dots, V_{\nu_N})$ of Conv_G, where the ν_i are the ordered elements of $\mathcal{P}_G$, without loss of generality starting from ν_1, transforms to a vertex of $\mathrm{Conv}_{G'}$ by the following piecewise-linear transformation $\Psi_{G,G'}$,

$$\Psi_{G,G'} : (V_{\nu_1}, V_{\nu_2}, \dots, V_{\nu_N}) \mapsto (V_{\nu_1'}, V_{\nu_2}, \dots, V_{\nu_N}), \text{ where} \quad (12.1)$$
$$V_{\nu_1'} = \min(V_{\nu_2} + V_{\nu_4}, V_{\nu_3} + V_{\nu_5}) - V_{\nu_1}.$$

Remark 12.2. We note that a statement analogous to Theorem 12.1 fails already for products $P_K P_J$, because while

$$\mathrm{val}_G(P_K P_J) = \mathrm{val}_G(P_K) + \mathrm{val}_G(P_J),$$

and $\psi_{G,G'}(\mathrm{val}_G(P_I)) = \mathrm{val}_{G'}(P_I)$ for $I = J, K$, by Theorem 12.1, we potentially have

$$\Psi_{G,G'}(\mathrm{val}_G(P_K P_J)) = \Psi_{G,G'}(\mathrm{val}_G(P_K) + \mathrm{val}_G(P_J)) \neq \mathrm{val}_{G'}(P_K) + \mathrm{val}_{G'}(P_J) = \mathrm{val}_{G'}(P_K P_J),$$

since the tropical cluster mutations are not linear.

Remark 12.3. Note that while $\Psi_{G,G'}$ sends the lattice points of Conv_G to the lattice points of $\mathrm{Conv}_{G'}$, it does not in general send the whole polytope Conv_G to the polytope $\mathrm{Conv}_{G'}$. This is again because $\Psi_{G,G'}$ is only piecewise linear. However, the Newton-Okounkov polytope Δ_G, which can be larger than Conv_G in view of Remark 12.2 (recall Section 8) will turn out to behave much better with respect to $\Psi_{G,G'}$.

Proof of Theorem 12.1. By Lemma 5.3 and Remark 5.4, we have an acyclic perfect orientation $\mathcal{O}$ of G whose set of boundary sources is $\{1, 2, \dots, n-k\}$. Therefore if we apply Theorem 5.8, our expression for the Plücker coordinate $P_{\max}$ is 1. Moreover, we have expressions for the other Plücker coordinates $P_K = P_K^G$ as flow polynomials, which are sums over pairwise-disjoint collections of self-avoiding walks in $\mathcal{O}$. The weight of each walk is the product of parameters x_μ, where μ ranges over all face labels to the left of a walk.

It is easy to see that the flow polynomials P_K^G and $P_K^{G'}$ are equal if G and G' differ by one of the moves (M2) or (M3): in either case, there is an obvious bijection between perfect orientations of both graphs involved in the move, and this bijection is weight-preserving.

Now suppose that G and G' differ by a square move. By Lemma 5.3, it suffices to compare perfect orientations $\mathcal{O}$ and $\mathcal{O}'$ of G and G' which differ as in Figure 7. Without loss of generality, G and G' are at the left and right, respectively, of Figure 7. (We should also consider the case that G is at the right and G' is at the left, but the proof in this case is analogous.) Recall that by Corollary 11.4, each flow polynomial P_K has a strongly minimal flow (see Definition 5.13) F_{min}, and hence $\mathrm{val}_G(P_K) = \mathrm{wt}(F_{min})$. The main step of the proof is to prove the following claim about how strongly minimal flows change under an oriented square move.

Claim. *Let G and $\mathcal{O}$ be as above, let K be an $(n-k)$-element subset of $\{1, \dots, n\}$, and let $F_{\min}$ be the strongly minimal flow from $\{1, \dots, n-k\}$ to K.*

(1) *Assuming the orientations in $\mathcal{O}$ locally around the face ν_1 are as shown in the left-hand side of Figure 7, then the restriction of $F_{\min}$ to the neighborhood of face ν_1 is as in the left-hand side of one of the six pictures in Figure 13, say picture I, where $I \in \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}\}$.*
(2) *If we let $F'_{\min}$ denote the flow obtained from $F_{\min}$ by the local transformation indicated in picture I, then $F'_{\min}$ is strongly minimal.*

Let us check (1). In theory, the restriction of $F_{\min}$ to the neighborhood of face ν_1 could be as in the left-hand side of any of the six pictures from Figure 13, *or* it could be as in Figure 14. However, if a flow locally looks like Figure 14, then it cannot be minimal – the single path shown in Figure 14 could be deformed to go around the other side of the face labeled ν_1, and that would result in a smaller weight. More specifically, the weight of a flow which locally looks like Figure 14, when restricted to coordinates $(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$, has valuation $(i+1, i+1, i+1, i, i+1)$, whereas the weight of its deformed version has valuation $(i, i+1, i+1, i, i+1)$, for some nonnegative integer i. This proves the first statement of the claim.

Now let us write $\mathrm{wt}(F_{\min}) = \prod_{\mu \in \mathcal{P}_G} x_\mu^{a_\mu}$, so that $(a_\mu)_{\mu \in \mathcal{P}_G} = \mathrm{val}_G(P_I)$. Suppose that the restriction of $F_{\min}$ to the neighborhood of face ν_1 looks as in picture I of Figure 13. Let $F'_{\min}$ be the flow in G' obtained

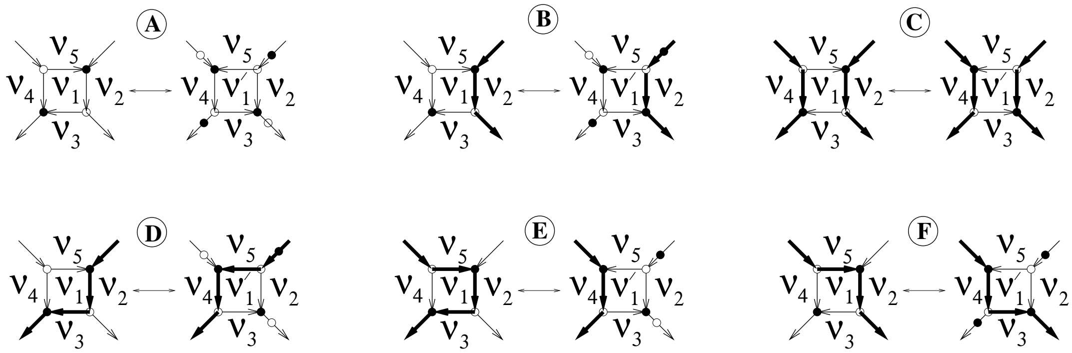


FIGURE 13. How minimal flows change in the neighborhood of face ν_1 as we do an oriented square move. The perfect orientations $\mathcal{O}$ and $\mathcal{O}'$ for G and G' are shown at the left and right of each pair, respectively. Note that in the top row, the flows do not change, but in the bottom row they do. Also note that the picture at the top left indicates the case that the flow is not incident to face ν_1.

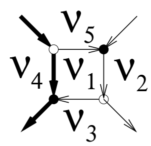


FIGURE 14. A path whose weight is not minimal.

from $F_{\min}$ by the local transformation indicated in picture I, and write $\mathrm{wt}(F'_{\min}) = \prod_{\mu \in \mathcal{P}_{G'}} x^{a'_\mu}$. (Clearly $F'_{\min}$ is indeed a flow in G'.) We need to show that $F'_{\min}$ is strongly minimal.

Let F' be some arbitrary flow in G', and write $(b'_\mu)_{\mu \in \mathcal{P}'_G} = \mathrm{val}_{G'}(\mathrm{wt}(F'))$. We need to show that $a'_\mu \le b'_\mu$ for all $\mu \in \mathcal{P}_{G'}$. We can assume that the restriction of F' to the neighborhood of face ν'_1 looks as in the right hand side of one of the six pictures in Figure 13, say picture J. A priori there is one more case (obtained from the right hand side of picture **B** by deforming the single path to go around ν'_1), but since this increases $b'_{\nu'_1}$, we don't need to consider it. Now let F be the flow in G obtained from F' by the local transformation indicated in picture J, and write $(b_\mu)_{\mu \in \mathcal{P}_G} = \mathrm{val}_G(\mathrm{wt}(F))$.

We already know, by our assumption on G, that $b_\mu \ge a_\mu$ for all $\mu \in \mathcal{P}_G$. Moreover it is clear from Figure 13 that

$$a'_\mu = \begin{cases} a_\mu & \text{if } \mu \ne \nu'_1 \\ a_{\nu_1} \quad \text{or} \quad a_{\nu_1} + 1 & \text{if } \mu = \nu'_1, \end{cases} \quad \text{and} \quad b'_\mu = \begin{cases} b_\mu & \text{if } \mu \ne \nu'_1 \\ b_{\nu_1} \quad \text{or} \quad b_{\nu_1} + 1 & \text{if } \mu = \nu'_1. \end{cases} \tag{12.2}$$

More specifically, $a'_{\nu_1} = a_{\nu_1} + 1$ (respectively, $b'_{\nu_1} = b_{\nu_1} + 1$) precisely when picture I (respectively, picture J) is one of the cases $\mathbf{D}, \mathbf{E}, \mathbf{F}$ from Figure 13.

From the cases above, it follows that $b'_\mu \ge a'_\mu$ for all $\mu \ne \nu'_1$ and $\mu \in \mathcal{P}_{G'}$. We need to check only that $b'_{\nu'_1} \ge a'_{\nu'_1}$. Since $b_{\nu_1} \ge a_{\nu_1}$, the only way to get $b'_{\nu'_1} < a'_{\nu'_1}$ is if $a'_{\nu_1} = a_{\nu_1} + 1$ and $b'_{\nu_1} = b_{\nu_1} = a_{\nu_1}$. In particular then $I \in \{\mathbf{D}, \mathbf{E}, \mathbf{F}\}$ and $J \in \{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$. So we need to show that each of these nine cases is impossible when $b_\mu \ge a_\mu$ and $b_{\nu_1} = a_{\nu_1}$.

Let us set $i = a_{\nu_1} = b_{\nu_1}$. If $I = \mathbf{D}$, then the vector $(a_{\nu_1}, a_{\nu_2}, a_{\nu_3}, a_{\nu_4}, a_{\nu_5})$ has the form $(i, i+1, i+1, i, i)$. If $I = \mathbf{E}$, the vector $(a_{\nu_1}, a_{\nu_2}, a_{\nu_3}, a_{\nu_4}, a_{\nu_5})$ has the form $(i, i+1, i+1, i, i+1)$. And if $I = \mathbf{F}$, the vector $(a_{\nu_1}, a_{\nu_2}, a_{\nu_3}, a_{\nu_4}, a_{\nu_5})$ has the form $(i, i+1, i, i, i+1)$.

Meanwhile, if $J = \mathbf{A}$, then $(b_{\nu_1}, b_{\nu_2}, b_{\nu_3}, b_{\nu_4}, b_{\nu_5}) = (i, i, i, i, i)$. If $J = \mathbf{B}$, then $(b_{\nu_1}, b_{\nu_2}, b_{\nu_3}, b_{\nu_4}, b_{\nu_5}) = (i, i+1, i, i, i)$. And if $J = \mathbf{C}$, then $(b_{\nu_1}, b_{\nu_2}, b_{\nu_3}, b_{\nu_4}, b_{\nu_5}) = (i, i+1, i, i-1, i)$.

In all nine cases, we see that we get a contradiction to the fact that $a_\mu \le b_\mu$ for all μ. To be precise, by looking at cases **A**, **B** and **C** we see that always $b_{\nu_3} = b_{\nu_5} = i$, while for a_μ we always have either $a_{\nu_3} = i+1$ or $a_{\nu_5} = i+1$, looking at **D**, **E** and **F**. This completes the proof of the claim.

Now it remains to check that the tropical cluster relation (12.1) is satisfied for each of the six cases shown in Figure 13. For example, in the top-middle pair shown in Figure 13, we have $a_{\nu_1} = a_{\nu_3} = a_{\nu_4} = a_{\nu_5} = a'_{\nu'_1} = i$, and $a_{\nu_2} = i+1$. Clearly we have $a_{\nu_1} + a'_{\nu'_1} = \min(a_{\nu_2} + a_{\nu_4}, a_{\nu_3} + a_{\nu_5})$. In the top-right pair, we have $a_{\nu_2} = i+2$, $a_{\nu_1} = a_{\nu_3} = a_{\nu_5} = i+1$, $a_{\nu_4} = i$, and $a'_{\nu'_1} = i+1$, which again satisfy (12.1). The other three cases can be similarly checked. This completes the proof of Theorem 12.1. □

13. Plücker coordinate valuations in terms of the rectangles network chart

In this section we work with a very special choice of plabic graph, namely $G = G^{\mathrm{rec}}_{k,n}$. In this case we can show explicitly that the polytopes Δ_G and Γ_G coincide, and that they are unimodularly equivalent to a Gelfand-Tsetlin polytope. Additionally, we will provide an explicit formula for the lattice points.

13.1. The polytope Conv_G for $G = G^{\mathrm{rec}}_{k,n}$. Recall that $\mathrm{Conv}_G := \mathrm{ConvexHull}(\mathrm{val}_G(L_1))$, see Definition 7.6, where L_1 is the span of the $P_\lambda/P_{\max}$. We now focus on computing the valuations $\mathrm{val}_G(P_\lambda)$ of the $P_\lambda/P_{\max}$; an explicit formula will be given in Proposition 13.4. Throughout this section we fix $G = G^{\mathrm{rec}}_{k,n}$.

Definition 13.1. We define a *GT tableau* to be a rectangular array of integers $\{V_{i\times j}\}$ (where $i \times j$ ranges over the nonempty rectangles contained in $\mathcal{P}_{k,n}$), which satisfy the following properties:

(1) Entries in the top row and leftmost column are at most 1.
(2) $V_{i\times j} \le V_{(i-1)\times(j-1)} + 1$.
(3) $V_{1\times 1} \ge 0$.
(4) Entries weakly increase from left to right in the rows, and from top to bottom in the columns.
(5) If $V_{i\times j} > 0$, then $V_{(i+1)\times(j+1)} = V_{i\times j} + 1$.

See Figure 15.

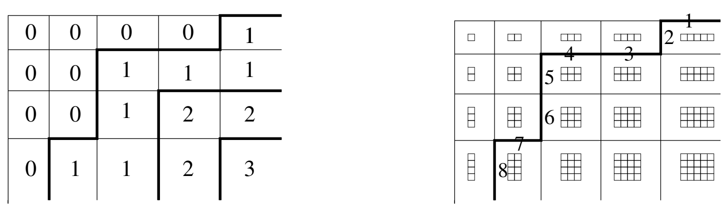


Figure 15. At the left: a GT tableau. At the right: the bijection showing how this tableau is identified with $\mathrm{val}_G(P_{2568})$.

Lemma 13.2. *Let $G = G^{\mathrm{rec}}_{k,n}$, and choose the perfect orientation $\mathcal{O}^{\mathrm{rec}}$ of $G^{\mathrm{rec}}_{k,n}$ shown in Figure 16. Then the flow polynomial $P^G_{\max} = P^G_{\{1,\dots,n-k\}}$ equals 1. The points $\mathrm{val}_G(P_J)$ for $J \in \binom{[n]}{n-k}$ can be encoded as, and are in bijection with GT tableaux. The bijection is given by partitioning the entries of a GT tableau according to their values using lattice paths, and then reading off J from the vertical labels of the northwest-most lattice path. In particular, for $J \ne J'$, $\mathrm{val}_G(P_J)$ and $\mathrm{val}_G(P_{J'})$ are distinct.*

Figure 15 shows the GT tableau associated to P_{2568}.

Proof. Since the source set is $I_{\mathcal{O}^{\mathrm{rec}}} = \{1, 2, \dots, n-k\}$, and the perfect orientation is acyclic, it follows that the flow polynomial $P^G_{\max}$ equals 1. Choose an arbitrary total order on the parameters $x_\mu \in \mathcal{X}\mathrm{Coord}_{\mathbb{X}}(G)$.

Recall that each flow polynomial P^G_J (which can be identified with a Plücker coordinate) is a sum over flows from $I_{\mathcal{O}^{\mathrm{rec}}} = \{1, 2, \dots, n-k\}$ to J. Since $\mathcal{O}^{\mathrm{rec}}$ is acyclic, each flow is just a collection of pairwise vertex-disjoint walks from $\{1, 2, \dots, n-k\} \setminus J$ to $J \setminus \{1, 2, \dots, n-k\}$ in $\mathcal{O}^{\mathrm{rec}}$. Note that if we write $\{1, 2, \dots, n-k\} \setminus J = \{i_1 > i_2 > \dots > i_\ell\}$ and write $J \setminus \{1, 2, \dots, n-k\} = \{j_1 < j_2 < \dots < j_\ell\}$, then any such

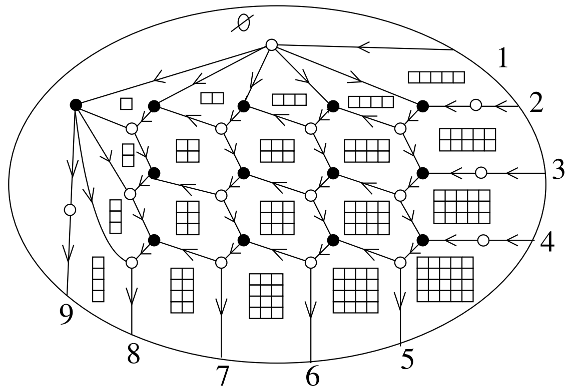


FIGURE 16. A perfect orientation $\mathcal{O}^{\text{rec}}$ of the reduced plabic graph $G^{\text{rec}}_{5,9}$. Note that the source set $I_{\mathcal{O}^{\text{rec}}} = \{1,2,3,4\}$. There is an obvious generalization of $\mathcal{O}^{\text{rec}}$ to any $G^{\text{rec}}_{k,n}$, which has source set $\{1,2,\dots,n-k\}$.

flow must consist of ℓ paths which connect i_1 to j_1, i_2 to j_2, …, and i_ℓ to j_ℓ. For example, in Figure 16, any flow used to compute $P_{\{2,5,6,8\}}$ must consist of three paths which connect 4 to 5, 3 to 6, and 1 to 8.

Recall that the weight $\text{wt}(q)$ of a path q is the product of the parameters x_μ where μ ranges over all face labels to the left of the path. Because of how the faces of $G^{\text{rec}}_{k,n}$ are arranged in a grid, we can define a partial order on the set of all paths from a given boundary source i to a given boundary sink j, with $q_1 \le q_2$ if and only if $\text{wt}(q_1) \le \text{wt}(q_2)$. In particular, among such paths, there is a unique *minimal* path, which "hugs" the southeast border of $G^{\text{rec}}_{k,n}$.

It is now clear that the strongly minimal flow F_J (whose existence is asserted by Corollary 11.4) from $\{1,2,\dots,n-k\} \smallsetminus J$ to $J \smallsetminus \{1,2,\dots,n-k\}$ is obtained by:

- choosing the minimal path q_1 in $\mathcal{O}^{\text{rec}}$ from i_1 to j_1;
- choosing the minimal path q_2 in $\mathcal{O}^{\text{rec}}$ from i_2 to j_2 which is vertex-disjoint from q_1;
- …
- choosing the minimal path q_ℓ in $\mathcal{O}^{\text{rec}}$ from i_ℓ to j_ℓ which is vertex-disjoint from $q_{\ell-1}$.

For example, when $J = \{2,5,6,8\}$, the strongly minimal flow F_J associated to J is shown at the left of Figure 17. At the right of Figure 17 we've re-drawn the plabic graph to emphasize the grid structure; this makes the structure of a strongly minimal flow even more transparent.

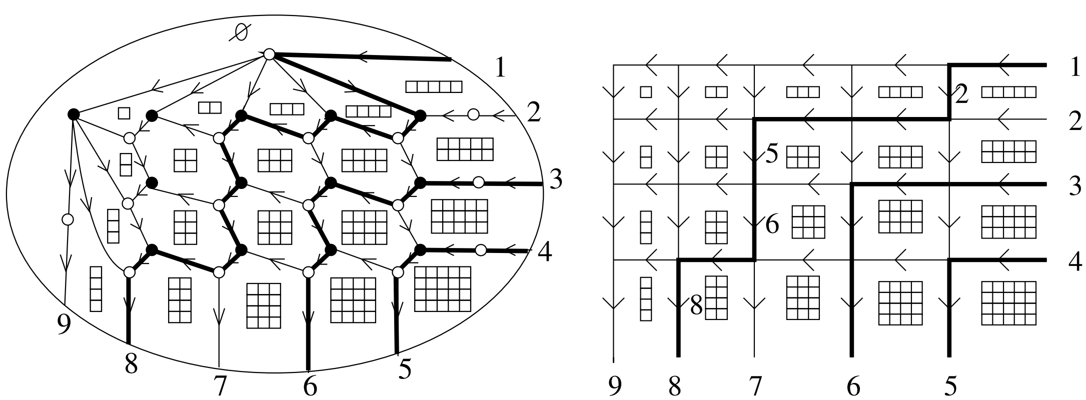


FIGURE 17. The strongly minimal flow associated to $J = \{2,5,6,8\}$. The associated GT tableau encoding the valuation is shown in Figure 15.

The exponent vector $\mathrm{val}_G(\mathrm{wt}(F_J)) = \mathrm{val}_G(P_J)$ of the weight of F_J shown in Figure 17 is depicted by the "tableau" in Figure 15. By inspection it is clear that these exponent vectors can be precisely encoded as GT tableaux, and are in bijection with them. Moreover, if one labels the steps of the northwest-most lattice path in F_J by the numbers from 1 to n, then there is a correspondence between the labels of the vertical steps and the destination set of the flow (namely, J), see Figure 17. In particular, the vertical step labeled j can be connected to the edge of the grid incident to $j \in J$ by a line of slope -1. □

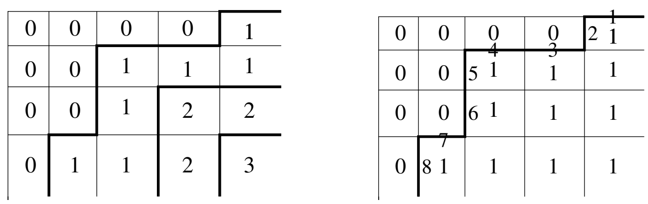


FIGURE 18. The "tableau", or exponent vector associated to the strongly minimal flow from Figure 17, along with the corresponding Gelfand-Tsetlin pattern.

Definition 13.3. Given two partitions λ and μ in $\mathcal{P}_{k,n}$, we let $\mu \smallsetminus \lambda$ denote the set of boxes remaining if we justify both μ and λ at the top-left of a $(n-k) \times k$ rectangle, then remove from μ any boxes that are in λ. We let $\mathrm{MaxDiag}(\mu \smallsetminus \lambda)$ denote the maximum number of boxes in $\mu \smallsetminus \lambda$ that lie along any diagonal (with slope -1) of the rectangle.

Proposition 13.4. *Let $G = G^{\mathrm{rec}}_{k,n}$ be the plabic graph defined in Section 7.1 (see Figure 8). Then*

$$\mathrm{val}_G(P_\lambda)_{i\times j} = \mathrm{MaxDiag}(i \times j \smallsetminus \lambda).$$

Before proving Proposition 13.4, we make several simple observations about the relationships between the faces of $G^{\mathrm{rec}}_{k,n}$, partitions, and strongly minimal flows.

Remark 13.5. Consider an $(n-k)$ by k rectangle R, with boxes labeled by rectangular Young diagrams as in the right of Figure 17 (for $k = 5$ and $n = 9$). Then if we place an i by j rectangle justified to the northwest of R, the region in its southeast corner will be labeled by the Young diagram $i \times j$.

Remark 13.6. Let $I \mapsto \lambda(I)$ denote the bijection from Section 2.3 between $(n-k)$-element subsets of $[n]$ and elements of $\mathcal{P}_{k,n}$. Then the topmost path in the strongly minimal flow for P_I cuts out the southeast border of $\lambda(I)$. For example, the right hand side of Figure 17 shows the strongly minimal flow for $P_{\{2,5,6,8\}}$. Note that the topmost path in the flow cuts out the partition $(4,2,2,1)$, which is the partition associated to $\{2,5,6,8\}$. This observation is already implicit in the proof of Lemma 13.2. Namely if one starts by labeling the vertical steps of the partition cut out by the topmost path in the strongly minimal flow (as is done in Figure 17) and then propagates each label southeast as far as possible, each label will end up on an edge incident to some destination $i \in I$ for the flow P_I.

Remark 13.7. Given a partition $\lambda \in \mathcal{P}_{k,n}$, let λ^c denote the Young diagram which is the complement of λ in the $(n-k)$ by k rectangle rotated by 180°. For $\check{\mathbb{X}}$ with its analogous associated network, and any $J \subset \binom{[n]}{k}$ interpreted as set of west steps of a partition $\mu = \mu(J)$, we have the following version of Remark 13.6. The topmost path in the strongly minimal flow for p_J cuts out the southeast border of the transpose of $\mu(J)^c$. See the right hand side of Figure 20 for an example.

Proof of Proposition 13.4. To compute $\mathrm{val}_G(P_\lambda)$ we use the strongly minimal flow for P_λ in $G = G^{\mathrm{rec}}_{k,n}$, which by Remark 13.6 cuts out the partition λ, see Figure 19. To compute the $i \times j$ component in $\mathrm{val}_G(P_\lambda)$, we need to compute the number of paths of the flow that are above the box b which is labeled by the partition $i \times j$. By Remark 13.5, this box is the southeast-most box in the i by j rectangle indicated in Figure 19. But now it is clear that the number of paths we are trying to compute is precisely $\mathrm{MaxDiag}(i \times j \smallsetminus \lambda)$. □

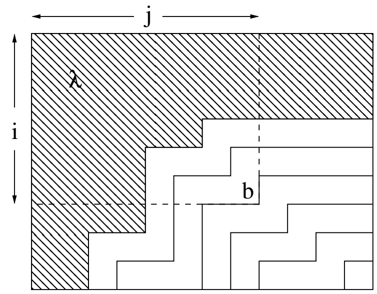


FIGURE 19

14. A Young diagram formula for Plücker coordinate valuations

In this section we prove the general Theorem 14.1, which gives an explicit formula for all leading terms of flow polynomials P_λ^G, that is, the valuations $\mathrm{val}_G(P_\lambda)$, when G is a reduced plabic graph of type $\pi_{k,n}$. We then use Theorem 14.1 to give explicit formulas for Plücker coordinates corresponding to frozen variables, see Section 14.2. Comparing with a result of Fulton and Woodward [FW04] (which was refined in the Grassmannian setting by [Pos05]) we find that the right-hand side of our formula has an interpretation in terms of the quantum multiplication in the quantum cohomology of the Grassmannian.

14.1. Valuations of Plücker coordinates.

Theorem 14.1. *Let G be any reduced plabic graph of type $\pi_{k,n}$ and $\lambda \in \mathcal{P}_{k,n}$. For any partition $\mu \in \mathcal{P}_G$,*

$$\mathrm{val}_G(P_\lambda)_\mu = \mathrm{MaxDiag}(\mu \smallsetminus \lambda),$$

where $\mathrm{MaxDiag}(\mu \smallsetminus \lambda)$ *is as in Definition 13.3.*

Remark 14.2. By [FW04], $\mathrm{MaxDiag}(\mu \smallsetminus \lambda)$ is equal to the smallest degree d such that q^d appears in the quantum product of two Schubert classes $\sigma_\mu \star \sigma_{\lambda^c}$ in the quantum cohomology ring $QH^*(\mathbb{X})$, when this product is expanded in the Schubert basis. See also [Yon03] and [Pos05]. Here σ_{λ^c} is the Poincaré dual Schubert class to σ_λ, compare Remark 13.7.

Note that Proposition 13.4 is precisely Theorem 14.1 in the special case of the rectangles cluster. We prove the theorem in general by explicitly constructing an element in $\mathrm{Trop}(\check{\mathbb{X}})$, which we think of as associated to P_λ by mirror symmetry.

Theorem 14.3. *Fix $\lambda \in \mathcal{P}_{k,n}$. There exists an element $x_\lambda(t) \in \check{\mathbb{X}}^\circ(\mathbf{K}_{>0})$ such that for any partition μ,*

$$\mathrm{Val}_{\mathbf{K}}(p_\mu(x_\lambda(t)) = \mathrm{MaxDiag}(\mu \smallsetminus \lambda).$$

Definition 14.4. To define the element $x_\lambda(t) \in \check{\mathbb{X}}^\circ(\mathbf{K}_{>0})$, we use the network parametrization - on the $\check{\mathbb{X}}$ side - associated to the grid shown at the left of Figure 20. All edges are directed left and down, but there are now k rows and $n-k$ columns in the grid. We make specific choices for network parameters labeling the regions, as follows. We rotate and reflect λ and place it in the southeast corner of the grid; then the boxes immediately northwest of inner and outer corners of λ are filled with t and t^{-1}, respectively. All other boxes receive the parameter 1. This gives rise to an element $x_\lambda(t) \in \check{\mathbb{X}}^\circ(\mathbf{K}_{>0})$, whose Plücker coordinates are computed as sums over flows, as in Theorem 5.8.

Remark 14.5. The element $x_\lambda(t)$ determines a zone or integral point $[x_\lambda(t)]$ in $\mathrm{Trop}(\check{\mathbb{X}})$, see Definition 10.6. Indeed $x_\lambda(t)$ is constructed to lie in $\check{\mathbb{X}}(\mathbf{L}_{>0})$.

Example 14.6. The right hand side of Figure 20 shows the strongly minimal flow for $p_{\{1,5,6,7,12,13\}}(x_\lambda(t)) = p_\mu(x_\lambda(t))$, where $\lambda = (4,3,3,3,2,1)$ and $\mu = (5,5,5,2,2,2,2)$. See Remark 13.7. The flow has weight t^3, because the path from 2 to 13 has weight t^2, and the path from 3 to 12 has weight t, and all other paths have weight 1. Meanwhile, the right hand side of Figure 21 shows another flow for $p_\mu(x_\lambda(t))$, which has weight t^2. This corresponds to the fact that $\mathrm{MaxDiag}(\mu \smallsetminus \lambda) = 2$.

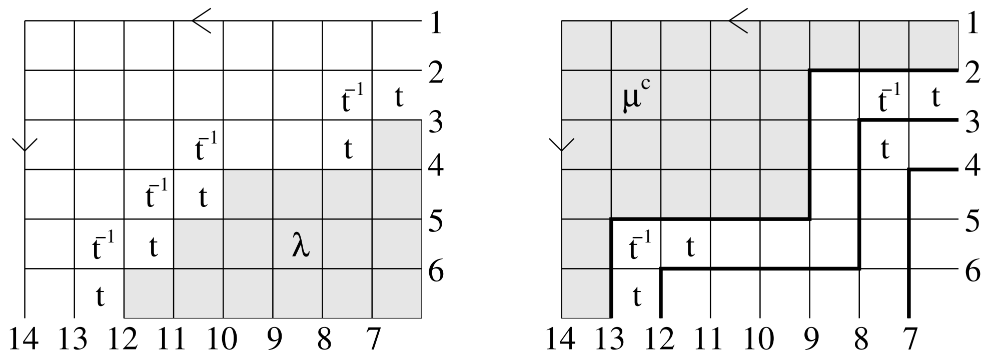


FIGURE 20. The picture on the left shows the rotated, reflected partition λ and the network coordinates used to define $x_\lambda(t)$, in the case that $k = 6$, $n = 14$, and $\lambda = (4,3,3,3,2,1)$. The picture on the right shows the strongly minimal flow for $p_{\{1,5,6,7,12,13\}}(x_\lambda(t)) = p_\mu(x_\lambda(t))$, where $\mu = (5,5,5,2,2,2,2)$. The shaded region labeled μ^c is actually the transpose of $\mu^c = (6,4,4,4,4,1,1,1)$. We suppress this in our notation.

Proof of Theorem 14.3. The strongly minimal flow F^0 contributing to $p_\mu(x_\lambda(t))$ is the flow shown in Figure 20, whose topmost path coincides with the southeast border of (the transpose of) μ^c. So the (reflected and rotated) partition μ consists of the boxes which are southeast of the topmost path of the flow. All other flows contributing to $p_\mu(x_\lambda(t))$ have the same starting and ending points as F^0 but now the paths are arbitrary pairwise non-intersecting paths consisting of west and south steps.

Let us call a path in the network *rectangular* if it consists of a series of west steps followed by south steps. Note that by construction, the weight of any path in the network associated to $x_\lambda(t)$ will be t^ℓ for some $\ell \geq 0$. Note that if a given path p from i to j encloses a box with t or t^{-1}, then any path p' from i to j which is weakly above p will have weight t^ℓ for $\ell \geq 1$. Moreover, the rectangular path from i to j will have weight precisely t.

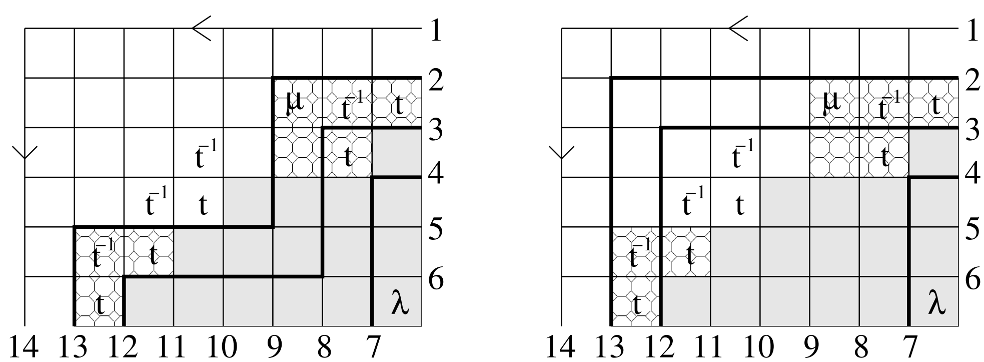


FIGURE 21. At the left we have the network parametrization used to define $x_\lambda(t)$, together with the strongly minimal flow F^0 associated to p_μ. The flow F at the right is obtained from F^0 by replacing the paths from 2 to 13 and from 3 to 12 by the corresponding rectangular paths.

Note that $\mathrm{Val}_{\mathbf{K}}(p_\mu(x_\lambda(t))) = \mathrm{Val}_{\mathbf{K}}(\mathrm{wt}(F))$, where F is the flow associated to p_μ whose weight is t^ℓ for ℓ as small as possible. By the observations of the previous paragraph, we can construct the desired flow F from the strongly minimal flow F^0 by replacing each path from i to j whose weight is *not* 1 by the rectangular path from i to j, see Figure 21. Then $\mathrm{wt}(F) = t^\ell$, where ℓ is the number of paths p in F^0 such that $\mathrm{wt}(p) \neq 1$. But the paths in F^0 with weight not equal to 1 are precisely the paths which enclose at least one box with t or t^{-1}. So $\mathrm{Val}_{\mathbf{K}}(p_\mu(x_\lambda(t)) = \ell$, where ℓ is the number of paths in F^0 which enclose at least one box with t or t^{-1}. It is not hard to see that this number is equal to $\mathrm{MaxDiag}(\mu \setminus \lambda)$. □

Proof of Theorem 14.1 . We want to show that for any reduced plabic graph G and any λ and μ,

$$\mathrm{val}_G(P_\lambda)_\mu = \mathrm{MaxDiag}(\mu \setminus \lambda). \tag{14.1}$$

By Proposition 13.4, we know that (14.1) is true when $G = G^{\mathrm{rec}}_{k,n}$ and μ is a rectangle. Combining this with Theorem 14.3, we obtain that if $G = G^{\mathrm{rec}}_{k,n}$, then

$$(\mathrm{val}_G(P_\lambda)_\mu)_{\mu \in \mathcal{P}_G} = (\mathrm{Val}_{\mathbf{K}}(p_\mu(x_\lambda(t))))_{\mu \in \mathcal{P}_G}. \tag{14.2}$$

But now if we apply a move to G, obtaining another plabic graph G', then Lemma 10.7 implies that the right-hand side of (14.2) transforms via the map $\Psi_{G,G'}$, while Theorem 12.1 implies that the left-hand side of (14.2) transforms via the map $\Psi_{G,G'}$. Therefore (14.2) holds for all plabic graphs G and all partitions $\mu \in \mathcal{P}_G$. Theorem 14.1 now follows from (14.2) and Theorem 14.3. □

14.2. Flow polynomials for frozen Plücker coordinates. In this section we describe the Plücker coordinates P_{μ_i} corresponding to the frozen vertices of our quivers.

Definition 14.7. Let $Q = (Q_0, Q_1)$ be an arbitrary quiver with no loops or 2-cycles, where Q_0 denotes the set of vertices of Q and Q_1 the set of arrows. Given $v \in \mathbb{Z}^{Q_0}$ and mutable vertex ν, we define the quantity

$$\mathring{v}_\nu := \sum_{\mu \to \nu} v_\mu - \sum_{v \leftarrow \mu'} v_{\mu'}, \tag{14.3}$$

where the summands correspond to arrows in Q to and from the vertex ν, respectively. We say that ν is *balanced* with respect to the pair (v, Q) if $\mathring{v}_\nu = 0$.

Lemma 14.8 below follows directly from the $\mathcal{X}$-cluster mutation formula (5.4).

Lemma 14.8. *Let $Q = (Q_0, Q_1)$ be a quiver as above, with $v \in \mathbb{Z}^{Q_0}$ and corresponding monomial x^v. Then the $\mathcal{X}$-mutation $\mathrm{Mut}^{\mathcal{X}}_\nu(x^v)$ at the vertex ν is a monomial if and only if $\mathring{v}_\nu = 0$. Moreover if it is a monomial then its new exponent vector v' is given by the (linear) formula*

$$v'_\eta = \begin{cases} (\sum_{\mu \to \nu} v_\mu) - v_\nu, & \eta = \nu, \\ v_\eta, & \eta \neq \nu. \end{cases} \tag{14.4}$$

□

Proposition 14.9. *Let G be an arbitrary $\mathcal{X}$-cluster seed of type $\pi_{k,n}$, with quiver $Q = Q(G)$ and set of vertices $\mathcal{P}_G$. Choose $j \in \{0, 1, \dots, n-1\}$. Then we have the following.*

1. *P_{μ_j} is a Laurent monomial when written in terms of the $\mathcal{X}$-seed G, i.e. $P_{\mu_j} = x^v$ for some $v \in \mathbb{Z}^{\mathcal{P}_G}$.*
2. *$\mathring{v}_\nu = 0$ for all mutable vertices ν in $\mathcal{P}_G$.*
3. *If G' is obtained from G by mutation at a mutable vertex ν, then when P_{μ_j} is written (as a Laurent monomial) in terms of the $\mathcal{X}$-seed G', its new exponent vector v' is obtained from v by (14.4).*

Proof. By Proposition 6.5, any $\mathcal{X}$-torus embeds into $\mathbb{X}^\circ$. Since $P_{\mu_i}/P_{\max}$ is regular on $\mathbb{X}^\circ$ it expands as a Laurent polynomial in terms of $\mathcal{X}$-cluster coordinates $\mathcal{X}\mathrm{Coord}_{\mathbb{X}}(G)$. Since $P_{\mu_i}/P_{\max}$ is nonvanishing by definition of $\mathbb{X}^\circ$ it follows that it must be given by a single Laurent monomial. Properties (2) and (3) follow from Lemma 14.8. □

Remark 14.10. While we used the embedding of $\mathcal{X}$ into $\mathbb{X}^\circ$ to give a quick proof that the frozen variables are Laurent monomials in any $\mathcal{X}$-torus, the same follows from a general result which we learned from Akhtar, which holds in any $\mathcal{X}$-cluster algebra constructed out of a quiver with no loops or 2-cycles. Namely Proposition 14.11 is a reformulation of [Akh17, Proposition 4.8] .

Proposition 14.11. *If x^v is a monomial on an $\mathcal{X}$-cluster torus such that v is balanced, i.e. $\mathring{v}_\mu = 0$ for all mutable vertices μ, then x^v stays monomial with balanced exponent vector under any sequence $\mathcal{X}$-mutations.*

15. The proof that $\Delta_G = \Gamma_G$

In this section we mostly work in the setting of arbitrary $\mathcal{X}$- and $\mathcal{A}$-seeds of type $\pi_{k,n}$. Recall that associated to any reduced plabic graph G of type $\pi_{k,n}$, we have both an $\mathcal{X}$-seed $(Q(G), \widetilde{\mathcal{X}\mathrm{Coord}}_{\mathbb{X}}(G))$ which determines a torus in $\mathbb{X}^\circ$, and an $\mathcal{A}$-seed $(Q(G), \widetilde{\mathcal{A}\mathrm{Coord}}_{\check{\mathbb{X}}}(G))$ which determines a torus in $\check{\mathbb{X}}^\circ$. And more generally, for any quiver mutation equivalent to $Q(G)$, we have an associated $\mathcal{X}$-seed and $\mathcal{A}$-seed and associated tori, which we continue to index by a letter G. Our main result is that for any choice of G, the Newton-Okounkov body Δ_G (which is defined in terms of the $\mathcal{X}$-seed associated to G) is equal to the superpotential polytope Γ_G (which is defined in terms of the $\mathcal{A}$-seed associated to G). Our proof starts by verifying this fact for $G = G^{\mathrm{rec}}_{k,n}$, proving along the way that in this case, Γ_G is isomorphic to a Gelfand-Tsetlin polytope (via a unimodular transformation). From this we deduce various properties of Γ_G including that $\Gamma_G = \Delta_G$ in the case where Γ_G is a lattice polytope. We then use the *theta function basis* of Gross, Hacking, Keel, and Kontsevich [GHKK14], as well as Corollary 10.16, which describes how the polytopes Γ^r_G mutate, to deduce that $\Gamma_G = \Delta_G$ in general and complete the proof.

15.1. The rectangles cluster, Gelfand-Tsetlin polytopes, and the integral case. In 1950 Gelfand and Tsetlin [GT50] introduced integral polytopes GT_ω associated to arbitrary dominant weights ω of GL_n, such that the lattice points of GT_ω parametrize a basis of the representation V_ω (the Gelfand-Tsetlin basis) and such that $GT_{r\omega} = rGT_\omega$. If $\omega = \omega_{n-r}$ this construction gives a polytope with $\binom{n}{k}$ lattice points, such that the number of lattice points in its r-th dilation agrees with the dimension of the irreducible representation $V_{r\omega_{n-k}}$. This number also equals the dimension of L_r (see (7.4)), and the degree r component of the homogeneous coordinate ring of $\mathbb{X}$. We start by explaining how the polytope $\Gamma^r_{G^{\mathrm{rec}}_{k,n}}$ is isomorphic to a Gelfand-Tsetlin polytope $GT_{\omega_{n-k}}$ via a unimodular transformation.

Definition 15.1 (Gelfand-Tsetlin polytope)**.** Let $GT_{r\omega_{n-k}} \subset \mathbb{R}^{\mathcal{P}_{G^{\mathrm{rec}}_{k,n}}}$ denote the polytope defined by

$$f_{i\times j} - f_{(i-1)\times j} \geq 0 \tag{15.1}$$
$$f_{i\times j} - f_{i\times(j-1)} \geq 0 \tag{15.2}$$
$$f_{1\times 1} \geq 0 \tag{15.3}$$
$$-f_{(n-k)\times k} \geq -r, \tag{15.4}$$

where the defining variables $f_{i\times j}$ range over all nonempty rectangles $i\times j$ contained in a $(n-k)\times k$ rectangle. This polytope is called the *Gelfand-Tsetlin polytope* for highest weight $r\omega_{n-k}$.

One often expresses Gelfand-Tsetlin polytopes in terms of *Gelfand-Tsetlin patterns*, triangular arrays of real numbers whose top row is fixed and whose rows interlace. Clearly $GT_{r\omega_{n-k}}$ is the set of all such Gelfand-Tsetlin patterns with top row $(0^k, r^{n-k})$. See Figure 22 for the example with $k = 3$ and $n = 5$. When $r = 1$ the polytope $GT_{\omega_{n-k}}$ has integer vertices, one for each Young diagram in $\mathcal{P}_{k,n}$.

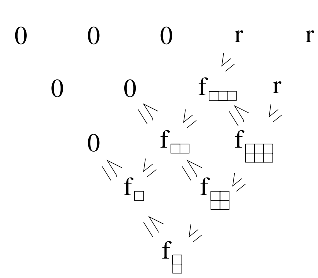


Figure 22. Gelfand-Tsetlin patterns for $GT_{r\omega_{n-k}}$ with $k = 3$ and $n = 5$. The convex hull of all such patterns is the polytope $GT_{r\omega_{n-k}}$.

The following lemma describes explicitly an isomorphism between the polytope $\Gamma^r_{G^{\mathrm{rec}}_{k,n}}$ and the Gelfand-Tsetlin polytope $GT_{r\omega_{n-k}}$. If one compares Figures 10 and 22, the isomorphism becomes quite transparent. An analogous transformation comes up in [AB04, Section 5.1].

Lemma 15.2. *The map* $F:\mathbb{R}^{\mathcal{P}_{G^{\mathrm{rec}}_{k,n}}}\to\mathbb{R}^{\mathcal{P}_{G^{\mathrm{rec}}_{k,n}}}$ *defined by*

$$(v_{i\times j})\mapsto(f_{i\times j})=(v_{i\times j}-v_{(i-1)\times(j-1)})$$

is a unimodular linear transformation, with inverse given by $v_{i\times j}=f_{i\times j}+f_{(i-1)\times(j-1)}+f_{(i-2)\times(j-2)}+\dots$. *Moreover,* $F(\Gamma^r_{G^{\mathrm{rec}}_{k,n}})=GT_{r\omega_{n-k}}$. *Therefore the polytope* $\Gamma^r_{G^{\mathrm{rec}}_{k,n}}$ *is isomorphic to the Gelfand-Tsetlin polytope* $GT_{r\omega_{n-k}}$ *by a unimodular linear transformation, and in particular has integer vertices.*

Proof. Using the formula (9.3) for the superpotential, we obtain the following inequalities defining $\Gamma_{G^{\mathrm{rec}}_{k,n}}$:

$$0\le v_{1\times 1} \tag{15.5}$$

$$v_{(n-k)\times k}-v_{(n-k-1)\times(k-1)}\le 1 \tag{15.6}$$

$$v_{(i-1)\times j}-v_{(i-2)\times(j-1)}\le v_{i\times j}-v_{(i-1)\times(j-1)}\quad\text{for } 2\le i\le n-k \text{ and } 1\le j\le k \tag{15.7}$$

$$v_{i\times(j-1)}-v_{(i-1)\times(j-2)}\le v_{i\times j}-v_{(i-1)\times(j-1)}\quad\text{for } 1\le i\le n-k \text{ and } 2\le j\le k \tag{15.8}$$

If we rewrite the inequalities (15.5) through (15.8) in terms of f-variables, we obtain the system of inequalities given by (15.1), (15.2), (15.3), and (15.4) which define the Gelfand-Tsetlin polytope $GT_{r\omega_{n-k}}$. □

Definition 15.3 (Integer decomposition property)**.** A polytope P is said to have the *integer decomposition property* (IDP), or be *integrally closed*, if every lattice point in the rth dilation rP of P is a sum of r lattice points in P, that is, $\mathrm{Lattice}(rP)=r\,\mathrm{Lattice}(P)$.

Lemma 15.4. *The polytopes* $GT_{\omega_{n-k}}$ *and* $\Gamma_{G^{\mathrm{rec}}_{k,n}}$ *have the integer decomposition property.*

Proof. This is well-known for $GT_{\omega_{n-k}}$ and can also be proved explicitly by an inductive argument on integral Gelfand-Tsetlin patterns. The result for $\Gamma_{G^{\mathrm{rec}}_{k,n}}$ now follows from Lemma 15.2. □

Proposition 15.5. *When* $G=G^{\mathrm{rec}}_{k,n}$*, the polytopes* Conv_G *and* Γ_G *coincide, and the lattice points of* $\mathrm{Conv}_G=\Gamma_G$ *are precisely the* $\binom{n}{k}$ *points* $\mathrm{val}_G(P_\lambda)$ *for* $\lambda\in\mathcal{P}_{k,n}$.

Proof. We write G for $G^{\mathrm{rec}}_{k,n}$. By definition, Conv_G is the convex hull of $\{\mathrm{val}_G(P_\lambda)\mid\lambda\in\mathcal{P}_{k,n}\}$. By Lemma 13.2, we have that for $\lambda\ne\lambda'$, $\mathrm{val}_G(P_\lambda)$ and $\mathrm{val}_G(P_{\lambda'})$ are distinct, so Conv_G contains at least $\binom{n}{k}$ lattice points.

We next show that $\mathrm{Conv}_G\subseteq\Gamma_G$. By Lemma 13.2, each point $\mathrm{val}_G(P_\lambda)$ can be encoded by a GT tableau T. If we apply the map F from Lemma 15.2 to T, it gets transformed into an $(n-k)\times k$ array of 0's and 1's with rows and columns weakly increasing. For example, Figure 18 shows both the tableau from Figure 15 (in "v-variables") and also its image under the map F (in "f-variables"). Therefore $F(T)$ is an integral Gelfand-Tsetlin pattern in $GT_{\omega_{n-k}}$, see Figure 22, and hence by Lemma 15.2, $T\in\Gamma_G$. This shows that $\mathrm{Conv}_G\subseteq\Gamma_G$.

By Lemma 15.2, the polytope Γ^r_G is an integral polytope with precisely $\dim V_{r\omega_{n-k}}$ lattice points. In particular, Γ_G is integral with precisely $\binom{n}{k}$ lattice points. It follows that $\mathrm{Conv}_G=\Gamma_G$, and the lattice points of $\mathrm{Conv}_G=\Gamma_G$ are precisely the $\binom{n}{k}$ points $\mathrm{val}_G(P_\lambda)$ for $\lambda\in\mathcal{P}_{k,n}$. □

Proposition 15.6. *For an arbitrary seed, indexed by G, the number of lattice points of the superpotential polytope* Γ^r_G *coincides with the dimension* $\dim V_{r\omega_{n-k}}$.

Proof. If $G=G^{\mathrm{rec}}_{k,n}$ then the statement follows from the analogous property of the Gelfand-Tsetlin polytope, because of Lemma 15.2. By Corollary 10.17, this cardinality is independent of G. □

Corollary 15.7. *For an arbitrary seed* $\check{\Sigma}^{\mathcal{A}}_G$*, the volume of the superpotential polytope* Γ_G *is given by*

$$\prod_{1\le i\le k}\frac{(k-i)!}{(n-i)!} \tag{15.9}$$

Proof. The Hilbert polynomial $h_{\mathbb{X}}(r)$ of the Grassmannian $\mathbb{X}$ in its Plücker embedding satisfies $h_{\mathbb{X}}(r) = \dim V_{r\omega_{n-k}}$ for $r >> 0$. And moreover the leading coefficient of $h_{\mathbb{X}}(r)$ is $\prod_{1\le i\le k} \frac{(k-i)!}{(n-i)!}$ [GW11]. But Proposition 15.6 implies that $\dim V_{r\omega_{n-k}}$ equals the number of lattice points in the r-th dilation of Γ_G, which implies that the Ehrhart polynomial of Γ_G equals $h_{\mathbb{X}}(r)$. Since the leading coefficient of the Ehrhart polynomial equals the volume of the corresponding polytope, the corollary follows. □

Corollary 15.8. *For arbitrary G, the superpotential polytope Γ_G and the Newton-Okounkov body Δ_G have the same volume.*

Proof. Using Proposition 15.6, it follows that the volume of Γ_G equals

$$\text{Volume}(\Gamma_G) = \lim_{r\to\infty} \frac{\dim V_{r\omega_{n-k}}}{r^{\dim(\mathbb{X})}}.$$

Meanwhile, it is a fundamental property of Newton-Okounkov bodies (associated to valuations with one-dimensional leaves) that their volume encodes the asymptotic dimension of the space of sections $H^0(\mathbb{X}, \mathcal{O}(rD))$ as $r \to \infty$. Explicitly we have by [LM09, Proposition 2.1] that

$$\text{Volume}(\Delta_G) = \limsup_{r\to\infty} \frac{\dim H^0(\mathbb{X}, \mathcal{O}(rD))}{r^{\dim(\mathbb{X})}}.$$

Since $H^0(\mathbb{X}, \mathcal{O}(rD))$ is isomorphic to the representation $V_{r\omega_{n-k}}$, the result follows. □

Corollary 15.9. *Suppose G is a reduced plabic graph of type $\pi_{k,n}$. The $\binom{n}{k}$ lattice points in Γ_G are precisely the valuations $\text{val}_G(P_\lambda)$ of Plücker coordinates.*

Proof. If G is the rectangles plabic graph this is the contents of Proposition 15.5. If we mutate the plabic graph G to another plabic graph G' by a square move, then the tropicalised cluster mutation transforms $\text{val}_G(P_\lambda)$ to $\text{val}_{G'}(P_\lambda)$ by Theorem 12.1. On the other hand the tropicalised cluster mutation gives a bijection between the lattice points of Γ_G and $\Gamma_{G'}$ by Corollary 10.17. □

Remark 15.10. Again when G is a reduced plabic graph of type $\pi_{k,n}$, one may use results of [PSW09] to prove that the polytope Conv_G has $\binom{n}{k}$ lattice points $\{\text{val}_G(P_J) \mid J \in \binom{[n]}{n-k}\}$; moreover, each of those lattice points is a vertex. To see this, recall that in [PSW09], the authors studied the *matching polytope* associated to a reduced plabic graph G, which is defined by taking the convex hull of *all* exponent vectors in the flow polynomials P_J^G from (5.3), where J runs over elements in $\binom{[n]}{n-k}$. It was shown there that every such exponent vector gives rise to a distinct vertex of the matching polytope. Since Conv_G is defined as the convex hull of a subset of the exponent vectors used to define the matching polytope, it follows that the elements of $\{\text{val}_G(P_J) \mid J \in \binom{[n]}{n-k}\}$ are vertices of Conv_G, and are all distinct.

Theorem 15.11. *Suppose G is a reduced plabic graph of type $\pi_{k,n}$ for which Γ_G is a lattice polytope. Then the Newton-Okounkov body Δ_G is equal to Γ_G, and these polytopes furthermore coincide with* Conv_G.

Proof. If Γ_G is a lattice polytope, then it is the convex hull of its lattice points. By Corollary 15.9 this implies $\Gamma_G = \text{Conv}_G$. On the other hand we have $\Delta_G \supseteq \text{Conv}_G$, by definition. So we get $\Delta_G \supseteq \Gamma_G$. But by Corollary 15.8 we know that Γ_G and Δ_G both have the same volume, and given any inclusion $A \supseteq B$ of convex bodies where A and B have the same volume it follows that $A = B$. □

15.2. The theta function basis. Recall that cluster $\mathcal{A}$- and $\mathcal{X}$-varieties are constructed by gluing together "seed tori" via birational maps known as cluster transformations; cluster varieties were introduced by Fock and Goncharov in [FG09] and are a more geometric point of view on the cluster algebras of Fomin and Zelevinsky [FZ02]. The cluster $\mathcal{A}$-variety is the geometric counterpart of a cluster algebra, while the cluster $\mathcal{X}$-variety corresponds to the y-seeds of Fomin and Zelevinsky [FZ07, Definition 2.9]. In this section we will assume that the reader has some familiarity with [GHK15] and [GHKK14]; in particular we will use the notation for cluster varieties from [GHK15, Section 2].

Note that the network charts for $\mathbb{X}^\circ$ in Section 5 and their further $\mathcal{X}$-mutations give $\mathbb{X}^\circ$ the structure of a cluster $\mathcal{X}$-variety, see Section 6. Meanwhile, the cluster charts for $\check{\mathbb{X}}^\circ$ in Section 4 give $\check{\mathbb{X}}^\circ$ the structure of a cluster $\mathcal{A}$-variety. See [Pos], [Sco06], and [MS16b, Section 1.1] for more details.

Theorem 15.12. *There is a theta function basis $\mathcal{B}(\mathbb{X}^\circ)$ for the coordinate ring $\mathbb{C}[\widehat{\mathbb{X}}^\circ]$ of the affine cone over the cluster $\mathcal{X}$-variety $\mathbb{X}^\circ$, which restricts to a theta function basis $\mathcal{B}(\mathbb{X})$ for the homogeneous coordinate ring $\mathbb{C}[\widehat{\mathbb{X}}]$ of the Grassmannian. And $\mathcal{B}(\mathbb{X})$ restricts to a basis $\mathcal{B}_r$ of the degree r component of the homogeneous coordinate ring, for every $r \in \mathbb{Z}_{\geq 0}$.*

Remark 15.13. We note that the degree r component of the homogeneous coordinate ring above is naturally isomorphic to L_r by the map which sends a degree r polynomial P in Plücker coordinates to $P/P_{\max}^r \in L_r$. We will use this isomorphism to identify $\mathbb{C}[\widehat{\mathbb{X}}]_r$ with L_r when convenient.

Proof. Gross-Hacking-Keel-Kontsevich [GHKK14, Theorem 0.3] showed that canonical bases of global regular "theta" functions exist for a formal version of cluster varieties, and in many cases (when "the full Fock-Goncharov conjecture holds"), these extend to bases for regular functions on the actual cluster varieties. They pointed out that the full Fock-Goncharov conjecture holds if there is a maximal green sequence for the cluster variety; in the case of $\mathbb{X}^\circ$, a maximal green sequence was found by Marsh and Scott, see [MS16a, Section 11]. Therefore we indeed have a theta function basis $\mathcal{B}(\mathbb{X}^\circ)$ for the coordinate ring $\mathbb{C}[\widehat{\mathbb{X}}^\circ]$ of the affine cone over the cluster $\mathcal{X}$-variety $\mathbb{X}^\circ$.

Note that there is also a theta function basis $\mathcal{B}(\mathbb{X})$ for the coordinate ring $\mathbb{C}[\widehat{\mathbb{X}}]$ of the affine cone over the Grassmannian; see [GHKK14, Section 9] for a discussion of how [GHKK14, Theorem 0.3] extends to partial compactifications of cluster varieties coming from frozen variables. Moreover we claim that $\mathcal{B}(\mathbb{X}) \subset \mathcal{B}(\mathbb{X}^\circ)$. This follows from [GHKK14, Proposition 9.4 and Corollary 9.17].

Finally $\mathcal{B}(\mathbb{X})$ restricts to a basis of L_r because it is compatible with the one-dimensional torus action (which is overall scaling in the Plucker embedding). □

We now prove Theorem 15.14, which says that for a cluster $\mathcal{X}$-variety, and an arbitrary choice of $\mathcal{X}$-chart, each theta basis element θ is pointed with respect to the $\mathcal{X}$-chart. In other words, θ can be written as a Laurent monomial multiplied by a polynomial with constant term 1 (cf. Definition 5.13) in the variables of the $\mathcal{X}$-chart. Theorem 15.14 follows from the machinery of [GHKK14], and we are grateful to Man-Wai (Mandy) Cheung, Sean Keel, and Mark Gross for their useful explanations on this topic.

Note that Theorem 15.14 confirms a conjecture of Fock and Goncharov, see [FG09, Conjecture 4.1, part 1], and also [GS16, page 41].

Theorem 15.14. *Fix a cluster $\mathcal{X}$-variety and an arbitrary $\mathcal{X}$-chart. Then every element of the theta function basis can be written as a pointed Laurent polynomial in the variables of the $\mathcal{X}$-chart. Moreover the exponents of the leading terms are all distinct.*

Proof. Elements of the theta function basis for $\mathcal{X}$ are constructed using a consistent *scattering diagram* $\mathfrak{D}^{\mathcal{X}}$ associated to the seed. In keeping with [GHKK14] we denote by N the character group of the $\mathcal{X}$-cluster torus of our chosen $\mathcal{X}$-cluster seed, which we embed as a lattice in $N_{\mathbb{R}} = N \otimes \mathbb{R}$. The $n \in N$ are interpreted as exponents of monomial functions on the $\mathcal{X}$-cluster torus.

The theta functions θ_n are indexed by lattice points $n \in N$, see [GHKK14, Definition 7.12], and

$$\theta_n = \sum_{\gamma} \mathrm{Mono}(\gamma),$$

where the sum is over all broken lines with initial exponent n. There is a monomial attached to each domain of linearity of a broken line, which is inductively computed based on which walls of the scattering diagram have been crossed; $\mathrm{Mono}(\gamma)$ is the monomial attached to the last domain of linearity. From the construction it is clear that if every function attached to each wall of $\mathfrak{D}^{\mathcal{X}}$ is positive, i.e. if it is a power series in $\mathbf{x}^n$ for n in the positive orthant N^+, then the element θ_n will be pointed with leading term $\mathbf{x}^n$, and the exponent vectors of leading terms of the θ_n's will in particular all be distinct.

In [GHKK14], the authors explain how to construct the scattering diagram for $\mathcal{X}$ from that for $\mathcal{A}_{\mathrm{prin}}$, which maps to $\mathcal{X}$. By [GHKK14, Construction 2.11], the walls of $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}$ have the form $(n,0)^\perp$ for $n \in N^+$. And by [GHKK14, Construction 7.11], the functions on walls of $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}$ are series in $\mathbf{z}^{(p^*(n),n)} = \mathbf{a}^{p^*(n)}\mathbf{x}^n$ for $n \in N^+$. As noted in [GHKK14, footnote 2, page 72], one can then obtain the scattering diagram $\mathfrak{D}^{\mathcal{X}}$ from $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}$ by intersecting each wall with $w^{-1}(0)$, where w is the weight map from tropical points of $\mathcal{A}_{\mathrm{prin}}$

to $\mathrm{Hom}(N,\mathbb{Z})$ [GHKK14, page 71], and replacing the series in $\mathbf{z}^{(p^*(n),n)} = \mathbf{a}^{p^*(n)}\mathbf{x}^n$ by the corresponding series in $\mathbf{x}^n$. Therefore each function attached to a wall of $\mathfrak{D}^{\mathcal{X}}$ is a power series in $\mathbf{x}^n$ for $n \in N^+$.

□

Lemma 15.15. *When $G = G^{\mathrm{rec}}_{k,n}$, for each lattice point $d \in \Gamma^r_G$, there is an element $\theta_d \in \mathcal{B}_r$ such that $\mathrm{val}_G(\theta_d) = d$.*

Proof. By Lemma 15.4, the polytope Γ_G has the integer decomposition property in the rectangles cluster case. Furthermore by Theorem 15.11, we have $\Delta_G = \mathrm{Conv}_G = \Gamma_G$, and hence $\mathrm{val}_G(L_r) \subset r\Gamma_G$. Therefore the lattice points in $\Gamma^r_G = r\Gamma_G$ are precisely the elements in $\mathrm{val}_G(L_r)$, since by Proposition 15.6 and Lemma 7.9, both sets have the same cardinality. Since the elements of $\mathcal{B}_r$ are a basis of L_r, and have distinct valuations by 15.14, it follows that for each lattice point $d \in \Gamma^r_G$, there is an element θ_d of $\mathcal{B}_r$, which when expressed in terms of the variables $\mathcal{X}\mathrm{Coord}_{\mathbb{X}}(G)$ of the $\mathcal{X}$-seed G, is pointed with leading term x^d. □

Lemma 15.16. *If G and G' index two $\mathcal{X}$-seeds which are connected by a single mutation, then we have a commutative diagram*

$$\begin{array}{ccc} & \mathcal{B}_r & \\ \mathrm{val}_G \swarrow & & \mathrm{val}_{G'} \searrow \\ \mathrm{val}_G(L_r) & \xrightarrow{\Psi_{G,G'}} & \mathrm{val}_{G'}(L_r), \end{array} \tag{15.10}$$

where $\Psi_{G,G'}$ is a bijection, the tropicalized cluster mutation *from Lemma 10.7.*

Proof. Since $\mathcal{B}_r$ is a basis of L_r and the elements have distinct leading terms, the maps val_G and $\mathrm{val}_{G'}$ are bijections. The fact that the diagram is commutative follows from the fact that the elements of $\mathcal{B}(\mathbb{X}^\circ)$ are parameterized by the tropical points of the $\mathcal{A}$-variety (see [GHKK14, (0.2)] and [GHK15, Conjecture 1.11] for this parametrization, as well as [FG06, (12.4) and (12.5)] for the mutation rule for tropical points of the $\mathcal{A}$-variety). Since the diagonal maps are bijections, $\Psi_{G,G'}$ is a bijection; see also Remark 10.12. □

Note that Lemma 15.16 would not hold if we replaced $\mathcal{B}_r$ by e.g. the standard monomials basis of L_r. Working with $\Psi_{G,G'}$ is a bit delicate, since the map is only piecewise linear (see Remark 12.3).

We now prove Theorem 15.17, the second main result of this paper.

Theorem 15.17. *Let G be any reduced plabic graph of type $\pi_{k,n}$, or more generally, any $\mathcal{X}$-seed G of type $\pi_{k,n}$. Then the Newton-Okounkov body Δ_G coincides with the superpotential polytope Γ_G. Moreover, the Newton-Okounkov body is a rational polytope.*

Proof. When $G = G^{\mathrm{rec}}_{k,n}$, we have from Lemma 15.15 that $\mathrm{val}_G(L_r) = \mathrm{Lattice}(\Gamma^r_G)$. By Lemma 15.16,

$$\Psi_{G,G'} : \mathrm{val}_G(L_r) \to \mathrm{val}_{G'}(L_r)$$

is a bijection, and by the proof of Corollary 10.17,

$$\Psi_{G,G'} : \mathrm{Lattice}(\Gamma^r_G) \to \mathrm{Lattice}(\Gamma^r_{G'})$$

is a bijection. Therefore, using the fact that all $\mathcal{X}$-seeds are connected by mutation, it follows that $\mathrm{val}_G(L_r) = \mathrm{Lattice}(\Gamma^r_G)$ for any $\mathcal{X}$-seed G of type $\pi_{k,n}$. Now since $\Gamma^r_G = r\Gamma_G$ (see Remark 9.11), we have

$$\Gamma_G = \overline{\mathrm{ConvexHull}\left(\bigcup_r \frac{1}{r}\,\mathrm{Lattice}(\Gamma^r_G)\right)} = \overline{\mathrm{ConvexHull}\left(\bigcup_r \frac{1}{r}\mathrm{val}_G(L_r)\right)} = \Delta_G \tag{15.11}$$

for any G of type $\pi_{k,n}$, where the first equality is as in Remark 7.4. □

In the plabic graph case we summarise our results as follows.

Corollary 15.18. *Let G be a reduced plabic graph of type $\pi_{k,n}$. Then Δ_G equals to Γ_G, and has precisely $\binom{n}{k}$ lattice points. These are the valuations of Plücker coordinates P_λ for $\lambda \in \mathcal{P}_{n,k}$, and they can be computed explicitly using the formula*

$$\mathrm{val}_G(P_\lambda)_\mu = \mathrm{MaxDiag}(\mu \smallsetminus \lambda),$$

where $\mathrm{MaxDiag}(\mu \smallsetminus \lambda)$ *is given in Definition 13.3. Here the* μ*'s run through* $\mathcal{P}_G$*.* □

This corollary is a combination of Theorem 14.1, Corollary 15.9, and Theorem 15.17. In Section 17 we will give an explicit description of Γ_G in terms of the plabic graph.

16. Khovanskii bases and toric degenerations

Under certain conditions the Newton Okounkov body construction can be used to obtain Khovanskii or SAGBI bases [KM16] and toric degenerations, see for example [Kav05], [Kav15] and [And13]. We will briefly review this connection as it applies in our setting.

Definition 16.1 (following [KM16, Definition 1])**.** Suppose R is a finitely generated $\mathbb{C}$-algebra with Krull dimension d and discrete valuation $\mathrm{val} : R \smallsetminus \{0\} \to \mathbb{Z}^d$ where we view $\mathbb{Z}^d$ as a group with a total ordering such that $v < v'$ implies $v + u < v' + u$. The *value semigroup* $S = S(R, \mathrm{val})$ of val is by definition the subsemigroup of $\mathbb{Z}^d$ which is the image of val. For each $v \in S$ define the subspaces

$$R_{\geq v} := \{f \in R \mid \mathrm{val}(f) \geq v\} \cup \{0\}, \qquad R_{>v} := \{f \in R \mid \mathrm{val}(f) > v\} \cup \{0\},$$

and define the associated graded algebra $\mathrm{gr}_{\mathrm{val}}(R) = \bigoplus_{v \in S} R_{\geq v}/R_{>v}$, graded over the semigroup S. For each nonzero f in R there is an element $\bar{f}$ in $\mathrm{gr}_{\mathrm{val}}(R)$, which lies in $R_{\geq v}/R_{>v}$ for $v = \mathrm{val}(f)$, and which is represented by f. A (finite) set $\mathcal{B} \subset R$ is called a (finite) *Khovanskii basis* for (R, val) if the image of $\mathcal{B}$ in the associated graded $\mathrm{gr}_{\mathrm{val}}(R)$ forms a set of algebra generators.

The example we have in mind for R is the homogeneous coordinate ring of $\mathbb{X}$ in *some* projective embedding. The valuation will be an extension of val_G which also incorporates the grading.

Remark 16.2. We will always assume that the valuation val has 1-dimensional leaves, that is, the graded components of $\mathrm{gr}_{\mathrm{val}}(R)$ are at most 1-dimensional. In this case we have that $\mathcal{B} \subset R$ is a Khovanskii basis if the set $\mathrm{val}(\mathcal{B})$ of valuations generates the semigroup S. This definition generalises the concept of a SAGBI basis, see also [KK08, Definition 5.24], as well as [BFF⁺16, Remark 4.9]. The terminology SAGBI stands for Subalgebra Analogue of Gröbner Basis for Ideals and originates from the case where R is a subalgebra of a polynomial ring. Note that a finite Khovanskii basis for R exists if and only if S is a finitely generated semigroup. The remarkable fact about a Khovanskii basis is that any element of R can be represented as a polynomial in the f_i by the *subduction algorithm*, see [KM16, Algorithm 3.8].

16.1. Let Y be an m-dimensional, irreducible projective variety, with a valuation $\mathrm{val} : \mathbb{C}(Y) \smallsetminus \{0\} \to \mathbb{Z}^m$ with one-dimensional leaves. Fix an ample divisor D on Y. We associate to (Y, D) the graded algebra

$$R = \bigoplus_{j=0}^{\infty} R^{(j)} = \bigoplus_{j=0}^{\infty} t^j H^0(Y, \mathcal{O}(jD)) \subset \mathbb{C}(Y)[t]. \tag{16.1}$$

We define an extended valuation $\overline{\mathrm{val}}$ on R, with value semigroup $\overline{S} \subseteq \mathbb{Z} \times \mathbb{Z}^m$ by setting

$$\overline{\mathrm{val}} : R \smallsetminus \{0\} \to \mathbb{Z} \times \mathbb{Z}^m, \tag{16.2}$$

$$\sum t^j f^{(j)} \mapsto \left(j_0, \mathrm{val}(f^{(j_0)})\right), \tag{16.3}$$

where $j_0 = \max\{j \mid f^{(j)} \neq 0\}$. Note that the projection to its first component gives $\overline{S}$ a $\mathbb{Z}_{\geq 0}$-grading.

Following [KM16], we choose an order for $\mathbb{Z} \times \mathbb{Z}^m$ (and hence $\overline{S}$) using a combination of the reverse order on $\mathbb{Z}$ and the standard lexicographical order on $\mathbb{Z}^m$. Namely $(r, v) < (r', v')$ if either $r > r'$ or $r = r'$ and $v < v'$. This order makes $\overline{S}$ a *maximum well ordered* poset, meaning that any subset of $\overline{S}$ has a maximal element. This property is needed for the subduction algorithm to terminate. See [KM16, Example 3.10].

We focus on the 'large enough' case where D is very ample and Y is projectively normal in the projective embedding $Y \hookrightarrow \mathbb{P}^d$ associated to D; therefore R is generated by $R^{(1)}$. Choose a $g \in H^0(Y, \mathcal{O}(1))$ such that D is the divisor of zeros of g. In this case the homogeneous coordinate ring $\mathbb{C}[\widehat{Y}]$ of Y is isomorphic to R via the map which sends $f \in \mathbb{C}[\widehat{Y}]_j$ to $t^j f/g^j \in R$, compare [Har77, II, Exercise 5.14].

More general versions of the following result can be found in [And13, Theorem 1], [Kav15, Section 7], and [Tei03]. We follow mostly [And13], though our conventions regarding the ordering < are reversed.

Proposition 16.3. *Let $Y, D, R, \overline{\mathrm{val}}$ and $\overline{S}$ be as above, where $\overline{\mathrm{val}}$ has one-dimensional leaves, D is very ample and Y is projectively normal in the associated projective embedding $Y \hookrightarrow \mathbb{P}^d$. Suppose that $\overline{S}$ is generated by its degree 1 part $\overline{S}^{(1)}$. Let C denote the cone spanned by $\overline{S}^{(1)}$, and Δ the polytope in $\mathbb{R}^m$ such that $\{1\} \times \Delta$ is the intersection of C with $\{1\} \times \mathbb{R}^m$. Assume Δ has the integer decomposition property.*

Then there exists a flat family $\mathcal{Y} \to \mathbb{A}^1$ embedded in $\mathbb{P}^d \times \mathbb{A}^1 \to \mathbb{A}^1$, such that the fiber over 0 is a normal, projective toric variety Y_0, while the other fibers are isomorphic to Y. Moreover, Δ is the moment polytope of Y_0 for its embedding into $\mathbb{P}^d$, and this embedding is projectively normal.

Remark 16.4. In contrast with the more general theorem [And13, Theorem 1], we have added the assumptions that the semigroup $\overline{S}$ is generated in degree 1, and the polytope Δ has the integer decomposition property. (These will be true in our application in Section 16.2.) If these assumptions are removed, then $\mathbb{P}^d$ may need to be replaced by weighted projective space, and the limit toric variety Y_0 may not be normal.

Proof. We sketch the construction of the toric degeneration, mostly following [And13]. The assumption on $\overline{S}$ implies that there exists a finite Khovanskii basis $\{\phi_{(1,\ell)} \mid \ell \in \mathcal{L}\}$ of R, where $\mathcal{L}$ denotes the lattice points of Δ and $\overline{\mathrm{val}}(\phi_{(1,\ell)}) = (1, \ell)$. Note that $|\mathcal{L}| = d+1$ and this Khovanskii basis is a vector space basis of $R^{(1)}$. The degeneration is obtained by applying (relative) Proj to a graded $\mathbb{C}[s]$-algebra $\mathcal{R}$ which is constructed from R as follows.

Consider the polynomial ring $A = \mathbb{C}[x_\ell; \ell \in \mathcal{L}]$, with the usual $\mathbb{Z}$-grading, as well as an extension of this grading to an $\overline{S}$-grading via $\deg(x_\ell) := (1, \ell)$. The maps $h : A \to R$ and $\overline{h} : A \to \mathrm{gr}_{\overline{\mathrm{val}}} R$ defined by

$$h(x_\ell) := \phi_{(1,\ell)}, \qquad \overline{h}(x_\ell) := \overline{\phi}_{(1,\ell)}$$

are homomorphisms of $\mathbb{Z}$-graded algebras and $\overline{S}$-graded algebras, respectively. The the maximum well-ordered property of the ordering on $\overline{S}$ implies that the kernel of h has a Gröbner basis $g_1, \dots, g_m$ whose $\overline{S}$-initial terms $\overline{g}_1, \dots, \overline{g}_m$ generate the kernel of $\overline{h}$. Moreover we can choose the g_i to be homogeneous (say of degree r_i) and $\overline{g}_i$ homogeneous (say of degree (r_i, v_i)). Then one can find a linear projection $\pi : \mathbb{Z} \times \mathbb{Z}^m \to \mathbb{Z}$ (see [And13]), such that the elements $\widetilde{g}_i \in A[s]$ defined by

$$\widetilde{g}_i := s^{-\pi(r_i, v_i)} g_i((s^{\pi(1,\ell)} x_\ell)_{\ell \in \mathcal{L}})$$

are of the form $\overline{g}_i + sA_{>(r_i,v_i)}$. Moreover $\mathcal{R} := A[s]/(\widetilde{g}_1, \dots, \widetilde{g}_m)$ is a flat $\mathbb{C}[s]$-algebra with $\mathcal{R}/s\mathcal{R} \cong A/(\overline{g}_1, \dots, \overline{g}_m) \cong \mathrm{gr}_{\overline{\mathrm{val}}}(R)$ and $\mathcal{R}[s^{-1}] \cong R \otimes \mathbb{C}[s, s^{-1}]$. We therefore obtain a family $\mathcal{Y}$ of projective varieties over $\mathbb{A}^1$ such that the fiber over 0 equals the projective toric variety with homogeneous coordinate ring $\mathrm{gr}_{\overline{\mathrm{val}}}(R)$, and all other fibers are isomorphic to Y. If we order the set $\mathcal{L}$, so $\mathcal{L} = \{\ell_1, \dots, \ell_{d+1}\}$, then the description of $\mathcal{R}$ gives rise to the embedding of $\mathcal{Y}$ into $\mathbb{P}^d \times \mathbb{A}^1$. Note that since $\overline{\mathrm{val}}$ has 1-dimensional leaves, $\mathrm{gr}_{\overline{\mathrm{val}}}(R) \cong \mathbb{C}[\overline{S}]$. Thus the zero fiber Y_0 in $\mathbb{P}^d$ has homogeneous coordinate ring $\mathbb{C}[\overline{S}]$. From its degree 1 part we see that the moment polytope of Y_0 is Δ. And since Δ has the integer decomposition property, it follows directly that Y_0 is projectively normal, and in particular also normal. □

16.2. Now we consider $Y = \mathbb{X}$. We choose an $\mathcal{X}$-cluster seed $\Sigma_G^{\mathcal{X}}$ of type $\pi_{k,n}$ and its associated valuation $\mathrm{val}_G : \mathbb{C}(\mathbb{X}) \setminus \{0\} \to \mathbb{Z}^{\mathcal{P}_G}$ with one-dimensional leaves (compare Lemma 7.9). Recall that $L_r = H^0(\mathbb{X}, \mathcal{O}(rD_{n-k}))$, and that by Theorem 15.17, $\Delta_G := \overline{\mathrm{ConvexHull}\left(\bigcup_r \frac{1}{r}\mathrm{val}_G(L_r)\right)}$ is a rational polytope.

Let $R := \bigoplus_j t^j L_j$ and consider the extended valuation $\overline{\mathrm{val}}_G : R \setminus \{0\} \to \mathbb{Z} \times \mathbb{Z}^{\mathcal{P}_G}$ as in (16.2). Note that R is isomorphic to the homogeneous coordinate ring of $\mathbb{X}$ by (7.4). The valuation $\overline{\mathrm{val}}_G$ is again a valuation with 1-dimensional leaves and we have the following result about R in our setting.

Lemma 16.5. *Given $(R, \overline{\mathrm{val}}_G)$ as above, we define the value semigroup*

$$\overline{S}_G := \{(r, v) \mid r \in \mathbb{Z}_{\geq 0}, v \in \mathrm{val}_G(L_r)\} \subseteq \mathbb{Z} \times \mathbb{Z}^{\mathcal{P}_G}. \tag{16.4}$$

Consider the cone $\mathrm{Cone}(G)$ in $\mathbb{R} \times \mathbb{R}^{\mathcal{P}_G}$ defined as the $\mathbb{R}_{\geq 0}$-span of vectors in $\{(1, w) \mid w$ is a vertex of $\Delta_G\}$. Then $\overline{S}_G$ equals the semigroup $\mathrm{Cone}(G) \cap (\mathbb{Z} \times \mathbb{Z}^{\mathcal{P}_G})$ consisting of lattice points of $\mathrm{Cone}(G)$. In particular the semigroup $\overline{S}_G$ is finitely generated, and hence we have a finite Khovanskii basis of R.

Proof. Clearly $\overline{S}_G \subseteq \mathrm{Cone}(G)$, as follows from the construction of Δ_G. The lemma says that conversely every lattice point in $\mathrm{Cone}(G)$ lies in $\overline{S}_G$, i.e. is of the form $(r, \mathrm{val}_G(f))$ for some $f \in L_r$. Equivalently if we fix r it says that the lattice points of $r\Delta_G$ agree with the image $\mathrm{val}_G(L_r)$ of the valuation map. But in the proof of Theorem 15.17 we saw that $\mathrm{val}_G(L_r) = \mathrm{Lattice}(r\Gamma_G)$ and $\Gamma_G = \Delta_G$. Thus we have shown that $\overline{S}_G$ is the semigroup of lattice points of $\mathrm{Cone}(G)$. Finally, $\mathrm{Cone}(G)$ is a rational convex cone by Theorem 15.17. Therefore by Gordan's lemma [Ful93] the semigroup of its lattice points (and therefore the semigroup $\overline{S}_G$) is finitely generated. This completes the proof of the lemma. □

It is well-known (see e.g. [CHHH14]) that for *any* rational polytope $\Delta \subset \mathbb{R}^m$, there is an $r \in \mathbb{Z}_{>0}$ such that $r\Delta$ has the integer decomposition property; this is an easy consequence of Gordan's lemma. Therefore we can make the following definition.

Definition 16.6. Let r_G denote the minimal positive integer such that the dilated polytope $r_G\Delta_G$ has the integer decomposition property.

Definition 16.7. Let $\mathbb{X}_{r_G} \subset \mathbb{P}(\mathrm{Sym}^{r_G}(\bigwedge^k \mathbb{C}^n))$ be the image of $\mathbb{X}$ after composing the Plücker embedding with the Veronese map of degree r_G. In other words $\mathbb{X}_{r_G}$ is the projective variety obtained via the embedding of $\mathbb{X}$ associated to the ample divisor $r_G D_{n-k}$. Let $\mathbb{C}[\widehat{\mathbb{X}}_{r_G}]$ denote the homogeneous coordinate ring of $\mathbb{X}_{r_G}$.

Definition 16.8. Associated to $\mathbb{X}_{r_G}$ we have

$$R_{r_G} = \bigoplus_{j=0}^{\infty} t^j H^0(Y, \mathcal{O}(j r_G D_{n-k})) \subset \mathbb{C}(\mathbb{X})[t],$$

with its extended valuation $\overline{\mathrm{val}}_{G,r_G}$ and the value semigroup

$$\overline{S}_{G,r_G} := \{(r,v) \mid r \in \mathbb{Z}_{\geq 0}, v \in \mathrm{val}_G(H^0(\mathbb{X}, \mathcal{O}(r\, r_G D_{n-k})))\}.$$

The semigroup $\overline{S}_{G,r_G}$ is also obtained by applying the map $(r,v) \to (\frac{1}{r_G} r, v)$ to $\overline{S}_G \cap (r_G\mathbb{Z}) \times \mathbb{Z}^{\mathcal{P}_G}$.

Note that $\mathbb{X}_{r_G}$ is still projectively normal, and R_{r_G} is isomorphic to $\mathbb{C}[\widehat{\mathbb{X}}_{r_G}]$. The associated Newton-Okounkov body is $\Delta_G(r_G D_{n-k}) = r_G\Delta_G$.

Lemma 16.9. *The semigroup $\overline{S}_{G,r_G}$ is generated by the finite set $\overline{S}^{(1)}_{G,r_G} = \{(1,v) \mid v \in \mathrm{Lattice}(r_G\Delta_G)\}$. In particular, for each lattice point $v \in r_G\Delta_G$ we may choose an element $\phi_v \in L_{r_G} \setminus \{0\}$ such that $\mathrm{val}_G(\phi_v) = v$. Then the corresponding set $\{\phi_{(1,v)} := t\phi_v \mid v \in \mathrm{Lattice}(r_G\Delta_G)\}$ is a finite Khovanskii basis of R_{r_G}, which lies in the $j = 1$ graded component.*

Proof. Let $(j,v) \in \overline{S}_{G,r_G}$. Then because $r_G\Delta_G$ has the integer decomposition property and v lies in its j-th dilation, we can write $v = \sum_{i=1}^j v_i$ where $v_i \in \mathrm{Lattice}(r_G\Delta_G)$. Then $(j,v) = \sum_{i=1}^j (1, v_i)$. Thus (j,v) is in the semigroup generated by $\overline{S}^{(1)}_{G,r_G}$. □

Corollary 16.10. *Suppose G is represented by a plabic graph and $r_G = 1$ (as in the case of $G = G^{k,n}_{\mathrm{rec}}$, see Lemma 15.4). Then the set $\{tP_\lambda/P_{\max} \mid \lambda \in \mathcal{P}_G\}$ is a Khovanskii basis of the algebra R.*

Proof. This corollary is a special case of Lemma 16.9, combined with Corollary 15.9. □

It now follows that associated to every seed $\Sigma_G^{\mathcal{X}}$ we obtain a flat degeneration of $\mathbb{X}$ to a toric variety.

Corollary 16.11. *Suppose $\Sigma_G^{\mathcal{X}}$ is an arbitrary $\mathcal{X}$-cluster seed of type $\pi_{k,n}$ and $r_G \in \mathbb{Z}_{>0}$ is as in Definition 16.6. Then we have a flat degeneration of $\mathbb{X}$ to the normal projective toric variety $\mathbb{X}_0$ associated to the polytope $r_G\Delta_G$ (i.e. to the Newton-Okounkov body associated to the rescaled divisor $r_G D_{n-k}$).*

Proof. By Lemma 16.9 the ring R_{r_G} has a finite Khovanskii basis which is contained in its $j = 1$ graded component. By Lemma 16.5 the image of this Khovanskii basis under $\overline{\mathrm{val}}_{G,r_G}$ is precisely the set of all of the lattice points of $\{1\} \times r_G\Delta_G$ (after adjusting according to Definition 16.8). By Definition 16.6 the polytope $\Delta = r_G\Delta_G$ has the integer decomposition property. Therefore the conditions of Proposition 16.3 are satisfied and we obtain a toric degeneration of $\mathbb{X}$ to the toric variety $\mathbb{X}_0$ associated to $r_G\Delta_G$. □

17. The cluster expansion of the superpotential and explicit inequalities for $\Gamma_G = \Delta_G$

Since Newton-Okounkov bodies are defined as a closed convex hull of infinitely many points, very often it is difficult to give a simple description of them. However, now that we have proved that $\Delta_G = \Gamma_G$, we have an inequality description of Δ_G coming from the cluster expansion of the superpotential W_q. In the case that G is a reduced plabic graph of type $\pi_{k,n}$, a combinatorial formula for the cluster expansion of W_q was given in [MR04, Section 12], which followed from the work of Marsh and Scott [MS16a]. We review this formula here and thus give the inequality description of $\Delta_G = \Gamma_G$ when G is a plabic graph.

17.1. The cluster expansion of W_q. Recall from (9.1) that

$$W_q = \sum_{i=1}^{n} q^{\delta^i_{n-k}} \frac{p_{\mu_i^\square}}{p_{\mu_i}}.$$

Fix a cluster associated to a plabic graph G. In order to give the cluster expansion of W_q it is enough to give the cluster expansion of each term $W_i = p_{\mu_i^\square}/p_{\mu_i}$.

Definition 17.1 (Edge weights)**.** We assign monomials from $\mathbb{C}[\mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G)]$ to edges of G as follows. Let v be the unique black vertex incident with an edge e. The *weight* w_e of e is defined to be the product of the Plücker coordinates labelling the faces of G which are incident with v but not with the rest of e (i.e. excluding the two faces on each side of e). (See Figure 23 for an illustration of the rule.) And the weight w_M of a matching M is the product of the weights of all edges in the matching.

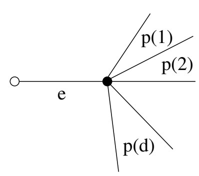


Figure 23. Weighting of an edge: $w_e = p(1)p(2)\cdots p(d)$.

Theorem 17.2 follows from the results of [MS16a].

Theorem 17.2 ([MR13, (12.2)])**.** *Fix a reduced plabic graph G of type $\pi_{k,n}$, and let J^i be the $(n-k)$-element subset $\{i+k+1, i+k+2, \dots, i-1\} \cup \{i+1\}$ (with indices considered modulo n as usual). Then we have that*

$$\frac{p_{\mu_i^\square}}{p_{\mu_i}} = \sum_M p_M, \qquad \text{where} \qquad p_M := \frac{w_M}{\prod_{p \in \mathcal{A}\mathrm{Coord}_{\check{\mathbb{X}}}(G)} p} p_{\mu_{i-1}} p_{\mu_{i+1}} p_{\mu_{i+2}} \dots p_{\mu_{i+k}}, \tag{17.1}$$

and the sum is over the set $\mathrm{Match}_G^{J^i}$ of all matchings M of G with boundary J^i, compare Section 11.

Example 17.3. Let $k = 3$, $n = 5$, and G the graph shown in Figure 24. We have $\mu_1 = \ydiagram{3}$, $\mu_2 = \ydiagram{3,3}$, $\mu_3 = \ydiagram{2,2}$, $\mu_4 = \ydiagram{1,1}$, $\mu_5 = \varnothing$. If $i = 2$ then there is a unique matching M of G with boundary $J^i = J^2 = \{1,3\}$ as shown at the left. This matching has weight $w_M = p_{\ydiagram{1}} p_{\ydiagram{2}}^2$, so $\frac{p_{\mu_i^\square}}{p_{\mu_i}} = \frac{p_{\ydiagram{2}}}{p_{\ydiagram{3,3}}}$. (Recall that $p_\varnothing = 1$.)

If $i = 3$, there are two matchings of G with boundary $J^i = J^3 = \{2,4\}$. The maximal matching M_3 with boundary J^3 is shown at the right of Figure 24 and it has weight $w_M = p_{\ydiagram{1}} p_{\ydiagram{2,2}} p_{\ydiagram{3}}$; so one of the two summands in $\frac{p_{\mu_i^\square}}{p_{\mu_i}}$ is $\frac{p_{\ydiagram{3}}}{p_{\ydiagram{2}}}$.

Recall the definition of the superpotential polytope, Definition 9.10, and the generalised superpotential polytope, Definition 9.13. We can now use Theorem 15.17 and Theorem 17.2 to write down the inequalities cutting out $\Gamma_G(r_1, \dots, r_n)$ and as a special case Γ_G^r.

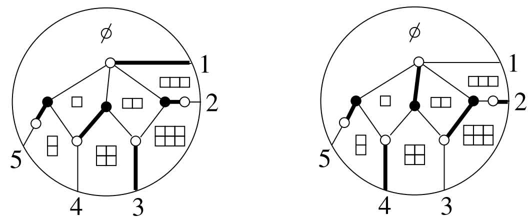


FIGURE 24. A graph G of type $\pi_{3,5}$ together with the unique matching with boundary $J^2 = \{1,3\}$ (left) and the maximal matching with boundary $J^3 = \{2,4\}$ (right).

Proposition 17.4. *Let $r_1, \dots, r_n \in \mathbb{R}$. The generalised superpotential polytope $\Gamma_G(r_1, \dots, r_n)$ is cut out by linear inequalities associated to matchings $M \in \mathrm{Match}_G^{J^i}$, where $1 \le i \le n$. Namely for $M \in \mathrm{Match}_G^{J^i}$, the associated inequality is*

$$\mathrm{Trop}_G(p_M) + r_i \ge 0. \tag{17.2}$$

By Theorem 15.17 which identifies the Newton-Okounkov body Δ_G with the superpotential polytope Γ_G, we obtain the following description of Δ_G.

Corollary 17.5. *The Newton-Okounkov body Δ_G is a polytope determined by certain linear inequalities associated to matchings $M \in \mathrm{Match}_G^{J^i}$, where $1 \le i \le n$. Namely for $M \in \mathrm{Match}_G^{J^i}$, the associated inequality is (17.2), where $r_i = 0$ for $i \ne n-k$ and $r_{n-k} = 1$.*

Example 17.6. We continue Example 17.3. When $i = 2 = n-k$ we have the term $qW_i = q\frac{p_{\mu_i^{\square}}}{p_{\mu_i}} = q\frac{p_{\square\square}}{p_{\boxplus\boxplus}}$ of W, which gives rise to the inequality $r + v_{\square\square} - v_{\boxplus\boxplus} \ge 0$. When $i = 3$ we have that one of the summands in $W_i = \frac{p_{\mu_i^{\square}}}{p_{\mu_i}}$ is $\frac{p_{\square\square\square}}{p_{\square\square}}$, which gives rise to the inequality $v_{\square\square\square} - v_{\square\square} \ge 0$. This matches up with our description of Γ_G from Example 9.12.

18. THE NEWTON-OKOUNKOV POLYTOPE $\Delta_G(D)$ FOR MORE GENERAL DIVISORS D

In this section we consider the Newton-Okounkov body of a general divisor D which is a linear combination of the boundary divisors D_j in $\mathbb{X}$, and we prove the analogue of $\Delta_G = \Gamma_G$.

Recall that we defined a polytope $\Gamma_G(r_1, \dots, r_n)$ using tropicalisation of the individual summands W_j of the superpotential, see Definition 9.13. The result below generalizes Theorem 15.17.

Theorem 18.1. *For the divisor $D = r_1 D_1 + r_2 D_2 + \dots + r_n D_n$ with $r_i \in \mathbb{Z}$, the associated Newton-Okounkov polytope is given by*

$$\Delta_G(D) = \Gamma_G(r_1, \dots, r_n). \tag{18.1}$$

Moreover unless $r := \sum r_j \ge 0$, both $\Delta_G(D)$ and $\Gamma_G(r_1, \dots, r_n)$ are the empty set.

One way to prove (18.1) is to try to mimic the proof of Theorem 15.17: to first prove it when $G = G_{k,n}^{\mathrm{rec}}$, and then to show that when one mutates away from G, the lattice points of both sides satisfy the tropical mutation formulas. For $\Gamma_G(r_1, \dots, r_n)$ this follows from Corollary 10.16, but for $\Delta_G(D)$ the mutation property requires more work. While one can complete the proof using this strategy, we instead deduce the theorem from Theorem 15.17: we show that changing the divisor from rD_{n-k} to $D = r_1 D_1 + \dots + r_n D_n$ with $r = r_1 + \dots + r_n$ translates both sides of (18.1) by the same vector, see Proposition 18.4 and Proposition 18.5.

Remark 18.2. If $r < 0$, the line bundle $\mathcal{O}(D) = \mathcal{O}(r)$ has no non-zero global sections, and hence $\Delta_G(D)$ is clearly the empty set. We demonstrate an analogous result for $\Gamma_G(r_1, \dots, r_n)$ in Proposition 18.6.

If $r = 0$ then $\mathcal{O}(D)$ is the structure sheaf $\mathcal{O}$ and $\Delta_G(D)$ consists of a single point. Namely

$$f_D := \prod_{j=1}^{n} P_{\mu_j}^{-r_j}$$

is a rational function on $\mathbb{X}$ (since $\sum r_j = 0$) and spans $H^0(\mathbb{X}, \mathcal{O}(D)) \cong \mathbb{C}$. Moreover $H^0(\mathbb{X}, \mathcal{O}(sD))$ is the one-dimensional vector space spanned by $(f_D)^s$. By the definition of $\Delta_G(D)$ we immediately obtain $\Delta_G(D) = \{v_D\}$, where $v_D = -\sum_j r_j \mathrm{val}_G(P_{\mu_j})$ is the valuation $\mathrm{val}_G(f_D)$.

In order to prove the theorem, we need the following lemma about valuations of frozen variables, i.e. the Plücker coordinates P_{μ_j} for $0 \le j \le n-1$.

Lemma 18.3. *Fix $j \in \{0, 1, \ldots, n-1\}$ and let $e = e^{(j)} = \mathrm{val}_G(P_{\mu_j})$. Then we have*

$$\mathrm{Trop}_G(p_{\mu_i^\square}/p_{\mu_i})(e^{(j)}) = \delta_{i,j} - \delta_{i,n-k}. \tag{18.2}$$

Proof. We check the identity for $\mathrm{Trop}_G(p_{\mu_i^\square}/p_{\mu_i})(e^{(j)})$ first in the case where G is a plabic graph and $\mathcal{P}_G$ contains $\mu_i^\square$. Indeed, in this case the identity follows easily from the max diag formula, Theorem 14.1. Now we can obtain any other seed from this one by a sequence of mutations. Since $e = \mathrm{val}_G(P_{\mu_j})$ mutates by the tropical $\mathcal{A}$-cluster mutation formula, see Proposition 14.9, this implies that the quantity $\mathrm{Trop}_G(p_{\mu_i^\square}/p_{\mu_i})(e)$ is independent of the choice of seed G. Thus the identity (18.2) holds in general. □

Proposition 18.4. *Let $D = \sum r_i D_i$ and $r := \sum_j r_j$. The Newton-Okounkov body $\Delta_G(D)$ is obtained from $\Delta_G(rD_{n-k})$ by translation. Explicitly, if $v_D := -\sum_j r_j \mathrm{val}_G(P_{\mu_j})$, we have*

$$\Delta_G(D) = \Delta_G(rD_{n-k}) + v_D. \tag{18.3}$$

Note that $\Delta_G(rD_{n-k}) = r\Delta_G$ if $r \ge 0$ and $\Delta_G(rD_{n-k}) = \varnothing$ if $r < 0$, see Remark 18.2.

Proof. We may suppose that $r := \sum_j r_j \ge 0$. To show that $\Delta_G(D) = r\Delta_G + v_D$ it suffices to check that for every $s \in \mathbb{Z}_{>0}$,

$$\frac{1}{s}\mathrm{val}_G(L_{sD}) = \frac{1}{s}\mathrm{val}_G(L_{sr}) + v_D. \tag{18.4}$$

However for any $D = \sum r_j D_j$ with $r = \sum r_j$ we have an isomorphism of vector spaces

$$m : L_r \to L_D \quad \text{given by} \quad f \mapsto f\frac{P_{\max}^r}{\prod_j P_{\mu_j}^{r_j}}.$$

This isomorphism shifts valuations and gives the equality $\mathrm{val}_G(L_D) = \mathrm{val}_G(L_r) + v_D$. If we replace D by sD, then the resulting equation for $\mathrm{val}_G(L_{sD})$ implies (18.4). This proves the desired formula for $\Delta_G(D)$. □

Proposition 18.5. *Let $r_1, \ldots, r_n \in \mathbb{R}$ and $r := \sum_j r_j$. Then $\Gamma_G(r_1, \ldots, r_n)$ is related to Γ_G^r by translation,*

$$\Gamma_G(r_1, \ldots, r_n) = \Gamma_G^r + v_D \ \textit{ where } v_D := -\sum_j r_j \mathrm{val}_G(P_{\mu_j}). \tag{18.5}$$

Proof. We want to show that the map $\mathbb{R}^{\mathcal{P}_G} \to \mathbb{R}^{\mathcal{P}_G}$ which sends v to $d = v + v_D$ bijectively takes Γ_G^r to $\Gamma_G(r_1, \ldots, r_n)$. Since $\Gamma_G(r_1, \ldots, r_n)$ is by definition the intersection of the sets $\mathrm{PosSet}^G_{(r_i)}(W_i) := \{d \mid \mathrm{Trop}_G(p_{\mu_i^\square}/p_{\mu_i})(d) + r_i \ge 0\}$, it suffices to show the analogous translation property for each such set.

Note that in general $\mathrm{Trop}_G(p_{\mu_i^\square}/p_{\mu_i})(d) = \min_M(\mathrm{Trop}_G(p_M)(d))$, where $p_{\mu_i^\square}/p_{\mu_i} = \sum_M p_M$ is the expansion of $p_{\mu_i^\square}/p_{\mu_i}$ as sum of Laurent monomials in the cluster variables associated to G. In the special case where $\mu_i^\square \in \mathcal{P}_G$ however, $\mathrm{Trop}_G(p_{\mu_i^\square}/p_{\mu_i})(d) = d_{\mu_i^\square} - d_{\mu_i}$ is linear.

Let us assume first that $\mu_i^\square \in \mathcal{P}_G$. In this case by linearity we have that, for any $v \in \mathbb{R}^{\mathcal{P}_G}$,

$$\mathrm{Trop}_G(p_{\mu_i^\square}/p_{\mu_i})(v + v_D) = \mathrm{Trop}_G(p_{\mu_i^\square}/p_{\mu_i})(v) + \mathrm{Trop}_G(p_{\mu_i^\square}/p_{\mu_i})(v_D) = \mathrm{Trop}_G(p_{\mu_i^\square}/p_{\mu_i})(v) - r_i + r\delta_{i,n-k}, \tag{18.6}$$

where we have evaluated $\mathrm{Trop}_G(p_{\mu_i^\square}/p_{\mu_i})(v_D)$ using Lemma 18.3. As a consequence

$$\mathrm{Trop}_G(p_{\mu_i^\square}/p_{\mu_i})(v + v_D) + r_i = \mathrm{Trop}_G(p_{\mu_i^\square}/p_{\mu_i})(v) + r\delta_{i,n-k}. \tag{18.7}$$

From (18.7) it follows that $v + v_D$ lies in $\mathrm{PosSet}^G_{(r_i)}(W_i)$ if and only if v lies in $\mathrm{PosSet}^G_{(r\delta_{i,n-k})}(W_i)$. Thus we have that, whenever $p_{\mu_i^\square}$ is a cluster variable in the $\mathcal{A}$-cluster associated to G,

$$\mathrm{PosSet}^G_{(r_i)}(W_i) = \mathrm{PosSet}^G_{(r\delta_{i,n-k})}(W_i) + v_D. \tag{18.8}$$

We would like to apply tropicalised $\mathcal{A}$-cluster mutation $\Psi_{G,G'}$ to both sides of (18.8) to obtain the analogous identity for arbitrary seeds. Let us now write $v_{D,G}$ instead of v_D to emphasise the dependence on G. Note that, since $v_{D,G}$ is a linear combination of elements of the form $\mathrm{val}_G(P_{\mu_i})$, the results of Section 14.2 imply that $v_{D,G}$ is *balanced* and transforms via tropicalised $\mathcal{A}$-cluster mutation if we mutate G. These two properties imply that for any $e \in \mathbb{R}^{\mathcal{P}_G}$,

$$\Psi_{G,G'}(e + v_{D,G}) = \Psi_{G,G'}(e) + v_{D,G'}. \tag{18.9}$$

On the other hand by Lemma 10.15,
(18.10)

$$\Psi_{G,G'}(\mathrm{PosSet}^G_{(r_i)}(W_i)) = \mathrm{PosSet}^{G'}_{(r_i)}(W_i) \quad \text{and} \quad \Psi_{G,G'}(\mathrm{PosSet}^G_{(r\delta_{i,n-k})}(W_i)) = \mathrm{PosSet}^{G'}_{(r\delta_{i,n-k})}(W_i).$$

From (18.9) and (18.10) put together, we obtain that the translation identity (18.8) is preserved under mutation. Thus (18.8) holds for all seeds (and all i).

As a consequence the polytope $\Gamma_G(r_1,\dots,r_n)$ is always the shift by $v_{D,G}$ of the polytope Γ_G^r. □

Proposition 18.6. *If $r := \sum r_j < 0$, then $\Gamma_G(r_1,\dots,r_n)$ is the empty set.*

Proof. By Proposition 18.5, $\Gamma_G(r_1,..,r_n)$ is related to Γ_G^r by a translation. Therefore it suffices to check that Γ_G^r is the empty set for $r < 0$. In the case where G is the rectangles cluster, Γ_G^r is unimodularly equivalent to the Gelfand-Tsetlin polytope (see Definition 15.1), which is clearly empty if $r < 0$, and a point if $r = 0$. Now we know from Corollary 10.16 that the polytopes Γ_G^r transform via tropical cluster mutation when we mutate G. Therefore Γ_G^r is also the empty set for a general seed. □

Proof of Theorem 18.1. If $r < 0$ the result follows from Remark 18.2 and Proposition 18.6. Now suppose $r \geq 0$. By Theorem 15.17, we have that $\Delta_G = \Delta_G(D_{n-k})$ and Γ_G coincide, which implies that $r\Delta_G = \Gamma_G^r$, see Remark 9.11. But now by Proposition 18.4 and Proposition 18.5, both $\Delta_G(D)$ and $\Gamma_G(r_1,\dots,r_n)$ are obtained from $r\Delta_G$ and Γ_G^r by translation by the same vector. □

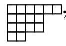
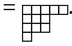
19. The highest degree valuation and Plücker coordinate valuations

Recall from Definition 7.1 that given an $\mathcal{X}$-seed G of type $\pi_{k,n}$, we defined a valuation $\mathrm{val}_G : \mathbb{C}(\mathbb{X}) \smallsetminus \{0\} \to \mathbb{Z}^{\mathcal{P}_G}$ using the lowest order term. When G is a plabic graph, the flow polynomials P_λ^G express the Plücker coordinates in terms of the $\mathcal{X}$-coordinates, and have strongly minimal and maximal terms, see Corollary 11.4. In Theorem 14.1, we gave an explicit formula for the Plücker coordinate valuations $\mathrm{val}_G(P_\lambda)$, such that the μ-th coordinate $\mathrm{val}_G(P_\lambda)_\mu$ is related to the smallest degree of q that appears when the quantum product of two Schubert classes $\sigma_\mu \star \sigma_{\lambda^c}$ is expanded in the Schubert basis.

In this section we briefly explain what is the analogue of Theorem 14.1 if we define our valuation in terms of the highest order term instead of the lowest order term. We will find that our formula is again connected to quantum cohomology, but this time to the *highest* degree of q that appears in a corresponding quantum product. In order to state our formula we first need to introduce some notation.

Definition 19.1. Let μ be a partition in $\mathcal{P}_{k,n}$, so μ lies in an $(n-k)\times k$ rectangle. We let $\mathrm{Diag}_0(\mu)$ denote the number of boxes in μ along the main diagonal (with slope -1).

Let us identify μ with the word $\omega(\mu) = (w_1,\dots,w_n)$ in $\{0,1\}^n$ obtained by reading the border of μ from southwest to northeast and associating a 0 to each horizontal step and a 1 to each vertical step. Then the cyclic shift S acts on partitions in $\mathcal{P}_{k,n}$ by mapping the partition corresponding to $(w_1,\dots,w_n)$ to the partition corresponding to $(w_2,\dots,w_n,w_1)$.

Example 19.2. Let $\mu =$ , viewed inside a 4×6 rectangle. Then $\omega(\mu) = (0,0,1,0,0,1,1,0,0,1)$, and $\mathrm{Diag}_0(\mu) = 3$. Applying the cyclic shift to $\omega(\mu)$ gives $(0,1,0,0,1,1,0,0,1,0)$, and hence $S(\mu) =$ .

For partitions in $\mathcal{P}_{k,n}$, $S^{n-k}(\varnothing) = S^{n-k}(1^{n-k}0^k) = 0^k1^{n-k} = \max$, where max is the $(n-k)\times k$ rectangle.

Theorem 19.3. *Let G be a reduced plabic graph of type $\pi_{k,n}$. Let $\mathrm{val}^G(P_\lambda) \in \mathbb{Z}^{\mathcal{P}_G}$ denote the exponent vector of the strongly* maximal *term of the flow polynomial P_λ^G. Then we have that*

$$\mathrm{val}^G(P_\lambda)_\mu = \mathrm{Diag}_0(\mu) - \mathrm{MaxDiag}(\lambda \smallsetminus S^{n-k}(\mu)). \tag{19.1}$$

Note that by [Pos05, Theorem 8.1], the right-hand side of (19.1) is equal to the largest degree d such that q^d appears in the quantum product of the Schubert classes $\sigma_\mu \star \sigma_{\lambda^c}$ in the quantum cohomology ring $QH^*(\mathbb{X})$, when this product is expanded in the Schubert basis.

We now sketch the proof of Theorem 19.3, which is analogous to the proof of Theorem 14.1.

Proof. Recall from Corollary 11.4 that each flow polynomial $P_\lambda = P_\lambda^G$ has a maximal flow; its exponent vector is precisely $\mathrm{val}^G(P_\lambda)$. Now, following the proof of Theorem 12.1, we show that when we mutate G at a square face, obtaining G', for any λ, the Plücker coordinate valuations $\mathrm{val}^G(P_\lambda)$ and $\mathrm{val}^{G'}(P_\lambda)$ are related by the tropicalized cluster mutation $\Psi^{G,G'}$. Here $\Psi^{G,G'}$ is defined the same way as $\Psi_{G,G'}$ except that we replace min by max. As in the proof of Theorem 12.1, the main step is to analyze how strongly maximal flows change under an oriented square move.

Next, we prove an analogue of Proposition 13.4, which gives the formula for Plücker coordinate valuations when $G = G^{\mathrm{rec}}_{k,n}$. Concretely, one can give a combinatorial proof that

$$\mathrm{val}^G(P_\lambda)_{i\times j} = \mathrm{Diag}_0(i \times j) - \mathrm{MaxDiag}(\lambda \smallsetminus S^{n-k}(i \times j)).$$

To complete the proof, we follow the proof of Theorem 14.1 and in particular Theorem 14.3, explicitly constructing an element $x^\lambda(t)$ of the Grassmannian over Laurent series, such that

$$\mathrm{Val}^{\mathbf{K}}(p_\mu(x^\lambda(t))) = \mathrm{Diag}_0(\mu) - \mathrm{MaxDiag}(\lambda \smallsetminus S^{n-k}(\mu)).$$

But now we have to work with Laurent series (or generalised Puiseux series) in t^{-1}, that is, series in t whose terms are bounded from above so that there always exists a maximal exponent. Then $\mathrm{Val}^{\mathbf{K}}(h(t))$ records the maximal exponent which occurs among the terms of $h(t)$. □

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Department of Mathematics, King's College London, Strand, London WC2R 2LS UK
E-mail address: `konstanze.rietsch@kcl.ac.uk`

Department of Mathematics, University of California at Berkeley, Berkeley, CA USA
E-mail address: `williams@math.berkeley.edu`