CLUSTER DUALITY AND MIRROR SYMMETRY FOR GRASSMANNIANS

K. RIETSCH AND L. WILLIAMS

Abstract. In this article we use the cluster structure on the Grassmannian and the combinatorics of plabic graphs to exhibit a new aspect of mirror symmetry for Grassmannians in terms of polytopes. For our $A$-model, we consider the Grassmannian $X = Gr_{n-k} \mathbb{C}^n$. The $B$-model is a Landau-Ginzburg model $(\breve{X}, W_q : \breve{X} \to \mathbb{C})$, where $\breve{X}$ is the complement of a particular anti-canonical divisor in a Langlands dual Grassmannian $\breve{X} = Gr_{k}((\mathbb{C}^\ast)^\ast)$, and the superpotential $W_q$ has a simple expression in terms of Plücker coordinates, see [MR13]. From a given plabic graph $G$ we obtain two coordinate systems: using work of Postnikov [Pos] and Talaska [Tal08] we have a positive chart $\Phi_G : (\mathbb{C}^\ast)^k(\mathbb{C}^{n-k}) \to X$ in our $A$-model, and using work of Scott [Sco06] we have a cluster chart $\Phi^\vee_G : (\mathbb{C}^\ast)^k(\mathbb{C}^{n-k}) \to \breve{X}$ in our $B$-model. To each positive chart $\Phi_G$ and choice of positive integer $r$, we associate a polytope $NO^r_G$, which is defined as the convex hull of a set of integer lattice points. This polytope is an example of a Newton-Okounkov polytope associated to the line bundle $O(r)$ on $X$. On the other hand, using the cluster chart $\Phi^\vee_G$ and the same positive integer $r$, we obtain a polytope $Q^r_G$ – described in terms of inequalities – by “tropicalizing” the composition $W_q \circ \Phi_G^\vee$. Our main result is that the polytopes $NO^r_G$ and $Q^r_G$ coincide.

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1. Introduction

Consider a polytope $P \subset \mathbb{R}^N$. There are naturally two, in a sense dual, perspectives on $P$. The first, which we call the “$A$-model perspective”, views $P$ as the polytope spanned as the convex hull of a collection of points. (For a lattice polytope we can naturally use the set of all lattice points contained in $P$.) On the other hand, $P$ can be viewed as the polytope defined by a collection of linear inequalities. We refer to this dual point of view as the “$B$-model perspective” on $P$.

These two perspectives on polytopes relate to mirror symmetry. Suppose the polytope $P$ is the moment polytope of a projective smooth toric Fano variety $X$, and $L$ denotes the associated $T$-equivariant ample line bundle on $X$. Then we may consider the “$A$-model perspective” on $P$ to be the construction of $P$ as the convex hull of the weights of a weight basis of $H^0(X, L)$ (which agree with the moment-map image of the torus-fixed points [Ati82, GS82]). For example if $X = \mathbb{C}P^2$ the standard basis of $H^0(X, L)$ is just given by the homogeneous coordinates $X, Y, Z$. Suppose $(t_1, t_2) \in T = (\mathbb{C}^\ast)^2$ acts by rescaling $X$ and $Y$. The integral points of the moment polytope $P \subset \mathbb{R}^2$ are then the weights $(1, 0), (0, 1)$ and $(0, 0)$. Following

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an idea of Okounkov [Oko96] these integral points can also be interpreted as image points of a valuation map, meaning here that they are read off the multi-degrees of the rational functions $x = X/Z$, $y = Y/Z$ and 1 which are the images of $X, Y, Z$ after embedding $H^0(\mathcal{X}, \mathcal{L})$ into $\mathbb{C}(x, y) = \mathbb{C}(\mathcal{X})$ (compare Example 1.1 and Section 6.1 in [LM09]). This point of view can be used to generalise the above construction of a polytope $\mathcal{P}$ associated to $\mathcal{X}$, to Grassmannians (and beyond). Thus for us the “A-model perspective” on the moment polytope of $\mathbb{C}P^2$ is that it is the convex hull of the lattice points $(1,0), (0,1)$ and $(0,0)$ in $\mathbb{R}^2$ obtained by looking at multi-degrees of global sections of the ample line bundle $\mathcal{L}$.

On the other side we have the mirror of the Fano variety $\mathcal{X}$, a Landau-Ginzburg model, which in the toric setting [Giv95] is a Laurent polynomial $L_q$ which is called the ‘superpotential’. From this mirror perspective the integer points defining the Newton polytope of $L_q$ naturally encode the inequalities defining $\mathcal{P}$. For example for $\mathbb{C}P^2$ the superpotential is $L_q = z_1 + z_2 + \frac{x}{z_1 z_2}$. Each summand of $L_q$ corresponds to a divisor in the compactification of $T = (\mathbb{C}^*)^2$ to $\mathbb{C}P^2$, and each such divisor gives rise to an inequality on $(a,b) \in \mathbb{Z}^2$ which characterises the exponents for which $x^a y^b Z \in \mathbb{H}^0(T, \mathcal{L})$ extends across the divisor as a regular function.

In our example the conditions are precisely

$$(1 \ 0) \begin{pmatrix} a \\ b \end{pmatrix} \geq 0, \quad (0 \ 1) \begin{pmatrix} a \\ b \end{pmatrix} \geq 0, \quad (-1 \ -1) \begin{pmatrix} a \\ b \end{pmatrix} \geq 0.$$ 

These inequalities can be interpreted as corresponding to the points $(1,0), (0,1), (-1,-1)$ of the Newton polytope of $L_q$, together with the choice of line bundle $\mathcal{L}$. This is in a sense the “B-model perspective” on the moment polytope $\mathcal{P}$.

We extend this point of view on mirror symmetry to the setting of Grassmannians, which are not toric but instead have a cluster structure [FZ02, Sco06]. The theme is that mirror symmetry can be seen as a way of switching perspectives between these two points of view on polytopes. Suppose $\mathcal{X} = Gr_{n-k}(\mathbb{C}^n)$, and set $N = \dim(\mathcal{X}) = k(n-k)$. As our A-model, we associate to an ample line bundle $\mathcal{L}_{r\omega_{n-k}}$ on $\mathcal{X}$ and a choice of transcendence basis of $\mathbb{C}(\mathcal{X})$ a polytope which is a Newton-Okounkov polytope, see [Oko96, Oko03, LM09, Kav11]. In our setting, the Newton-Okounkov polytope is constructed as the convex hull of a set of lattice points in $\mathbb{R}^N$. The integral points of the polytope parametrise a basis of the space of global sections of the line bundle. We focus our attention particularly on coordinate systems on $\mathcal{X}$ associated to plabic graphs, via a construction due to Postnikov [Pos] and Talaska [Tal08] using networks. We use these coordinate systems to associate to each plabic graph a set of $N$-dimensional polytopes, one for every ample line bundle $\mathcal{L}_{r\omega_{n-k}}$.

In mirror symmetry the ‘mirror’ of the Grassmannian $\mathcal{X}$ is a Landau-Ginzburg model, which is most naturally and simply described as the pair $(\hat{\mathcal{X}}, W_q)$, where $\hat{\mathcal{X}}$ is the complement of a particular anticanonical divisor in the Langlands dual Grassmannian $\mathcal{X}$, and $W_q$ is a regular function on $\hat{\mathcal{X}}$, see [MR13]. (We note that $(\hat{\mathcal{X}}, W_q)$ is isomorphic to a Lie-theoretic version introduced earlier in [Rie08] and the Langland-Ginzburg model can be used to recover the quantum cohomology ring of $\mathcal{X}$ [Rie08] as well as the Dubrovin-Givental connection [MR13]. In our B-model Grassmannian $\mathcal{X}$ we again consider coordinate systems associated to plabic graphs; this time they are cluster charts for the cluster algebra structure on the Grassmannian [FZ02, Sco06]. We ‘tropicalize’ the superpotential $W_q$ to associate a family of polytopes to such a cluster chart. Namely, the polytopes are obtained as follows. Consider the restriction of $W_q$ to a chosen cluster torus $T_\mathcal{C}$. This gives rise to a Laurent polynomial in cluster coordinates $(p_k)_{k \in \mathcal{C}}$. To obtain the polytope corresponding to the (co)weight $r\omega_{n-k}$ set $q = t^r$ and evaluate $W_q$ on $\mathbb{C}((t))$-valued points of $T_\mathcal{C}$, i.e. let the coordinates be $p_k \in \mathbb{C}((t))$. The condition on the cluster coordinates $p_k$, that the valuation of $W_q$ is nonnegative, defines a lattice polytope with vertices in a space $\mathbb{Z}^N$ of valuations of cluster coordinates. In this way the superpotential $W_q$ encodes precisely and succinctly a set of polytopes associated to the cluster chart, moreover it describes them in terms of linear inequalities as in the “B-model perspective”.

Our main result, Theorem 5.9, says that the polytopes associated to a plabic graph in the A-model agree with those in the B-model. This result in a sense identifies mirror dual families of torus charts in the A-model and B-model Grassmannian.
There is a further similarity to the toric setting. Each summand of the superpotential $W_q$ (as expressed in (5.1)) corresponds to a component of a particular anticanonical divisor $D$ in $X$, by the mirror theorem from [MR13] in the equivariant setting, see also [Ric15]. We therefore conjecture a refinement of our theorem relating the $A$-model and $B$-model coordinates associated to a plabic graph. Suppose $X_i \subset X$ is the divisor corresponding to the $i$-th summand $w_i$ of $W_q$. We obtain a subset of the $B$-model inequalities by tropicalising $w_i|T_C$ (for the cluster torus $T_C \subset \mathbb{X}^n$ associated to the plabic graph). We then conjecture that this subset of inequalities precisely describes necessary and sufficient conditions for a local section $s \in H^0(X \setminus D, L_{r\omega_{n-i}})$ to extend as regular section along the associated divisor $X_i$ (in terms of valuations, or multi-degrees, of $s$ calculated using the network coordinates on $X$ associated to the same plabic graph via the $A$-model). We plan to investigate this in a separate work.

We now give some more background and make connections with existing theories. Firstly, our work here is closely related to Fock and Goncharov’s notions of cluster $A$ and $X$-varieties. On the $A$-model side, our positive charts are instances of charts for a cluster $X$-variety, and accordingly they transform via [FG09, Equation 13]. On the $B$-model side, our cluster charts are instances of charts for a cluster $A$-variety, and accordingly they transform via [FG09, Equation 14]. We expect there to be a natural enhancement of this theory of Langlands dual cluster $X$ and $A$-varieties along the lines of our result here (involving the superpotential of one encoding Newton-Okounkov polytopes of a partial compactification of the other) which could be added to the conjectures in [GS13, Section 1.5.3].

Note that on the $B$-model side the idea of relating the superpotential to a polytope in the context of mirror symmetry for flag varieties has come up in previous works of Batyrev, Ciocan-Fontanine, Kim and van Straten [BCFKvS99, BCFKvS98], in connection with toric degenerations of partial flag varieties, and later also in [NNU10]. The construction of polytopes we use on the $B$-model side appeared in Berenstein and Kazhdan’s theory of geometric crystals [BK07], and afterwards [FG09, GHKK14]. Remarkably, the introduction of geometric crystals was unconnected to mirror symmetry, although the Lie-theoretic superpotential $\mathcal{F}_P$ for $G/P$ defined in [Rie08] already appeared (albeit with a different purpose) in [BK07, (1.14) and (2.1)]. In the case where $G/P$ is a Grassmannian, $\mathcal{F}_P$ was proved to be isomorphic to $W_q$ in [MR13]. Although this isomorphism exists, we focus here on the $W_q$ version of the superpotential and torus charts which are very special for Grassmannians, and there is no further mention of $\mathcal{F}_P$ or geometric crystals.

On the $A$-model side, in the full flag variety case, toric degenerations (as used to construct Laurent polynomial mirrors in [BCFKvS99]) were related to Newton-Okounkov polytopes in [Kav11]. Moreover Gelfand-Tsetlin and string polytopes were shown to arise in that setting [Oko98, Kav11]. To our knowledge, apart from Newton-Okounkov polytopes giving rise to toric degenerations [And13], there has not yet been any link made between Newton-Okounkov polytopes and mirror symmetry.

We mention also that the $G/B$ theory of geometric crystals has been extended by the work of Goncharov and Shen [GS13] to so-called configurations of flags, with associated theorems about bases parametrised by tropical points, and conjectures about mirror symmetry involving these spaces and superpotentials on them. More generally Gross, Hacking, Keel and Kontsevich [GHKK14] introduced a superpotential in a wider setting of ‘cluster varieties with principal coefficients’, whose tropicalisation can tautologically parametrise regular functions on a ‘mirror’ cluster variety, see also [Mag15]. It would be interesting to understand how the mirror $\langle \mathbb{X}, W_q \rangle$ relates to this theory.

This project originated out of the observation that Gelfand-Tsetlin polytopes appear naturally in both the $B$-model and $A$-model of a Grassmannian, using a transcendence basis as input data. It also arose out of the wish to better understand the superpotential for Grassmannians from [MR13].

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2. Notation for the Grassmannian

For our $A$-model let $X$ be the Grassmannian of $(n - k)$-planes in $\mathbb{C}^n$. We will denote its dimension by $N = k(n - k)$. An element of $X$ can be represented as the column-span of a full-rank $(n - k) \times (n - k)$ matrix modulo right multiplication by nonsingular $(n - k) \times (n - k)$ matrices. Let $\{[n]_{n-k}\}$ be the set of all $(n - k)$-element subsets of $[n] = \{1, \ldots, n\}$. For $J \in \{[n]_{n-k}\}$, let $P_J(A)$ denote the maximal minor of an $n \times (n - k)$ matrix $A$ located in the row set $J$. The map $A \mapsto (P_J(A))$, where $J$ ranges over $\{[n]_{n-k}\}$, induces the Plücker embedding $X \rightarrow \mathbb{P}^{\binom{n}{n-k}-1}$, and the $P_J$ are called Plücker coordinates.

We also think of $X$ as a homogeneous space for the group $GL_n(\mathbb{C})$, acting on the left. We fix the standard pinning of $GL_n$ consisting of upper and lower-triangular Borel subgroups $B_+, B_-$, maximal torus $T$ in the intersection, and simple root subgroups $x_i(t)$ and $y_i(t)$ given by exponentiating the standard upper and lower-triangular Chevalley generators $e_i, f_i$ with $i = 1, \ldots, n-1$. We denote the Lie algebra of $T$ by $\mathfrak{h}$ and we have fundamental weights $\omega_i \in \mathfrak{h}^*$ as well as simple roots $\alpha_i \in \mathfrak{h}^*$. For our $X = Gr_{n-k}(\mathbb{C}^n)$ there is a natural identification between $H^2(X)$ and the subspace of $\mathfrak{h}^*$ spanned by $\omega_{n-k}$, under which $\omega_{n-k}$ is identified with the first Chern class $c_1(O(1))$ of the ample line bundle on $X$ associated to the Plücker embedding. In other words $\omega_{n-k}$ corresponds to the hyperplane class in the Plücker embedding.

Let $(\mathbb{C}^n)^\ast$ denote the vector space which is dual to $\mathbb{C}^n$. We think of elements of $\mathbb{C}^n$ as column vectors and of elements of $(\mathbb{C}^n)^\ast$ as row vectors. We then let $\tilde{X} = Gr_k((\mathbb{C}^n)^\ast)$ be the ‘mirror dual’ Grassmannian of $k$-planes in the vector space $(\mathbb{C}^n)^\ast$. An element of $\tilde{X}$ can be represented as the row-span of a full-rank $k \times n$ matrix $M$. This new Grassmannian is considered to be a homogeneous space via a right action by the Langlands dual group $GL_k^\vee(\mathbb{C})$ (which is isomorphic to $GL_n(\mathbb{C})$, but we distinguish the two groups nevertheless). For this group we use the same notations as introduced in the preceding paragraph for $GL_n$, but with an added superscript $\vee$. In the Langlands dual setting the $r\omega_{n-k}$ that corresponded to a line bundle on $X$ can now be considered (as $(r\omega_{n-k})^\vee$) to represent a one-parameter subgroup of $T^\vee$, or element of $T^\vee(C((t)))$, if $t$ is the parameter. Note that the Plücker coordinates of $\tilde{X}$ are naturally parametrised by $\{[n]_k\}$; for every $k$-subset $I$ in $[n]$ the Plücker coordinate $p_I$ is associated to the $k \times k$ minor of $M$ with column set given by $I$.

We can also index Plücker coordinates, both of $X$ and $\tilde{X}$, conveniently using Young diagrams. Let $\mathcal{P}_{k,n}$ denote the set of Young diagrams fitting in an $(n-k) \times k$ rectangle. There is a natural bijection between $\mathcal{P}_{k,n}$ and $\{[n]_k\}$, defined as follows. Let $\mu$ be an element of $\mathcal{P}_{k,n}$, justified so that its top-left corner coincides with the top-left corner of the $(n-k) \times k$ rectangle. The south-east border of $\mu$ is then cut out by a path from the northeast to southwest corner of the rectangle, which consists of $k$ west steps and $(n-k)$ south steps. After labeling the $n$ steps by the numbers $\{1, \ldots, n\}$, we map $\mu$ to the labels of the south steps. This gives a bijection from $\mathcal{P}_{k,n}$ to $\{[n]_k\}$. If we use the labels of the west steps instead, we get a bijection from $\mathcal{P}_{k,n}$ to $\{[n]_k\}$. Therefore the elements of $\mathcal{P}_{k,n}$ index the Plücker coordinates $p_\mu$ on $X$ and simultaneously the Plücker coordinates on $\tilde{X}$, which we denote by $p_\mu$.

The totally positive Grassmannian $X_{>0} := Gr_{n-k}(\mathbb{R}^n)$ is the subset of the Grassmannian of $(n-k)$-planes in $\mathbb{R}^n$ consisting of elements such that all Plücker coordinates are strictly positive (equivalently, all Plücker coordinates are strictly negative). This definition is equivalent to Lusztig’s original definition [Lus94] of the totally positive part of a generalized partial flag variety $G/P$ applied in the Grassmannian case.

3. Plabic graphs and the Grassmannian

In this section we review Postnikov’s notion of plabic graphs [Pos], which we will then use to define positive charts and cluster charts for the Grassmannian.

**Definition 3.1.** A plabic graph\(^2\) is an undirected graph $G$ drawn inside a disk (considered modulo homotopy) with $n$ boundary vertices on the boundary of the disk, labeled $b_1, \ldots, b_n$ in clockwise order, as well as some colored internal vertices. These internal vertices are strictly inside the disk and are colored in black and white. Moreover, each boundary vertex $b_i$ in $G$ is incident to a single edge.

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\(^2\)"Plabic" stands for “planar bi-colored.”
See Figure 1 for an example of a plabic graph.

\[\begin{array}{c}
\begin{array}{c}
\text{\textbullet} \quad 1 \\
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array}\]

\textbf{Figure 1.} A plabic graph

There is a natural set of local transformations (moves) of plabic graphs, which we now describe. Note that we will always assume that a plabic graph $G$ has no isolated components (i.e. every connected component contains at least one boundary vertex). We will also assume that $G$ is \textit{leafless}, i.e. if $G$ has an internal vertex of degree 1, then that vertex must be adjacent to a boundary vertex.

(M1) SQUARE MOVE. If a plabic graph has a square formed by four trivalent vertices whose colors alternate, then we can switch the colors of these four vertices.

\[\begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array}\]

\textbf{Figure 2.} Square move

(M2) UNICOLORED EDGE CONTRACTION/UNCONTRACTION. If a plabic graph contains an edge with two vertices of the same color, then we can contract this edge into a single vertex with the same color. We can also uncontract a vertex into an edge with vertices of the same color.

\[\begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array}\]

\textbf{Figure 3.} Unicolored edge contraction

(M3) MIDDLE VERTEX INSERTION/REMOVAL. If a plabic graph contains a vertex of degree 2, then we can remove this vertex and glue the incident edges together; on the other hand, we can always insert a vertex (of any color) in the middle of any edge.

\[\begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array}\]

\textbf{Figure 4.} Middle vertex insertion/ removal

(R1) PARALLEL EDGE REDUCTION. If a plabic graph contains two trivalent vertices of different colors connected by a pair of parallel edges, then we can remove these vertices and edges, and glue the remaining pair of edges together.

\textbf{Definition 3.2.} Two plabic graphs are called \textit{move-equivalent} if they can be obtained from each other by moves (M1)-(M3). The \textit{move-equivalence class} of a given plabic graph $G$ is the set of all plabic graphs which are move-equivalent to $G$. A leafless plabic graph without isolated components is called \textit{reduced} if there is no graph in its move-equivalence class to which we can apply (R1).
Definition 3.3. Let $G$ be a reduced plabic graph as above with boundary vertices $b_1, \ldots, b_n$. The trip $T_i$ from $b_i$ is the path obtained by starting from $b_i$ and traveling along edges of $G$ according to the rule that each time we reach an internal black vertex we turn (maximally) right, and each time we reach an internal white vertex we turn (maximally) left. This trip ends at some boundary vertex $b_{\pi(i)}$. In this way we associate a trip permutation $\pi_G = (\pi(1), \ldots, \pi(n))$ to each reduced plabic graph $G$, and we say that $G$ has type $\pi_G$.

As an example, the trip permutation associated to the reduced plabic graph in Figure 1 is $(3, 4, 5, 1, 2)$.

Remark 3.4. Let $\pi_{k,n} = (n-k+1, n-k+2, \ldots, n, 1, 2, \ldots, n-k)$. In this paper we will be particularly concerned with reduced plabic graphs whose trip permutation is $\pi_{k,n}$. Note that the trip permutation of a plabic graph is preserved by the local moves (M1)-(M3), but not by (R1). For reduced plabic graphs the converse holds, namely it follows from [Pos, Theorem 13.4] that any two reduced plabic graphs with trip permutation $\pi_{k,n}$ are move-equivalent.

Next we use the trips to label each face of a reduced plabic graph by a partition.

Definition 3.5. Let $G$ be a reduced plabic graph of type $\pi_{k,n}$. Note that each trip $T_i$ partitions the disk containing $G$ into two parts: the part on the left of $T_i$, and the part on the right. Place an $i$ in each face of $G$ which is to the left of $T_i$. After doing this for all $1 \leq i \leq n$, each face will contain an $(n-k)$-element subset of $\{1, 2, \ldots, n\}$. Finally we identify that $(n-k)$-element subset with the corresponding Young diagram in $P_{k,n}$. We let $\overline{\mathcal{P}}_G$ denote the set of Young diagrams inside $P_{k,n}$ associated in this way to $G$. Note that $\overline{\mathcal{P}}_G$ always contains $\emptyset$ as the partition labeling a boundary region. We therefore set $P_G := \overline{\mathcal{P}}_G \setminus \{\emptyset\}$.

Figure 6 shows the labeling of each face of our running example by a Young diagram in $P_{k,n}$ (here $k = 3$ and $n = 5$).

Figure 6. A plabic graph with trip permutation $\pi_{3,5}$, with faces labeled by Young diagrams in $P_{3,5}$. Here $P_G = \{\emptyset, \ldots, \{\}\}$.}

3.1. Positive charts for $X$ from plabic graphs. We now fix a reduced plabic graph $G$ of type $\pi_{k,n}$. We will explain how to construct from it a positive chart for $X = Gr_{n-k}(\mathbb{C}^n)$. First we need to introduce perfect orientations and flows.

Definition 3.6. A perfect orientation $\mathcal{O}$ of a plabic graph $G$ is a choice of orientation of each of its edges such that each black internal vertex $u$ is incident to exactly one edge directed away from $u$; and each white internal vertex $v$ is incident to exactly one edge directed towards $v$. A plabic graph is called perfectly orientable if it admits a perfect orientation. The source set $I_{\mathcal{O}} \subset [n]$ of a perfect orientation $\mathcal{O}$ is the set of $i$ for which $b_i$ is a source of $\mathcal{O}$ (considered as a directed graph). Similarly, if $j \in T_{\mathcal{O}} := [n] - I_{\mathcal{O}}$, then $b_j$ is a sink of $\mathcal{O}$. 
Each reduced plabic graph $G$ of type $\pi_{k,n}$ will have precisely $N+1$ faces, where $N = k(n-k)$, and each perfect orientation of $G$ will have a source set of size precisely $n-k$ [Pos]. Let $\overline{G} = \{x_\mu \mid \mu \in \overline{P}_G\}$ be a set of parameters which are indexed by the Young diagrams $\mu$ labeling faces of $G$. So $|\overline{G}| = N+1$.

For $J$ a set of boundary vertices with $|J| = |I_O|$, a flow $F$ from $I_O$ to $J$ is a collection of self-avoiding walks and self-avoiding cycles in $O$, all pairwise vertex-disjoint, such that the sources of these walks are $I_O - (I_O \cap J)$ and the destinations are $J - (I_O \cap J)$.

Note that each self-avoiding walk (respectively, cycle) in $O$ partitions the faces of $G$ into those which are on the left and those which are on the right of the walk (respectively, cycle). We define the weight of each such walk or cycle to be the product of parameters $x_\mu$, where $\mu$ ranges over all face labels to the left of the walk or cycle. And we define the weight $\text{wt}(F)$ of a flow $F$ to be the product of the weights of all walks and cycles contained in the flow.

Fix a perfect orientation $O$ of a reduced plabic graph $G$ of type $\pi_{k,n}$. Given $J$ a $(n-k)$-element subset of $\{1, 2, \ldots, n\}$, we define
\begin{equation}
P_J = \sum_F \text{wt}(F),
\end{equation}
where $F$ ranges over all flows from $I_O$ to $J$. \footnote{Note that we are abusing notation here, in that we previously used $P_J$ to denote Plücker coordinates. However, Theorem 3.8 proves that the $P_J$ defined in (3.1) are in fact formulas for the Plücker coordinates of a point in the Grassmannian $X$.}

**Example 3.7.** We continue with our running example from Figure 7. There are two flows $F$ from $I_O$ to $\{2, 4\}$, and $P_{\{2,4\}} = x_\mu x_\nu x_\tau x_\upsilon x_\mu x_\nu x_\upsilon x_\tau x_\mu x_\nu x_\upsilon x_\tau$. There is one flow from $I_O$ to $\{3, 4\}$, and $P_{\{3,4\}} = x_\mu x_\nu x_\upsilon x_\tau x_\mu x_\nu x_\upsilon x_\tau x_\mu x_\nu x_\upsilon x_\tau x_\mu x_\nu x_\upsilon x_\tau x_\mu x_\nu x_\upsilon x_\tau x_\mu x_\nu x_\upsilon x_\tau$.

Theorem 3.8 below follows from the work of Postnikov [Pos] and Talaska [Tal08]. The statement that we give is essentially the one from [Tal08].

**Theorem 3.8.** [Tal08, Theorem 1.1] Let $G$ be a reduced plabic graph of type $\pi_{k,n}$, and choose a perfect orientation $O$. Recall that $G$ has precisely $N+1$ faces, where $N = k(n-k)$, one of which is labeled by the empty partition, $\varnothing$, and the remaining of which are labeled by the set $P_G \subset P_{k,n}$. If we let the parameters $x_\mu \in \overline{G}$ vary over all positive real numbers, subject to the condition that $\prod_\mu x_\mu = 1$, then we get an injective map
\[ \Phi_G : (\mathbb{R}_{>0})^{P_G} \to \mathbb{P}(\mathbb{C}^{n-1}) \]
defined by
\begin{equation}
\{x_\mu\}_{\mu \in P_G} \mapsto \{P_J\}_{J \in \binom{[n]}{k-1}},
\end{equation}
where any occurrence of $x_\varnothing$ has been replaced by $(\prod_{\mu \in \varnothing} x_\mu)^{-1}$ in (3.1). Moreover, the image of $\Phi_G$ is precisely the totally positive Grassmannian $X_{k,n} = Gr_{n-k}(\mathbb{R}^n)$ in its Plücker embedding.

**Remark 3.9.** Given a reduced plabic graph $G$, the choice of perfect orientation $O$ only affects the formulas for Plücker coordinates $P_J$ up to a common scalar. In the construction of $\Phi_G$ we will typically choose an acyclic perfect orientation $O$ which has source set $I_O = \{1, \ldots, n-k\}$, as in the example in Figure 7. (Lemma 7.2 implies that such a perfect orientation exists.) It is convenient to work with such a perfect orientation because then we have $P_{\{1, \ldots, n-k\}} = 1$.
Remark 3.10. In this paper we will use the map from (3.2) but allow our parameters to take on nonzero complex values. Abusing notation, we also refer to this map as $\Phi_G$, and call it a positive chart. In fact $\Phi_G$ has a rational inverse, as follows from [Pos, Section 7], see also [Tal11]. The positive chart $\Phi_G$ is therefore an injective map onto a dense open subset of $X = Gr_{n-k}(\mathbb{C}^n)$.

Definition 3.11. Let $\Delta_G$ denote the set of $N$ parameters,

$$\Delta_G := \{x_\mu \mid \mu \in \mathcal{P}_G\} = \Delta_G \setminus \{x_\emptyset\}.$$

By Remark 3.10 we can view the parameters $\Delta_G$ as rational functions on $X$ which restrict to coordinates on the open torus $\Phi_G((\mathbb{C}^*)^{\mathcal{P}_G})$ in $X$. Therefore we can think of $\Delta_G$ as a transcendence basis of $\mathbb{C}(X)$.

Example 3.12. We continue with our running example from Figure 7. The formulas for the Plücker coordinates are:

$$
\begin{align*}
P_{(1,2)} &= 1 \\
P_{(1,3)} &= x^{1,2} \\
P_{(1,4)} &= x^{1,3} \\
P_{(1,5)} &= x^{1,4} \\
P_{(2,3)} &= x^{2,3} \\
P_{(2,4)} &= x^{2,4} (1 + x^{3,5}) \\
P_{(2,5)} &= x^{2,5} (1 + x^{3,4} + x^{4,5}) \\
P_{(3,4)} &= x^{3,4} (1 + x^{4,5}) \\
P_{(3,5)} &= x^{3,5} (1 + x^{4,5}) \\
P_{(4,5)} &= x^{4,5} (1 + x^{4,5}) \\
\end{align*}
$$

This gives us a positive chart $\Phi_G : (\mathbb{C}^*)^{\mathcal{P}_G} \to \mathbb{P}^9$ for the Grassmannian $Gr_2(\mathbb{C}^5)$. Note that $x_\emptyset$ does not appear in the above formulas (and no substitution was necessary) since the region labeled by $\emptyset$ is to the right of every path from $I_G$ to $[n] \setminus I_G$. It is not hard to invert the map $\Phi_G$ and express the $x_\mu$ as rational functions in the Plücker coordinates, thus describing $\Delta_G$ as a subset of $\mathbb{C}(X)$.

Definition 3.13. We say that a monomial $\prod_{\mu} x_\mu^{a_\mu}$ in a polynomial is strongly minimal if for every other monomial $\prod_{\mu} x_\mu^{b_\mu}$ occurring in the polynomial, we have $a_\mu \leq b_\mu$ for all $\mu$. Consider the case of a Plücker coordinate $P_J$, a plabic graph $G$ and perfect orientation with source set $\{1, \ldots, n-k\}$. Then $P_J$ expressed in terms of $\Delta_G$ is a sum over flows from $\{1, \ldots, n-k\}$ to $J$. We call a flow from $\{1, \ldots, n-k\}$ to $J$ strongly minimal if its weight monomial is strongly minimal in $P_J$.

Remark 3.14. In Example 3.12, each Plücker coordinate $P_{(i,j)}$ has a strongly minimal term. See Remark 6.4 for a more general statement.

3.2. Cluster charts for $\hat{X}$ from plabic graphs. In this section we again fix a reduced plabic graph $G$ of type $\pi_{k,n}$. But now we will use it to construct a cluster chart for $\hat{X}^0 \subset \hat{X} = Gr_k((\mathbb{C}^n)^*)$.

Recall from Definition 3.5 that we have a labeling of each face of $G$ by some Young diagram in $\mathcal{P}_{k,n}$. We now interpret each Young diagram in $\mathcal{P}_{k,n}$ as a $k$-element subset of $\{1,2,\ldots,n\}$, see Section 2. It follows from [Sco06] and [Pos] that the collection $\hat{C}_G$ of Plücker coordinates indexed by these $k$-element subsets is a cluster for the cluster algebra associated to the homogeneous coordinate ring of the affine cone over $\hat{X}$. In particular, these Plücker coordinates are algebraically independent, and any Plücker coordinate for $\hat{X}$ can be written as a Laurent polynomial in the variables from $\hat{C}_G$ with nonnegative coefficients. Choosing the normalization of Plücker coordinates on $\hat{X}^0$ such that $p_\emptyset = p_{\{1,\ldots,k\}} = 1$, we get a map

$$\Phi_G^0 : (\mathbb{C}^*)^{\mathcal{P}_G} \to \hat{X}^0 \subset \hat{X}$$
which we call a \textit{cluster chart}, which satisfies $p_\nu(\Phi^X_G((t_\mu)_\mu)) = t_\nu$ for $\nu \in \mathcal{P}_G$. Here $\mathcal{P}_G$ is as in Definition 3.5.

We let

\[ C_G = \left\{ \frac{p_\mu}{p_\nu} \mid p_\mu \in \overline{C}_G \setminus \{p_\nu\} \right\}. \]

When it is clear that we are setting $p_\emptyset = 1$ we may write $C_G = \{p_\mu \mid \mu \in \mathcal{P}_G\}$.

**Example 3.15.** We continue our example from Figure 6. The Plücker coordinates labeling the faces of $G$ are $\overline{C}_G = \{p_{(1,2,3)}, p_{(1,2,4)}, p_{(1,3,4)}, p_{(2,3,4)}, p_{(1,2,5)}, p_{(1,4,5)}, p_{(3,4,5)}\}$.

\section{The $A$-model (Newton-Okounkov) Polytope $NO^*_G$}

**Definition 4.1.** Suppose $X$ is a Grassmannian and we have Plücker coordinates on $X$. Suppose furthermore that $\Delta$ is a transcendence basis of $\mathbb{C}(X)$. We call $\Delta$ a positive transcendence basis for the Grassmannian $X$ if each Plücker coordinate is a rational function in the elements of $\Delta$ with coefficients which are all nonnegative.

**Example 4.2.** The set $\Delta_G$ defined in Section 3.1, Definition 3.11, and the cluster $C_G$ defined in Section 3.2 are positive transcendence bases for $X$ and $\mathcal{X}$, respectively.

In the $A$-model, for $X$, we choose a reduced plabic graph $G$ with trip permutation $\pi_{k,n}$ and use the corresponding positive transcendence basis $\Delta_G$ to define a \textit{Newton-Okounkov type polytope} associated to ample line bundle on $X$. In order to do this we need to define a valuation on $\mathbb{C}(X)$.

**Definition 4.3.** Given a reduced plabic graph $G$ of type $\pi_{k,n}$, we fix a total order $<$ on the parameters $x_\mu \in \Delta_G$, and extend this to a term order on monomials in the $\Delta_G$ which is lexicographic with respect to $<$. More precisely, suppose we write $\Delta_G = \{x_{\mu_1}, \ldots, x_{\mu_N}\}$ and choose the total order $x_{\mu_1} < x_{\mu_2} < \cdots < x_{\mu_N}$. Then to compare two monomials in lexicographic order, we first compare exponents of $x_{\mu_1}$, and in case of equality we compare exponents of $x_{\mu_2}$, and so forth. We use this to define a map

\[ \text{val}_G : \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^{\mathcal{P}_G} \]

as follows. Let $f$ be a polynomial in the Plücker coordinates for $X = Gr_{n-k}(\mathbb{C}^n)$. We use Theorem 3.8 and Definition 3.11 to write $f$ uniquely as a Laurent polynomial in $\Delta_G$. We then choose the lexicographically minimal term $\prod_{\mu \in \mathcal{P}_G} x_{\mu}^{a_\mu}$ and define $\text{val}_G(f)$ to be the associated exponent vector $(a_\mu)_\mu \in \mathbb{Z}^{\mathcal{P}_G}$. For $(f/g) \in \mathbb{C}(X) \setminus \{0\}$ (here $f, g \in \mathbb{C}[X]$ are polynomials in the Plücker coordinates), we define $\text{val}_G(f/g) = \text{val}_G(f) - \text{val}_G(g)$.

**Remark 4.4.** When $G$ is understood, we will often write $\text{val}$ in place of $\text{val}_G$.

We fix $\lambda = r\omega_{n-k}$ with $r \in \mathbb{Z}_{>0}$, or equivalently the ample line bundle $L_r := L_{r\omega_{n-k}}$ on $X$. The global sections $H^0(X, L_r) \cong V_{r\omega_{n-k}}$ embed into $\mathbb{C}(X)$ with image the linear span

\[ L_r = \left\{ \frac{M}{P_{(r_{1,\ldots,n-k})}} \mid M \in \mathcal{M}_r \right\}, \]

where $\mathcal{M}_r$ is the set of all degree $r$ monomials in the Plücker coordinates.

**Definition 4.5.** We define the $A$-model polytope $NO^*_G \subset \mathbb{Z}^{\mathcal{P}_G}$ to be the convex hull of the points obtained by taking the image of $L_r$ under the valuation $\text{val}_G$.

**Remark 4.6.** It is not hard to see that the polytope $NO^*_G$ is the $r$th dilation of $NO^*_G$, see Lemma 6.2. Moreover $NO^*_G$ is an example of a \textit{Newton-Okounkov polytope} [Oko96, Oko98, KK12, LM09]. Note that although we used a total order $<$ on the parameters in order to define $NO^*_G$, the polytope turns out not to depend on the choice of total order in our setting, and that choice will not enter into our proofs.

**Example 4.7.** We now take $r = 1$ and compute the polytope $NO^*_G$ associated to Example 3.12. Computing the valuation of each Plücker coordinate we get the result shown in Table 1.

Therefore the polytope $NO^*_G$ is the convex hull of the set of points $\{(0,0,0,0,0,0), (1,0,0,0,0,0), (1,1,0,0,0,0), (1,1,1,0,0,0), (1,0,0,1,0,0), (1,1,0,1,0,0), (1,1,1,1,0,0), (2,1,0,1,1,0), (2,1,1,1,1,0)\}$.
5. The $B$-model (superpotential) polytope $Q'^G$

Let $\hat{X} = Gr_k((\mathbb{C}^n)^*)$ be the mirror Grassmannian. We begin by recalling the definition from [MR13] of the superpotential $W_q$ and of the subvariety $\hat{X}^o \subset \hat{X}$ where $W_q$ is regular.

**Definition 5.1.** Set $J_i := [i + 1, i + k]$, interpreted cyclically as a subset of $[n]$. Then we define $\hat{X}^o$ to be the complement of the divisor $D = \bigcup_{i=1}^{n-k} \{p_{J_i} = 0\}$ associated to the Plücker coordinates indexed by the $J_i$,

$$\hat{X}^o := \hat{X} \setminus D = \{ x \in \hat{X} | p_{J_i}(x) \neq 0 \; \forall i \in [n]\}.$$ 

Furthermore set $J_i^+ := [i + 1, i + k - 1] \cup \{i + k + 1\}$. The superpotential $W_q$ is the regular function on $\hat{X}^o$ defined by

$$W_q := \sum_{m=1}^{n-k-1} \frac{p_{J_i}}{p_{J_m}} + q \frac{p_{J_{i+k}}} {p_{J_{n-k}}} + \sum_{m=n-k+1}^{n} \frac{p_{J_i}}{p_{J_m}},$$

depending on the extra parameter $q$. We may also index the Plücker coordinates by Young diagrams, so let $\mu_i$ be the Young diagram corresponding to $J_i$. Then when $i \leq n - k$, we have that $\mu_i$ is the rectangular $i \times k$ Young diagram, and when $i \geq n - k$, it is the rectangular $(n-k) \times (n-i)$ Young diagram. For $i = n - k$, the Young diagram $\mu_i^+$ associated to $J_i^+$ is the unique diagram in $P_{k,n}$ obtained by adding a box to $\mu_i$. For $i = n - k$ it is the rectangular $(n-k-1) \times (k-1)$ Young diagram obtained from $\mu_{n-k}$ by removing a rim hook.

**Remark 5.2.** Let $\mathcal{A}$ denote the cluster algebra associated to the homogeneous coordinate ring of (the affine cone over) $\hat{X}^o$ [Sco06] embedded via the Plücker embedding. It follows from (5.1) that $W_q$ lies in $\mathcal{A}[q]$: (5.1) has numerators which are Plücker coordinates (hence cluster variables), and the only Plücker coordinates which occur in the denominator are frozen variables. However, the expression (5.1) is not a cluster expansion, as the Plücker coordinates in (5.1) are not algebraically independent and hence cannot be contained in a single cluster. Proposition 5.3 below gives the cluster expansion for $W_q$ in terms of the ‘rectangles cluster’.

<table>
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<tr>
<th>Plücker</th>
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Table 1. The valuations $val_G(P_J)$ of the Plücker coordinates
Proposition 5.3 ([MR13]). If we let $i \times j$ denote the Young diagram which is a rectangle with $i$ rows and $j$ columns, then on the subset of $\tilde{X}^o$ where all $p_{i\times j} \neq 0$, the superpotential $W_q$ equals

$$W_q = \frac{p_{1\times 1}}{p_\emptyset} + \sum_{i=2}^{n-k} \sum_{j=1}^{k} \frac{p_{i\times j}}{p_{i-1\times j}} \frac{p(i-2)\times(j-1)}{p(i-1)\times(j-1)} + q \frac{p(n-k-1)\times(k-1)}{p(n-k)\times k} + \sum_{i=1}^{n-k} \sum_{j=2}^{k} \frac{p_{i\times j}}{p_{i-1\times j}} \frac{p(i-1)\times(j-2)}{p(i-1)\times(j-1)} p_{i\times(j-1)}.$$  

Here of course if $i$ or $j$ equals 0, then $p_{i\times j} = p_\emptyset$.

The Laurent polynomial (5.2) can be encoded in a diagram (shown in Figure 8 for $k = 3$ and $n = 5$). Namely it is the Laurent polynomial obtained by summing over all the arrows the Laurent monomials

![Figure 8. The diagram defining the superpotential.](image)

obtained by dividing the expression at the head by the expression at the tail of the arrow. So in this example, we have

$$W_q = \frac{p_\emptyset}{p_\emptyset} + \frac{p_{p_3}}{p_{p_3}} + \frac{p_{p_3}}{p_{p_3}} + \frac{p_{p_3}}{p_{p_3}} + \frac{p_{p_3}}{p_{p_3}} + \frac{p_{p_3}}{p_{p_3}} + \frac{p_{p_3}}{p_{p_3}} + \frac{p_{p_3}}{p_{p_3}} + \frac{p_{p_3}}{p_{p_3}} + q \frac{p_{p_3}}{p_{p_3}},$$

where the $p_{i\times j}$ are the Plücker coordinates for $\tilde{X}$. We note that the quiver underlying the diagram was introduced by [BCFKvS00] to encode the EHX Laurent polynomial superpotential [EHX97] associated to a Grassmannian (in the vein of Givental’s quiver for the full flag variety [Giv97]). It was related to the rectangles cluster later in [MR13]. For arbitrary $k$ and $n$ the quiver is built from a grid of size $(n-k) \times k$, by appending an extra node at the top left and an extra node at the bottom right, labeled 1 and $q$, respectively. Horizontal arrows are oriented to the right, and vertical arrows are oriented down. The numerators in the nodes in the $i$th row of the grid are labeled by the rectangles of sizes $i \times j$ where $j$ ranges from 1 to $k$. The denominator of a given node is obtained from the rectangle in its numerator by removing a rim hook from the rectangle.

Definition 5.4. Let $\mathcal{K} = \mathbb{C}(t)).$ We may consider $\mathcal{K}$-valued points of the Grassmannian $\tilde{X}$ and its open subvariety $\tilde{X}^o$, denoted $\tilde{X}(\mathcal{K})$ and $\tilde{X}^o(\mathcal{K})$, respectively. If $\mathcal{K}_+ \subset \mathbb{R}(t))$ is defined to be the semifield of Laurent series with leading coefficient in $\mathbb{R}_{>0}$, then $\tilde{X}(\mathcal{K}_+)$ makes sense (compare [Lus94]) as the analogue of the totally positive Grassmannian in this setting. We also write $\tilde{X}^o(\mathcal{K}_+)$ instead of $\tilde{X}(\mathcal{K}_+)$, since the $p_{i\times j}$, by virtue of having leading coefficient in $\mathbb{R}_{>0}$, are automatically nonzero on points of $\tilde{X}(\mathcal{K}_+)$.  

Definition 5.5. Let $r$ be a positive integer. We have the superpotential $W_{t^r} : \tilde{X}^o(\mathbb{C}(t)) \to \mathbb{C}(t))$ defined by replacing $q$ by $t^r$ and working over the field $\mathcal{K} = \mathbb{C}(t))$.  

Remark 5.6. Let $G$ be a reduced plabic graph of type $\pi_{k,n}$. Since $W_{t^r}$ lies in $A[t^r]$ (see Remark 5.2), when one rewrites $W_{t^r}$ in terms of an arbitrary cluster $\mathcal{C}_G$, the resulting expression is a Laurent polynomial with coefficients in $\mathcal{K}_+$. The superpotential $W_{t^r}$ restricts to a map $W_{t^r}^* : \tilde{X}^o(\mathcal{K}_+) \to \mathcal{K}_+$.  

We now use the superpotential $W_{t^r}$ and a choice of plabic graph $G$ to define a polytope $Q_G^r \subset \mathbb{R}^N$.  

Definition 5.7. Let $G$ be a reduced plabic graph of type $\pi_{k,n}$ and let $\mathcal{P}_G$ denote the set of nonempty partitions in $\mathcal{P}_{k,n}$ which index the cluster variables $\mathcal{C}_G$. We normalize the Plücker coordinates by setting $p_\emptyset = 1$. Choose a positive integer $r$, and write the superpotential $W_{t^r}$ in terms of the Plücker coordinates
from $C_G$; by Remark 5.6, the resulting expression $W_\nu (C_G)$ is a Laurent polynomial with coefficients in the positive part $K_+$ of $\mathbb{C}(t)$.

Given a Laurent polynomial $h$ in $C_G$ with coefficients in $K_+$, the tropicalization $\text{Trop}(h): \mathbb{Z}^{P_G} \rightarrow \mathbb{Z}$ is a piecewise linear function which is defined as follows. First we set

$$\text{Trop} \left( c(t) \prod_{\mu \in C_G} r_{\mu}^{m_\mu} \right) (v) = m + \sum_{\mu \in P_G} a_{\mu} v_{\mu}$$

for each Laurent monomial $c(t) \prod_{\mu \in C_G} r_{\mu}^{m_\mu}$ with $c(t) = a_m t^m + \sum_{j > m} a_j t^j$ with $a_m > 0$. Then we require that if $f$ and $g$ are two Laurent polynomials in the variables $C_G$ with coefficients in $K_+$, then $\text{Trop}(f + g) = \min(\text{Trop}(f), \text{Trop}(g))$.

Applying this construction to the superpotential yields the map

$$\text{Trop}(W_\nu (C_G)) : \mathbb{Z}^{P_G} \rightarrow \mathbb{Z}.$$

The inequality

$$\text{Trop}(W_\nu (C_G)) \geq 0$$

then defines a polytope $Q^r_G$ in $\mathbb{Z}^{P_G}$, which we call the B-model polytope associated to $G$ and $r \omega_{n-k}$. Note that our polytope is defined by a set of inequalities, one for each Laurent monomial $f$ that occurs in $W_\nu (C_G)$: namely, $\text{Trop}(f) \geq 0$.

**Example 5.8.** Let $G$ be the graph of Figure 1. The superpotential $W_\nu$ from (5.3) is already written in terms of the corresponding cluster $C_G$. We obtain the following inequalities:

\[
\begin{align*}
0 & \leq v_{0} - v_{0} \\
0 & \leq v_{1} - v_{0} \\
0 & \leq v_{1} + v_{2} - t v_{1} \\
0 & \leq v_{0} - v_{2} \\
0 & \leq v_{0} - t v_{0} \\
0 & \leq v_{1} + v_{2} - t v_{1} \\
0 & \leq r + v_{0} - v_{0}
\end{align*}
\]

These inequalities define a polytope $Q^r_G$. One may check that when $r = 1$, the polytope $Q^r_G$ defined by these inequalities coincides with the polytope $NO^1_G$ from Example 4.7.

Our main result is the following.

**Theorem 5.9.** Let $G$ be a reduced plabic graph of type $\pi_{k,n}$. Then for any positive integer $r$, the A-model (Newton-Okounkov) polytope $NO^r_G$ obtained from the positive chart $\Phi_G$ for $X = Gr_{n-k}(\mathbb{C}^n)$ coincides with the B-model polytope $Q^r_G$ obtained from the cluster chart $\Phi_G^r$ and superpotential $W_\nu$ on $\hat{X}^0 \subset \hat{X} = Gr_k((\mathbb{C}^n)^*)$.

**Remark 5.10.** Recall that the positive chart associated to a plabic graph $G$ is by construction a map to $X$, or indeed,

$$\Phi_G : (\mathbb{C}^*)^{P_G} \rightarrow X \setminus D,$$

for the anticanonical divisor $D = \bigcup_{i=1}^{n} \{ P_{[i+1,i+n-k]} = 0 \}$. The cluster $C_G = \{ p_\mu \mid \mu \in P_G \}$ on the other hand more naturally defines a (rational) map in the other direction,

$$\Phi_G^r : \hat{X} \setminus D \rightarrow (\mathbb{C}^*)^{P_G}$$

(5.5)
Our Theorem 5.9 suggests that we should consider the maps (5.4) and (5.5) as being ‘the same’, much the same way that the co-characters $X_\ast(T)$ of a torus $T$ are identified with the characters $X_\ast(T^\vee)$ of the dual torus $T^\vee$.

6. **The A-model and B-model polytopes associated to the all-rectangles charts coincide**

Our strategy for proving Theorem 5.9 is as follows. In this section, we will prove the theorem for a particularly nice plabic graph $G_{\text{rec}}^{k,n}$, whose face labels consist of all of the rectangles contained in $P_{k,n}$.

Then in Section 7, we will show that if Theorem 5.9 holds for a given reduced plabic graph $G$, then it holds for a graph $G'$ obtained from $G$ by performing one of the moves (M1), (M2), (M3) from Section 3. Since all plabic graphs of type $\pi_{k,n}$ are connected via moves (see Remark 3.4), this will complete the proof.

We start by defining a particular reduced plabic graph $G_{\text{rec}}^{k,n}$ with trip permutation $\pi_{k,n}$. This is a reduced plabic graph whose internal faces are arranged into an $(n-k) \times k$ grid pattern, as shown in Figure 9. When one uses Definition 3.5 to label faces by Young diagrams, one obtains the labeling of faces by rectangles which is shown in the figure. The generalization of this figure for arbitrary $k$ and $n$ should be clear. Moreover, the plabic graph $G_{\text{rec}}^{k,n}$ has a nice perfect orientation $O_{\text{rec}}$, which is shown in Figure 10.

The source set is $\{1, 2, \ldots, n-k\}$.

![Figure 9. The reduced plabic graph $G_{\text{rec}}^{5,9}$ with trip permutation $\pi_{5,9}$, with faces labeled by the rectangles in $P_{5,9}$.](image)

**Remark 6.1.** It is easy to check that the plabic graph $G_{\text{rec}}^{k,n}$ is reduced, using e.g. the criterion of [KW14, Theorem 10.5].

**Lemma 6.2.** For any reduced plabic graph $G$ of type $\pi_{k,n}$ and any positive integer $r$, the A-model polytope $NO_G^r$ is the $r$th dilation of the polytope $NO_G^1$, i.e. $NO_G^r = \{ rx \mid x \in NO_G^1 \}$.

**Proof.** It follows from the fact that $\text{val}_G(\bigcap_j P_j) = \sum_j (\text{val}_G(P_j))$ and the definition of $NO_G$, that $NO_G^r$ is the Minkowski sum of $r$ copies of $NO_G$. But now recall that for any convex body $A$ and positive real numbers $\lambda$ and $\mu$, we have that $(\lambda + \mu)A = \lambda A + \mu A$ [Sch14, Remark 1.1.1], where the sum on the right-hand side is the Minkowski sum. The statement of the lemma follows. \hfill $\square$

**Proposition 6.3.** For any positive integer $r$, the A-model polytope $NO_G^r_{\text{rec}}$ is contained in the B-model polytope $Q_{\text{rec}}^r$.

**Proof.** We start by observing that the labeling of the nodes in Figure 8 corresponds to the labeling of the faces of the plabic graph $G_{\text{rec}}^{k,n}$. More specifically, the Young diagrams in the numerators in Figure 8 are precisely the Young diagrams appearing in the faces of $G_{\text{rec}}^{k,n}$.
Lemma 6.2 and from the fact that if $r$ will treat the case obtained by: which “hugs” the southeast border of $G \setminus J$ (6.4) a partial order on the set of all paths from a given boundary source to the left of the path. Because of how the faces of $P$ each Plücker coordinate $x$ Choose an arbitrary total order on the parameters $r_{1,\ldots,n-k}$. Note that the source set $I_{G_{k,n}} = \{1, 2, 3, 4\}$. There is an obvious generalization of $O_{G_{k,n}}$ to any $G_{k,n}^\text{rec}$, which has source set $\{1, 2, \ldots, n-k\}$.

Using the formula (5.2) for the superpotential, we obtain the following inequalities defining $Q^r_{G_{k,n}}$:

\begin{align}
0 & \leq v_1 x_1 \\
(6.2) & \quad v_{(n-k)k} - v_{(n-k-1)(k-1)} \leq r \\
(6.3) & \quad v_{i(j-1)} - v_{i(j-2)} \leq v_{i(j-1)} - v_{(i-1)(j-1)} \quad \text{for } 2 \leq i \leq n-k \text{ and } 1 \leq j \leq k \\
(6.4) & \quad v_{i(j-1)} - v_{i(j-2)} \leq v_{i(j-1)} - v_{(i-1)(j-1)} \quad \text{for } 1 \leq i \leq n-k \text{ and } 2 \leq j \leq k
\end{align}

To describe $NO_{G_{k,n}}^\text{rec}$, we use the perfect orientation $O_{G_{k,n}}$ shown in Figure 10. Since the source set is $I_{G_{k,n}} = \{1, 2, \ldots, n-k\}$, and $G_{k,n}^\text{rec}$ is acyclic, it follows that the Plücker coordinate $P_{(1, \ldots, n-k)}$ equals 1. Choose an arbitrary total order on the parameters $r_{i,k,n}$ $\in \Delta_{G_{k,n}^\text{rec}}$. We can compute $NO_{G_{k,n}}^\text{rec}$ by writing down each Plücker coordinate $P_j$ in terms of the plabic chart $\Delta_{G_{k,n}^\text{rec}}^{\text{rec}}$ as in (3.1), then computing the convex hull of the points $\{\text{val}(M) \mid M \in \mathcal{M}_r\}$ with $\mathcal{M}_r$ the set of degree $r$ monomials in the Plücker coordinates.

Now we claim that we can reduce the proof of Proposition 6.3 to the case that $r = 1$. This follows from Lemma 6.2 and from the fact that if $r_0$ vectors $v^1, v^2, \ldots, v^n$ satisfy the inequalities (6.1) through (6.4) for $r = 1$, then their sum satisfies the inequalities for $r = r_0$. Therefore throughout the rest of the proof we will treat the case $r = 1$.

Recall that each Plücker coordinate $P_j$ is a sum over flows from $I_{G_{k,n}} = \{1, 2, \ldots, n-k\}$ to $J$. Since $G_{k,n}^\text{rec}$ is acyclic, each flow is just a collection of pairwise vertex-disjoint walks from $\{1, 2, \ldots, n-k\}$ to $J \setminus \{1, 2, \ldots, n-k\}$ in $O_{G_{k,n}}$. Note that if we write $\{1, 2, \ldots, n-k\} \setminus J = \{i_1 > i_2 > \cdots > i_\ell\}$ and write $J \setminus \{1, 2, \ldots, n-k\} = \{j_1 < j_2 < \cdots < j_\ell\}$, then any such flow must consist of $\ell$ paths which connect $i_1$ to $j_1$, $i_2$ to $j_2$, and $\ldots$, and $i_\ell$ to $j_\ell$. For example, in Figure 10, any flow used to compute the Plücker coordinate $P_{(2,5,6,8)}$ must consist of three paths which connect 4 to 5, 3 to 6, and 1 to 8.

Recall that the weight $\text{wt}(q)$ of a path $q$ is the product of the parameters $x_\mu$ where $\mu$ ranges over all face labels to the left of the path. Because of how the faces of $G_{k,n}^\text{rec}$ are arranged in a grid, we can define a partial order on the set of all paths from a given boundary source $i$ to a given boundary sink $j$, with $q_1 \leq q_2$ if and only if $\text{wt}(q_1) \leq \text{wt}(q_2)$. In particular, among such paths, there is a unique minimal path, which “hugs” the southeast border of $G_{k,n}^\text{rec}$.

It’s now clear that the flow from $\{1, 2, \ldots, n-k\} \setminus J$ to $J \setminus \{1, 2, \ldots, n-k\}$ with minimal valuation is obtained by:

- choosing the minimal path $q_1$ in $O_{G_{k,n}}^{\text{rec}}$ from $i_1$ to $j_1$;
- choosing the minimal path $q_2$ in $O_{G_{k,n}}^{\text{rec}}$ from $i_2$ to $j_2$ which is vertex-disjoint from $q_1$;
- \ldots
- choosing the minimal path $q_{\ell}$ in $O_{G_{k,n}}^{\text{rec}}$ from $i_{\ell}$ to $j_{\ell}$ which is vertex-disjoint from $q_{\ell-1}$.

![Figure 10. A perfect orientation $O_{G_{k,n}}^{\text{rec}}$ of the reduced plabic graph $G_{k,n}^{\text{rec}}$. Note that the source set $I_{G_{k,n}} = \{1, 2, 3, 4\}$. There is an obvious generalization of $O_{G_{k,n}}^{\text{rec}}$ to any $G_{k,n}^{\text{rec}}$, which has source set $\{1, 2, \ldots, n-k\}$.](image-url)
We call this the minimal flow $F_J$ associated to $J$. For example, when $J = \{2, 5, 6, 8\}$, the minimal flow $F_J$ associated to $J$ is shown at the left of Figure 11. At the right of Figure 11 we’ve re-drawn the plabic graph to emphasize the grid structure; this makes the structure of a minimal flow even more transparent.

![Figure 11. The minimal flow associated to $J = \{2, 5, 6, 8\}$.](image)

Now we have that $NO_{\mathcal{G}_{e,n}}^k$ is the convex hull of the points $\text{val} (\text{wt}(F_J))$ as $J$ varies over elements of $\binom{[n]}{n-k}$. Here $\text{val}$ applied to a monomial is simply its exponent vector, which lies in $\mathbb{Z}^{P_{k,n}}$, with coordinates indexed by the nonempty rectangles in $P_{k,n}$. It remains to check that each point $\text{val} (\text{wt}(F_J))$ satisfies each of the inequalities (6.1), (6.2) (with $r = 1$), (6.3), and (6.4). So for example, the exponent vector of the weight of the minimal flow shown in Figure 11 is depicted by the “tableau” in Figure 12.

![Figure 12. The “tableau”, or exponent vector associated to the minimal flow from Figure 11.](image)

Since each $\text{wt}(F_J)$ is a monomial in the parameters $x_{\mu}$, it follows that each point $\text{val}(\text{wt}(F_J))$ lies in $\mathbb{Z}_{\geq 0}^N$. Therefore (6.1) is always satisfied.

Now we consider (6.2). Consider an arbitrary minimal flow $F_J$, and let $V_{i \times j}$ denote the coordinates of $\text{val}(\text{wt}(F_J))$; here $i \times j$ ranges over all nonempty rectangles in $P_{k,n}$. We need to show that $V_{(n-k)\times k} \leq V_{(n-k-1)\times (k-1)} + 1$. But this is true, because in any flow, the number of paths which have rectangle $(n-k)\times k$ to the left is at most one more than the number of paths which have rectangle $(n-k-1)\times (k-1)$ to the left. More specifically, $V_{(n-k)\times k} - V_{(n-k-1)\times (k-1)}$ equals the number of paths in the flow which use the edge separating the faces labeled by rectangles $(n-k)\times k$ and $(n-k-1)\times (k-1)$, and it’s only possible to use that edge in a single path. Therefore (6.2) holds for all points $\text{val}(\text{wt}(F_J))$.

Now we consider (6.3). We need to show that

$$V_{(i-1)\times (j-1)} + V_{(i-1)\times j} \leq V_{i \times j} + V_{(i-2)\times (j-1)}$$

for any minimal flow $F_J$. Let us partition the paths comprising the flow $F_J$ into four types:

- Type 1: those for which the rectangle $i \times j$ is to the left, but not the other three rectangles involved in (6.5).
- Type 2: those for which the rectangles $i \times j$ and $(i-1) \times j$ are to the left, but not the other two rectangles involved in (6.5).
• Type 3: those for which the rectangles \((i-1) \times (j-1), (i-1) \times j,\) and \(i \times j\) are to the left, but not the rectangle \((i-2) \times (j-1)\).

• Type 4: those for which all rectangles involved in (6.5) are to the left.

Note that for any path \(q\) of Type 2 or 4, \(\text{val}(\text{wt}(q))\) contributes equally to both sides of the inequality (6.5). So we can ignore such paths. Now consider a path \(q\) of Type 3. There can be at most one such path in a flow, because such a path must use the edge between the faces labeled by \((i-1) \times (j-1)\) and \((i-2) \times (j-1)\). Such a path \(q\) will contribute a +1 to the left-hand side of (6.5) (and nothing to the right-hand side). However, if such a path \(q\) is present, then by the minimality of the flow \(F_J\), there must be another path of Type 1, which will contribute a +1 to the right-hand side of (6.5) (and nothing to the left-hand side). Therefore (6.5) follows.

The proof of (6.4) is analogous to the proof of (6.3).

Remark 6.4. In the previous proof, it follows from the explicit description of the minimal flow \(F_J\) that for any other flow \(F'_J\) from \(s_{G_{k,n}}\) to \(J\), we have that \(\text{wt}(F'_J)\) is a product of \(\text{wt}(F_J)\) and a monomial in the parameters. Therefore the minimal flow is strongly minimal, and the Plücker coordinate \(P_J\) has a strongly minimal term (namely \(\text{wt}(F_J)\)), in the sense of Definition 3.13.

To show that the polytopes \(Q_{G_{k,n}}^{\text{rec}}\) and \(NO_{G_{k,n}}^{\text{rec}}\) coincide, there are several approaches. Now that we know that \(NO_{G_{k,n}}^{\text{rec}} \subseteq Q_{G_{k,n}}^{\text{rec}}\), one approach is to show that both polytopes are integral, and then show that they both have the same number of lattice points. On the A-model side, the Okounkov lemma [Oko97] implies that the number of integral points in \(NO_{G_{k,n}}^{\text{rec}}\) equals the dimension of the space of sections of the corresponding line bundle. On the B-model side, one can then show that \(Q_{G_{k,n}}^{\text{rec}}\) is affinely isomorphic to the Gelfand-Tsetlin polytope (by an isomorphism preserving the number of lattice points), whose integer points count exactly the same thing.

However, to keep our arguments self-contained, we can also give a direct proof that the two polytopes coincide, see Lemma 6.6. We start by showing in Lemma 6.5 that the B-model polytope has integer vertices.\(^4\) In this case our polytope is essentially dual to the one considered in [BCFKvS98], and so integrality may also be deduced from [BCFKvS98, Theorem 3.13] (see also [BCFKvS00, Corollary 2.2.4]), using the fact that the dual of a reflexive polytope is reflexive, in particular integral (see [Bat94, Definition 4.1.5] and [Bat94, Theorem 4.1.6]).

Lemma 6.5. The polytope \(Q_{G_{k,n}}^{\text{rec}}\) has integer vertices.

Proof. Let \(f_{ij} = v_{ij} - v_{(i-1)(j-1)}\). If we rewrite the inequalities (6.1) through (6.4) in terms of the \(f\)-variables, we obtain the following system of inequalities defining \(Q_{G_{k,n}}^{\text{rec}}:\)

\[
\begin{align*}
(6.6) & \quad f_{ij} - f_{i(j-1)} \geq 0 \\
(6.7) & \quad f_{ij} - f_{i(j-1)} \geq 0 \\
(6.8) & \quad f_{1 \times j} \geq 0 \\
(6.9) & \quad -f_{(n-k) \times k} \geq -r.
\end{align*}
\]

If we write these inequalities in the form \(Cf \geq b\), where \(C\) is the matrix of coefficients and \(f\) is the vector of \(f\)-variables, we claim that \(C\) is a totally unimodular matrix, i.e. every square nonsingular submatrix has determinant \(\pm 1\). To see this, note that the rows of \(C\) corresponding to (6.6) and (6.7) can be interpreted as the incidence matrix of a directed graph, which is known to be totally unimodular [Sch86]. The row of \(C\) corresponding to (6.8) is a unit vector, i.e. all entries are 0 except for one which is 1. And (6.9) corresponds to a row of \(C\) which is the negative of a unit vector.

Since \(C\) is totally unimodular, the polytope \(Q_{G_{k,n}}^{\text{rec}}\) has integer vertices in terms of the \(f\)-variables [Sch86]. But then it also has integer vertices in terms of the \(v\)-variables. \(\square\)

\(^4\)In the first version of this paper, we implicitly assumed but didn’t prove that this polytope is integral. We would like to thank Steven Karp, who pointed out this gap to us within 14 hours of the paper being posted to the arXiv, and at the same time supplied the proof that we give of Lemma 6.5.
Proposition 6.6. For any positive integer r, the B-model polytope \(Q_{G_{k,n}}^r\) is contained in the A-model polytope \(\text{NO}_{G_{k,n}}^r\).

Proof. As before, we claim that Proposition 6.6 follows from the polytope \(v_{6.2}, (6.3), (6.4)\) (for nonempty rectangles contained in \(P_{v_{k,n}}\) we obtain \(v\) entries weakly increase from top to bottom in the columns is analogous. using (6.3) gives \(\leq 2\) square with entries all \(\geq 1\) will either have some \(v\) coincide. In this section we will show that if Theorem 5.9 holds for a given reduced plabic graph \(G\), then

So to prove Proposition 6.6, we need to show that a point \(v_{i,j}\) which satisfies the inequalities (6.1), (6.2), (6.3), (6.4) (for \(r = 1\), must necessarily satisfy conditions (1), (2), (3), (4), and (5) above.

To prove (1), note that (6.4) for \(i = 1\) gives \(v_{1,x_1} \leq v_{1,x_2} \leq \cdots \leq v_{1,x_k}\). And (6.3) for \(j = k\) gives \(v_{1,x_k} \leq v_{2,x_k} - v_{1,x(k-1)} \leq v_{3,x_k} - v_{2,x(k-1)} \leq \cdots \leq v_{(n-k)+x_k} - v_{(n-k-1)+x(k-1)}\). Finally since (6.2) implies that \(v_{(n-k)+x_k} - v_{(n-k-1)+x(k-1)} \leq 1\), we have that \(v_{1,x_1} \leq v_{1,x_2} \leq \cdots \leq v_{1,x_k} \leq 1\), which implies that entries in the top row are at most 1. The proof that entries in the leftmost column are at most 1 is similar, so we have (1).

To prove (2), note that (6.4) gives \(v_{i,x_j} - v_{i+1,x(j-1)} \leq v_{i+1,x(j+1)} - v_{i,x(j+1)} \leq \cdots \leq v_{i,x(k-1)}\). Then using (6.3) gives \(v_{i,x_k} - v_{i+1,x(j-1)} \leq v_{i+1,x(k-1)} - v_{i,x(k-1)} \leq \cdots \leq v_{(n-k)+x_k} - v_{(n-k-1)+x(k-1)}\). Finally using (6.2) we obtain \(v_{i,x_j} - v_{i+1,x(j-1)} \leq 1\), which proves (2).

Clearly (3) follows from (6.1).

We will use induction to prove (4). Note that in our proof of (1), we already showed that entries in the top row of the tableau are weakly increasing; this is the base case of our induction. Now suppose that \(v_{i,x_1} \leq v_{i,x_2} \leq \cdots \leq v_{i,x_k}\) for some \(i \geq 1\). Using (6.4), we have that \(v_{i+1,x(j-1)} \leq v_{i+1,x(j+1)} - v_{i,x(j+1)} - v_{i,x(j-2)}\) (for \(2 \leq j \leq k\)). By the inductive hypothesis, \(v_{i,x(j-1)} - v_{i,x(j-2)}\) is nonnegative, which implies that \(v_{i+1,x(j-1)} \leq v_{i,x(j-1)}\). This completes the proof that entries weakly increase from left to right in the rows; the proof that entries weakly increase from top to bottom in the columns is analogous.

To prove (5), we use contradiction. We know from (4) that \(v_{i+1,x(j-1)} \geq v_{i,x_j}\). So let us suppose that \(v_{i,x_j} = \ell > 0\) but \(v_{i+1,x(j-1)} = v_{i,x_j}\). It follows from (4) that we must then have a two-by-two square in our tableau whose entries are all \(\ell\), i.e., \(v_{i,x_j} = v_{i+1,x(j-1)} = v_{i+1,x(j+1)} = v_{i,x(j+1)} = \ell\). Now from (6.4) we have \(v_{i+1,x(j-1)} - v_{i,x(j-1)} \leq v_{i+1,x(j-1)} - v_{i,x(j-1)}\), and since \(v_{i,x(j-1)} - v_{i+1,x(j-1)}\), we must that \(v_{i+1,x(j-1)} = \ell\). Similarly using (6.3) we find that \(v_{i+1,x(j-1)} = \ell\). The same type of argument shows that \(v_{i,x(j-1)} = \ell\). So we have another two-by-two square in our tableau whose entries are all \(\ell\), but it is shifted diagonally northwest from our first two-by-two square. If we propagate this argument, we will eventually obtain a two-by-two square with entries all \(\ell\), which is either aligned at the top or to the left of the tableau. In other words, we will either have \(v_{i,x_j} = v_{i+1,x(j-1)} = v_{i+2,x(j+1)} = \ell\) (for some \(j\)), or \(v_{i+1,x(j+1)} = v_{i+1,x(j-1)} = v_{i,x(j+1)} = v_{i,x(j-1)} = \ell\) (for some \(i\)). Without loss of generality let us assume that the former is true. But then using (6.3) and (6.4) we have that \(v_{i,x_j} \leq v_{i,x(j-1)} \leq v_{i,x(j+1)} - v_{i,x(j-1)} \) and so \(v_{i,x_j} \leq v_{i,x(j+1)} - v_{i,x(j-1)}\). But this is a contradiction, as \(v_{i,x_j} = v_{i,x(j+1)} = \ell > 1\).

\[\square\]

7. The A-model and B-model polytopes always coincide

In Section 6 we showed that the A-model and B-model polytopes associated to the plabic graph \(G_{k,n}\) coincide. In this section we will show that if Theorem 5.9 holds for a given reduced plabic graph \(G\), then
it holds for a graph \( G' \) obtained from \( G \) by performing one of the moves (M1), (M2), (M3) from Section 3. Since all plabic graphs of type \( \pi_{k,n} \) are connected via moves (see Remark 3.4), this will complete the proof of Theorem 5.9.

7.1. How the A-model polytope changes when we apply a move to \( G \).

**Theorem 7.1.** Suppose that \( G \) and \( G' \) are reduced plabic graphs of type \( \pi_{k,n} \), which are related by a single move. If \( G \) and \( G' \) are related by one of the moves (M2) or (M3), then the polytopes \( \text{NO}_{G'} \subset \mathbb{R}^N \) and \( \text{NO}_{G} \subset \mathbb{R}^N \) are identical. If \( G \) and \( G' \) are related by the square move (M1), then the polytopes \( \text{NO}_{G} \) and \( \text{NO}_{G'} \) differ by a tropicalized cluster mutation. More specifically, let \((V_{\mu_1}, V_{\mu_2}, \ldots, V_{\mu_N})\) be the coordinates of \( \text{NO}_G \), where the \( \mu_i \) are the non-empty partitions labeling the faces of \( G \). Without loss of generality, suppose we do a square move at the face labeled by \( \mu_1 \) in Figure 13. Then \( \text{NO}_{G'} \) is obtained from \( \text{NO}_G \) by the following piecewise-linear transformation from \( \mathbb{R}^N \) to \( \mathbb{R}^N \):

\[
(V_{\mu_1}, V_{\mu_2}, \ldots, V_{\mu_N}) \mapsto (V_{\mu'_1}, V_{\mu_2}, \ldots, V_{\mu_N}), \quad \text{where} \quad V_{\mu'_1} = \min(V_{\mu_2} + V_{\mu_4}, V_{\mu_3} + V_{\mu_5}) - V_{\mu_1}.
\]

Since \( V_{\mu_1} \) and \( V_{\mu'_1} \) satisfy the relation

\[
V_{\mu_1} + V_{\mu'_1} = \min(V_{\mu_2} + V_{\mu_4}, V_{\mu_3} + V_{\mu_5}),
\]

we call this map a tropicalized cluster mutation.

![Figure 13](image)

The following lemma, which appeared in [PSW07],\(^5\) will be helpful in the proof of Theorem 7.1.

**Lemma 7.2.** [PSW07, Lemma 3.2 and its proof] Each reduced plabic graph \( G \) has an acyclic perfect orientation \( O \). Moreover, we may choose \( O \) so that the set of boundary sources \( I \) is the index set for the lexicographically minimal non-vanishing Plücker coordinate on the corresponding cell. (In particular, if \( G \) is of type \( \pi_{k,n} \), then we can choose \( O \) so that \( I = \{1, \ldots, n - k\} \).) Then given another reduced plabic graph \( G' \) which is move-equivalent to \( G \), we can transform \( O \) into a perfect orientation \( O' \) for \( G' \), such that \( O' \) is also acyclic with boundary sources \( I \), using oriented versions of the moves (M1), (M2), (M3). Up to rotational symmetry, we will only need to use the oriented version of the move (M1) shown in Figure 14.

![Figure 14](image)

**Remark 7.3.** In the course of the proof of Theorem 7.1 we also show that each Plücker coordinate \( P_K \), expressed in terms of \( \Delta_G \) for some \( G \) as above, has a strongly minimal term, in the sense of Definition 3.13. In particular, the polytope \( \text{NO}_G \) does not depend on the ordering on \( \Delta_G \) used in Definition 4.3.

We now prove Theorem 7.1.

---

\(^5\)The published version of [PSW07], namely [PSW09], did not include the lemma, because it turned out to be unnecessary.
Proof. By Lemma 7.2, we may choose an acyclic perfect orientation $\mathcal{O}$ of $G$ whose set of boundary sources is $\{1, 2, \ldots, n - k\}$. Therefore if we apply Theorem 3.8, our expression for the Plücker coordinate $P_{\{1, \ldots, n-k\}}$ is 1. Moreover, we have expressions for the other Plücker coordinates in terms of the weights of flows, which are pairwise-disjoint collections of self-avoiding walks in $\mathcal{O}$. The weight of each walk is the product of parameters $x_\mu$, where $\mu$ ranges over all face labels to the left of a walk.

By Lemma 6.2, $\text{NO}_G^r$ is the $r$th dilation of $\text{NO}_G^1$. Therefore it suffices to prove the theorem for $\text{NO}_G^1$.

It is easy to see that the polytopes $\text{NO}_G^1$ and $\text{NO}_G^1$ are identical if $G$ and $G'$ differ by one of the moves (M2) or (M3): in either case, there is an obvious bijection between perfect orientations of both graphs involved in the move, and this bijection is weight-preserving.

Now suppose that $G$ and $G'$ differ by a square move. By Lemma 7.2, it suffices to compare perfect orientations $\mathcal{O}$ and $\mathcal{O}'$ of $G$ and $G'$ which differ as in Figure 14. Without loss of generality, $G$ and $G'$ are at the left and right, respectively, of Figure 14. (We should also consider the case that $G$ is at the right and $G'$ is at the left, but the proof in this case is analogous.) The main step of the proof is to prove the following claim about existence of strongly minimal flows, compare Definition 3.13.

Claim. Let $K$ be a subset of $\{1, \ldots, n\}$ and let us suppose that in $G$, for the perfect orientation $\mathcal{O}$, there exists a strongly minimal flow $F_{\min}$ from $\{1, \ldots, n-k\}$ to $K$.

1. Assuming the orientations in $\mathcal{O}$ locally around the face $\mu_1$ are as shown in the left-hand side of Figure 14, then the restriction of $F_{\min}$ to the neighborhood of face $\mu_1$ is as in the left-hand side of one of the six pictures in Figure 15, say picture 1, where $I \in \{A, B, C, D, E, F\}$.

2. There is a strongly minimal flow $F_{\min}'$ in $G'$, and it is the flow obtained from $F_{\min}$ by the local transformation indicated in picture $I$.

If a strongly minimal flow exists this means that $\text{val}_G(P_K)$ is the valuation of the weight of this flow and is a strongly minimal monomial in $P_K$. Recall from Remark 6.4 that the assumption of the claim holds for the plabic graph $G_{k,n}^c$ and any $K$. Namely a strongly minimal flow exists. Therefore once we have checked the claim, it will follow from Lemma 7.2 that $P_K$, expressed in terms of the coordinates $\Delta_G$, always has a strongly minimal monomial, for any choice of reduced plabic graph $G$ of type $\pi_{k,n}$. The valuation $\text{val}_G(P_K)$ then agrees with the valuation of this monomial.

\[
\begin{array}{cccc}
A & B & C & D \\
\begin{array}{c}
\mu_3 \\
\mu_4 \\
\mu_1 \\
\mu_2 \\
\end{array} & \begin{array}{c}
\mu_1 \\
\mu_2 \\
\mu_3 \\
\mu_4 \\
\end{array} & \begin{array}{c}
\mu_1 \\
\mu_2 \\
\mu_3 \\
\mu_4 \\
\end{array} & \begin{array}{c}
\mu_1 \\
\mu_2 \\
\mu_3 \\
\mu_4 \\
\end{array} \\
\begin{array}{c}
\mu_3 \\
\mu_4 \\
\mu_1 \\
\mu_2 \\
\end{array} & \begin{array}{c}
\mu_1 \\
\mu_2 \\
\mu_3 \\
\mu_4 \\
\end{array} & \begin{array}{c}
\mu_1 \\
\mu_2 \\
\mu_3 \\
\mu_4 \\
\end{array} & \begin{array}{c}
\mu_1 \\
\mu_2 \\
\mu_3 \\
\mu_4 \\
\end{array} \\
\end{array}
\]

Figure 15. How minimal flows change in the neighborhood of face $\mu_1$ as we do an oriented square move. The perfect orientations $\mathcal{O}$ and $\mathcal{O}'$ for $G$ and $G'$ are shown at the left and right of each pair, respectively. Note that in the top row, the flows do not change, but in the bottom row they do. Also note that the picture at the top left indicates the case that the flow is not incident to face $\mu_1$.

Let us check (1). In theory, the restriction of $F_{\min}$ to the neighborhood of face $\mu_1$ could be as in the left-hand side of any of the six pictures from Figure 15, or it could be as in Figure 16. However, if a flow locally looks like Figure 16, then it cannot be minimal – the single path shown in Figure 16 could be deformed to go around the other side of the face labeled $\mu_1$, and that would result in a smaller weight. More specifically, the weight of a flow which locally looks like Figure 16, when restricted to coordinates...
Now let us write \( \text{wt}(F_{\min}) = \prod_{\mu \in P_G} x_{\mu}^{\nu_{\mu}} \), so that \( (a_{\mu})_{\mu \in P_G} = \text{val}_G(P_I) \). Suppose that the restriction of \( F_{\min} \) to the neighborhood of face \( \mu_1 \) looks as in picture \( I \) of Figure 15. Let \( F'_{\min} \) be the flow in \( G' \) obtained from \( F_{\min} \) by the local transformation indicated in picture \( I \), and write \( \text{wt}(F'_{\min}) = \prod_{\mu \in P_{G'}} x_{\mu}^{\nu_{\mu}} \). (Clearly \( F'_{\min} \) is indeed a flow in \( G' \).) We need to show that \( F'_{\min} \) is strongly minimal.

Let \( F' \) be some arbitrary flow in \( G' \), and write \( (b_{\mu})_{\mu \in P_{G'}} = \text{val}_{G'}(\text{wt}(F')) \). We need to show that \( a_{\mu} \leq b_{\mu} \) for all \( \mu \in P_{G'} \). We can assume that the restriction of \( F' \) to the neighborhood of face \( \mu'_1 \) looks as in the right-hand side of one of the six pictures in Figure 15, say picture \( J \). A priori there is one more case (obtained from the right-hand side of picture \( B \) by deforming the single path to go around \( \mu'_1 \)), but since this increases \( b_{\mu'} \), we don’t need to consider it. Now let \( F \) be the flow in \( G \) obtained from \( F' \) by the local transformation indicated in picture \( J \), and write \( (b_{\mu})_{\mu \in P_G} = \text{val}_G(\text{wt}(F)) \).

We already know, by our assumption on \( G \), that \( b_{\mu} \geq a_{\mu} \) for all \( \mu \in P_G \). Moreover it is clear from Figure 15 that

\[
(7.2) \quad a'_{\mu} = \begin{cases} 
    a_{\mu} & \text{if } \mu \neq \mu'_1, \\
    \text{or } a_{\mu} + 1 & \text{if } \mu = \mu'_1,
\end{cases}
\quad \text{and} \quad b'_{\mu} = \begin{cases} 
    b_{\mu} & \text{if } \mu \neq \mu'_1, \\
    \text{or } b_{\mu} + 1 & \text{if } \mu = \mu'_1.
\end{cases}
\]

More specifically, \( a'_{\mu} = a_{\mu} + 1 \) (respectively, \( b'_{\mu} = b_{\mu} + 1 \)) precisely when picture \( I \) (respectively, picture \( J \)) is one of the cases \( D, E, F \) from Figure 15.

From the cases above, it follows that \( b_{\mu}' \geq a_{\mu}' \) for all \( \mu \neq \mu'_1 \) and \( \mu \in P_{G'} \). We need to check only that \( b_{\mu'_1}' \geq a_{\mu'_1}' \). Since \( b_{\mu_1} \geq a_{\mu_1} \), the only way to get \( b_{\mu'_1}' < a_{\mu'_1}' \) is if \( a_{\mu_1}' = a_{\mu_1} + 1 \) and \( b_{\mu_1}' = b_{\mu_1} = a_{\mu_1} \). In particular, then \( I \in \{D, E, F\} \) and \( J \in \{A, B, C\} \). So we need to show that each of these nine cases is impossible when \( b_{\mu_1} \geq a_{\mu_1} \) and \( b_{\mu_1} = a_{\mu_1} \).

Let us set \( i = a_{\mu_1} = b_{\mu_1} \). If \( I = D \), then the vector \( (a_{\mu_1}, a_{\mu_2}, a_{\mu_3}, a_{\mu_4}, a_{\mu_5}) \) has the form \( (i, i+1, i+1, i, i) \). If \( I = E \), the vector \( (a_{\mu_1}, a_{\mu_2}, a_{\mu_3}, a_{\mu_4}, a_{\mu_5}) \) has the form \( (i, i+1, i+1, i, i+1) \). And if \( I = F \), the vector \( (a_{\mu_1}, a_{\mu_2}, a_{\mu_3}, a_{\mu_4}, a_{\mu_5}) \) has the form \( (i, i+1, i, i, i+1) \).

Meanwhile, if \( J = A \), then \( (b_{\mu_1}, b_{\mu_2}, b_{\mu_3}, b_{\mu_4}, b_{\mu_5}) = (i, i, i, i, i) \). If \( J = B \), then \( (b_{\mu_1}, b_{\mu_2}, b_{\mu_3}, b_{\mu_4}, b_{\mu_5}) = (i, i+1, i, i, i) \). And if \( J = C \), then \( (b_{\mu_1}, b_{\mu_2}, b_{\mu_3}, b_{\mu_4}, b_{\mu_5}) = (i, i+1, i, i+1) \).

In all nine cases, we see that we get a contradiction to the fact that \( a_{\mu_1} \leq b_{\mu_1} \) for all \( \mu_1 \). To be precise, by looking at cases \( A, B \) and \( C \) we see that always \( b_{\mu_3} = b_{\mu_5} = i \), while for \( a_{\mu_1} \) we always have either \( a_{\mu_3} = i+1 \) or \( a_{\mu_5} = i+1 \), looking at \( D, E \) and \( F \).

This completes the proof of the claim. We know now that if we have a Plücker coordinate \( P_K \) expressed as a sum of monomials in any \( \Delta_G \), then the lexicographically-minimal term is always also strongly minimal (in particular the ordering on \( \Delta_G \) doesn’t matter), and when we perform a square move, the lexicographically-minimal term changes as in Figure 15.

Now it remains to check that the tropical cluster relation (7.1) is satisfied for each of the six cases shown in Figure 15. For example, in the top-middle pair shown in Figure 15, we have \( a_{\mu_1} = a_{\mu_3} = a_{\mu_4} = a_{\mu_5} = a_{\mu'_1} = i \), and \( a_{\mu_2} = i+1 \). Clearly we have \( a_{\mu_1} + a_{\mu'_1} = \min(a_{\mu_3} + a_{\mu_4} + a_{\mu_5} + a_{\mu'_1}) \). In the top-right pair, we have \( a_{\mu_2} = i+2, a_{\mu_1} = a_{\mu_3} = a_{\mu_5} = i+1, a_{\mu_4} = i, \) and \( a_{\mu'_1} = i+1 \), which again satisfy (7.1). The other three cases can be similarly checked. This completes the proof. \( \square \)
7.2. How the B-model polytope changes when we apply a move to $G$. Recall from Definition 5.7 that the B-model polytope $Q_G^r \subset \mathbb{R}^N$ is defined by $\text{Trop}(W_r(C_G)) \geq 0$. More specifically, given a reduced plabic graph $G$, we write the superpotential $W_r$ as a Laurent polynomial in the Plücker coordinates from $C_G$, and get one inequality from each Laurent monomial $f$ of $W_r(C_G)$: namely, $\text{Trop}(f) \geq 0$.

**Theorem 7.4.** Suppose that $G$ and $G'$ are reduced plabic graphs of type $\pi_{k,n}$, which are related by a single move. If $G$ and $G'$ are related by one of the moves (M2) or (M3), then the polytopes $Q_G^r \subset \mathbb{R}^N$ and $Q_{G'}^r \subset \mathbb{R}^N$ are identical. If $G$ and $G'$ are related by the square move (M1), then the polytopes $Q_G^r$ and $Q_{G'}^r$ differ by a tropicalized cluster mutation. More specifically, let $(v_{\mu_1}, v_{\mu_2}, \ldots, v_{\mu_N})$ be the coordinates of $Q_G^r$, where the $\mu_i$ are the non-empty partitions labeling the faces of $G$. Without loss of generality, suppose we do a square move at the face labeled by $\mu_1$ in Figure 13. Then $Q_{G'}^r$ is obtained from $Q_G^r$ by the following piecewise-linear transformation from $\mathbb{R}^N$ to $\mathbb{R}^N$:

$$(v_{\mu_1}, v_{\mu_2}, \ldots, v_{\mu_N}) \mapsto (v'_{\mu_1}, v_{\mu_2}, \ldots, v_{\mu_N}),$$

where

$$v'_{\mu_1} = \min(v_{\mu_2} + v_{\mu_4}, v_{\mu_3} + v_{\mu_5}) - v_{\mu_1}.$$

**Proof.** Recall from Definition 3.5 that we use trips to label every face of a reduced plabic graph $G$ of type $\pi_{k,n}$ by a Young diagram in $\mathcal{P}_{k,n}$. We then identify each Young diagram with a Plücker coordinate and obtain a collection $C_G$ of Plücker coordinates for the Grassmannian $\mathbb{X} = Gr_k(\mathbb{C}^n)$ associated to $G$. Clearly if $G$ and $G'$ are related by one of the moves (M2) or (M3), then the set of face labels will not change. Therefore it is immediate that $Q_G^r = Q_{G'}^r$.

If $G$ and $G'$ are related by a square move (M1), it is well-known (and easy to verify) that the Plücker coordinates labeling the faces satisfy the following relation (see Figure 13 for notation):

$$(7.3) \quad p_{\mu_1} p_{\mu'_1} = p_{\mu_2} p_{\mu_4} + p_{\mu_3} p_{\mu_5}.$$ 

This relation is a three-term Plücker relation, and also a cluster transformation. It follows from (7.3) that

$$\text{Trop}(p_{\mu_1}) + \text{Trop}(p_{\mu'_1}) = \min \left( \text{Trop}(p_{\mu_2}) + \text{Trop}(p_{\mu_4}), \text{Trop}(p_{\mu_3}) + \text{Trop}(p_{\mu_5}) \right),$$

and hence

$$(7.4) \quad v_{\mu_1} + v_{\mu'_1} = \min(v_{\mu_2} + v_{\mu_4}, v_{\mu_3} + v_{\mu_5}).$$

Therefore if we substitute $p_{\mu_2} p_{\mu_4} p_{\mu_3} p_{\mu_5}$ for $p_{\mu_1}$ into our expression for the superpotential $W_r(C_G)$, obtaining a new Laurent polynomial $W_r(C_{G'})$, the effect on the polytope $Q_G^r$ will be to substitute the quantity $\min(v_{\mu_2} + v_{\mu_4}, v_{\mu_3} + v_{\mu_5}) - v_{\mu'_1}$ for $v_{\mu_1}$. This completes the proof of Theorem 7.4. \hfill \square

7.3. Proof of Theorem 5.9. We can now put together all the ingredients from the previous sections to complete the proof of Theorem 5.9.

**Proof.** By Propositions 6.3 and 6.6, for any positive integer $r$, and for the special choice of plabic graph $G_{k,n}$, we have that the A-model and B-model polytopes coincide, namely $\text{NOO}_{G_{k,n}} = Q_{G_{k,n}}$. By Remark 3.4, one can get from the plabic graph $G_{k,n}$ to any other reduced plabic graph of type $\pi_{k,n}$ by a sequence of moves. Finally if we compare Theorem 7.1 with Theorem 7.4, we see that if $G$ and $G'$ are plabic graphs which are related by a single move, then the piecewise-linear transformation relating $\text{NOO}_G$ to $\text{NOO}_{G'}$ is the same as the piecewise-linear transformation relating $Q_G^r$ to $Q_{G'}^r$. It follows that for any reduced plabic graph of type $\pi_{k,n}$, we have that $\text{NOO}_G = Q_G^r$. \hfill \square

**References**


**Department of Mathematics, King’s College London, Strand, London WC2R 2LS UK**

E-mail address: konstanze.rietsch@kcl.ac.uk

**Department of Mathematics, University of California at Berkeley, Berkeley, CA USA**

E-mail address: williams@math.berkeley.edu