# Introduction to Cluster Algebras
# Chapters 4–5

(preliminary version)

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# Preface

This is a preliminary draft of Chapters 4–5 of our forthcoming textbook *Introduction to cluster algebras*, joint with Andrei Zelevinsky (1953–2013). Chapters 1–3 have been posted as `arXiv:1608:05735`. We expect to post additional chapters in the not so distant future.

Comments and suggestions are welcome.

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# Contents

*Chapter 4*

# New patterns from old

This chapter provides several methods for obtaining new seed patterns (or new cluster algebras) from existing ones.

## 4.1. Restrictions and embeddings of quivers and matrices

We begin by discussing some purely combinatorial constructions involving mutations of quivers or matrices—but not clusters or seeds.

**Definition 4.1.1.** Let $\tilde{B}$ be an $m \times n$ extended skew-symmetrizable matrix. For a subset $I \subset [1, m]$, consider the matrix $\tilde{B}_I$ obtained from $\tilde{B}$ by restricting to the row set $I$ and to the column set $I \cap [1, n]$. It is easy to see that $\tilde{B}_I$ is again an extended skew-symmetrizable matrix. We say that $\tilde{B}_I$ is *obtained from* $\tilde{B}$ *by restriction to* $I$. More generally, we say that a matrix $\tilde{B}'$ is obtained from $\tilde{B}$ by restriction if $\tilde{B}'$ can be identified with a matrix $\tilde{B}_I$ as above under an appropriate relabeling of rows and columns.

Note that if $\tilde{B} = \tilde{B}(Q)$ is the extended exchange matrix of a quiver $Q$, then $\tilde{B}_I = \tilde{B}(Q_I)$ is the extended exchange matrix of the quiver $Q_I$, where $Q_I$ is obtained from $Q$ by taking the subset of vertices of $Q$ indexed by $I$ along with all the arrows in $Q$ that connect the vertices in $I$. Such a quiver $Q_I$ is called a *full* (or *induced*) *subquiver* of $Q$. The vertices in $Q_I$ inherit the property of being frozen or mutable from the ambient quiver $Q$.

The following property is easy to check.

**Lemma 4.1.2.** *Mutation of matrices/quivers commutes with restriction. More precisely, if* $\tilde{B}_I$ *is the restriction of an extended skew-symmetrizable matrix* $\tilde{B}$ *to a subset* $I$*, and* $k \in I$ *is mutable, then* $\mu_k(\tilde{B}_I) = (\mu_k(\tilde{B}))_I$*.*

**Definition 4.1.3.** We say that a property (of extended skew-symmetrizable matrices) is *hereditary* if it is preserved under restriction: for any matrix $\tilde{B}$ which has this property, the same holds true for all its submatrices $\tilde{B}_I$. For quivers, a property is hereditary if it is inherited by the full subquivers of any quiver which has that property.

We are interested in hereditary properties which are preserved under mutations. The first example of this kind is directly obtained from Lemma 4.1.2.

**Proposition 4.1.4.** *Finite mutation type is a hereditary property.*

In particular, an extended exchange matrix obtained by restriction from an extended exchange matrix of finite mutation type (cf. Definition 2.6.9) has finite mutation type.

We will show later that the property of being mutation-equivalent to an orientation of a (possibly disconnected, simply laced) Dynkin diagram is hereditary, see Remark 5.10.9. See also Theorem 8.4.1 for a version of this statement that includes extended Dynkin diagrams.

**Example 4.1.5.** Recall that an acyclic quiver is one containing no oriented cycles. A quiver with no frozen vertices is called *mutation-acyclic* if it is mutation equivalent to an acyclic quiver. It was shown in [**3**], using the machinery of quiver representations, that the property of being mutation-acyclic is hereditary. It would be interesting to find an elementary proof.

In light of Example 4.1.5, it is natural to consider the unoriented analogue of the notion of mutation-acyclicity.

**Remark 4.1.6.** A quiver is called *arborizable* if it is mutation-equivalent to an orientation of a forest (i.e., an undirected simple graph with no cycles).

Unfortunately, arborizability is not a hereditary property. A counterexample is given in Figure 4.1.

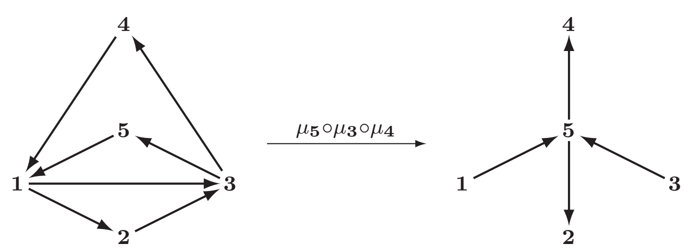



**Figure 4.1.** The quiver $Q$ shown on the left is arborizable since it is mutation equivalent to the quiver $Q'$ on the right. On the other hand, the full subquiver $Q_I$ on the vertex set $I = \{1, 2, 3\}$ is not arborizable: $Q_I$ is of finite mutation type, and its mutation class $[Q_I]$ consists of two quivers (up to isomorphism) neither of which is an orientation of a tree.

**Lemma 4.1.7.** *For mutation classes* $\mathbf{Q}$ *and* $\mathbf{Q}'$*, the following are equivalent:*

(i) *there exist* $\tilde{B} \in \mathbf{Q}$ *and* $\tilde{B}' \in \mathbf{Q}'$ *such that* $\tilde{B}$ *is obtained from* $\tilde{B}'$ *by restriction;*

(ii) *for any* $\tilde{B} \in \mathbf{Q}$*, there exists* $\tilde{B}' \in \mathbf{Q}'$ *such that* $\tilde{B}$ *is obtained from* $\tilde{B}'$ *by restriction.*

**Proof.** The equivalence of (i) and (ii) follows from Lemma 4.1.2. □

The notion of restriction descends to a partial order on the set of mutation classes of extended exchange matrices, as follows.

**Definition 4.1.8.** Let $\mathbf{Q}$ and $\mathbf{Q}'$ be mutation classes. We say that $\mathbf{Q}$ is *embeddable* into $\mathbf{Q}'$, and write $\mathbf{Q} \leq \mathbf{Q}'$, if either of the two equivalent conditions (i)–(ii) in Lemma 4.1.7 hold. In particular, when $\mathbf{Q}$ and $\mathbf{Q}'$ are mutation classes of quivers, we say that $\mathbf{Q}$ is *embeddable* into $\mathbf{Q}'$, if there exist quivers $Q \in \mathbf{Q}$ and $Q' \in \mathbf{Q}'$ such that $Q$ is a full subquiver of $Q'$.

**Example 4.1.9.** In Figure 2.11, the mutation class of each quiver (including any orientation of each of the Dynkin diagrams shown) is embeddable into the mutation class of every quiver appearing in the rows below. We revisit (and extend) this example in Remark 6.22.

Recall that $[\tilde{B}]$ denotes the mutation class of an extended exchange matrix $\tilde{B}$.

**Remark 4.1.10.** Fix a mutation class $\mathbf{R}$. Then embeddability into $\mathbf{R}$ is a hereditary property. More precisely, if $[\tilde{B}] \leq \mathbf{R}$, then $[\tilde{B}_I] \leq \mathbf{R}$ for any matrix $\tilde{B}_I$ obtained by restriction from $\tilde{B}$.

**Remark 4.1.11.** The equivalent conditions (i)–(ii) in Lemma 4.1.7 do not imply the condition

(iii) for any $\tilde{B}' \in \mathbf{Q}'$, there exists $\tilde{B} \in \mathbf{Q}$ such that $\tilde{B}$ is obtained from $\tilde{B}'$ by restriction.

To see this, consider the quivers $Q$ and $Q'$ shown in Figure 4.1. Let $\mathbf{Q} = [Q_I]$ with $I = \{1, 2, 3\}$ and $\mathbf{Q}' = [Q']$. Then (i)–(ii) hold while (iii) fails. (Note that that the mutation sequence relating the two quivers in Figure 4.1 uses mutations at vertices outside of $I$.)

**Problem 4.1.12.** Let $T$ and $T'$ be finite trees, and let $\mathbf{T}$ and $\mathbf{T}'$ denote the mutation classes containing their respective orientations. Is it true that $\mathbf{T}$ is embeddable into $\mathbf{T}'$ if and only if $T$ can be obtained from $T'$ by contracting some edges?

**Remark 4.1.13.** Given mutation classes $\mathbf{Q}$ and $\mathbf{R}$, the problem of deciding whether $\mathbf{Q}$ is embeddable into $\mathbf{R}$ is generally very hard (unless $\mathbf{R}$ is finite).

For example, for any positive integer $k$, there is no known algorithm to determine whether the two-vertex *Kronecker quiver*

$$\bullet \overset{\longrightarrow}{\underset{\longrightarrow}{\cdots}} \bullet \tag{4.1}$$

with $k$ arrows is embeddable into the mutation class of a given quiver $R$. That is, for any $k$, there is no known general method to determine whether $R$ can be mutated to a quiver containing two vertices connected by $k$ arrows.

## 4.2. Seed subpatterns and cluster subalgebras

**Definition 4.2.1.** Let $(\tilde{\mathbf{x}}, \tilde{B})$ be a seed of rank $n$, and let $x_i \in \tilde{\mathbf{x}}$ be a cluster variable. *Freezing* (at the index $i$; or, of the variable $x_i$) is a transformation of the seed that reclassifies $i$ and $x_i$ as frozen, and accordingly removes the $i$th column from the exchange matrix $\tilde{B}$. (In addition, this would typically require a change of indexing, provided we want to keep using the smaller indices $1, \dots, n-1$ for the mutable variables.) More generally, we can freeze any subset of cluster variables. The order of freezing does not matter.

In the quiver case, freezing at a subset of mutable vertices amounts to reclassifying all these vertices, and the corresponding cluster variables, as frozen; and then removing all arrows connecting frozen vertices to each other.

**Lemma 4.2.2.** *Freezing commutes with seed mutation.*

**Proof.** This property is straightforward from the definitions. □

It is natural to try to extend the operation of restriction (to a full subquiver, or more generally to a submatrix of the exchange matrix) to the level of seeds. Note however that restriction does *not*, generally speaking, commute with seed mutation—because an exchange relation for a cluster variable associated with a vertex in a subquiver may well involve variables coming from outside the subquiver. This observation explains the additional constraints appearing in Definition 4.2.3 below.

**Definition 4.2.3.** Let $(\tilde{\mathbf{x}}, \tilde{B})$ be a seed, and let $I \subset [1, m]$ be a subset of indices such that $b_{ik} = 0$ for any $i \notin I$ and $k \in I \cap [1, n]$. (In other words, none of the variables $x_i$, for $i \notin I$, appears on the right-hand side of an exchange relation $x_k\, x'_k = \cdots$, for $k \in I$.) We then define the *restricted seed* $(\tilde{\mathbf{x}}_I, \tilde{B}_I)$ with the extended cluster $\tilde{\mathbf{x}}_I = (x_i)_{i \in I}$ and the extended exchange (sub)matrix $\tilde{B}_I$, cf. Definition 4.1.1. In the case where $\tilde{B}$ comes from a quiver $Q$, this is equivalent to requiring that there are no arrows between $I \cap [1, n]$ and $[1, m] \setminus I$.

**Example 4.2.4.** Let $Q$ and $Q' = \mu_k(Q)$ be the quivers shown in Figure 2.1 on the left and on the right, respectively. The subset $I = \{a, q, k\}$ does not satisfy the conditions in Definition 4.2.3 (neither for $Q$, nor for $Q'$) since the vertex $k \in I$ is connected by arrows to vertices not in $I$. On the other hand, if we first freeze $k$, then we can restrict to $I$ in $Q'$ (but not in $Q$).

**Lemma 4.2.5.** *Passing to a restricted seed commutes with seed mutation.*

It is easy to see that repeated applications of freezing and restriction produce an outcome that can be achieved by a single application of freezing followed by restriction.

**Definition 4.2.6.** Let $\Sigma$ be a seed. Freeze some subset of cluster variables, as in Definition 4.2.1, to obtain a seed $\Sigma'$. After that, apply the construction in Definition 4.2.3 to the seed $\Sigma'$ to obtain a restricted seed $\Sigma''$. We then say that the seed pattern defined by $\Sigma''$ is a *seed subpattern* of the seed pattern defined by $\Sigma$; and the cluster algebra associated to $\Sigma''$ is a *cluster subalgebra* of the cluster algebra associated to $\Sigma$.

Put simply, a cluster subpattern can be viewed as a part of the original pattern in which we are only allowed to exchange cluster variables labeled by a particular subset of indices, and in which we discard (some of) the coefficient variables that do not appear at all in the resulting exchange relations.

Note that if passing to a seed subpattern (resp., cluster subalgebra) involves freezing at least one cluster variable, then its rank is smaller than the rank of the original pattern (resp., cluster algebra).

**Example 4.2.7.** Let $\mathbf{P}_{n+3}$ be a convex polygon whose vertices are labeled $\{1, 2, \ldots, n+3\}$ in clockwise order. As explained in Section 2.2, each triangulation $T$ of $\mathbf{P}_{n+3}$ gives rise to a seed $(\tilde{x}(T), Q(T))$ whose mutable variables are the Plücker coordinates $P_{ij}$ labeled by the diagonals of $T$, and whose frozen variables are the Plücker coordinates labeled by the sides of $\mathbf{P}_{n+3}$. We thus obtain a seed pattern and a cluster algebra $R_{2,n+3}$ of rank $n$.

Let $S = \{s_1 < \cdots < s_\ell\}$ be a subset of $\{1, 2, \ldots, n\}$, with $\ell \geq 4$, and let $\mathbf{P}_S$ be the convex polygon on the vertex set $S$. Let $T$ be a triangulation of $\mathbf{P}_{n+3}$ that includes the sides of the polygon $\mathbf{P}_S$, and consequently contains a triangulation $T_S$ of $\mathbf{P}_S$. Take the seed $(\tilde{x}(T), Q(T))$, and freeze all the cluster variables in $\tilde{x}(T)$ corresponding to the sides of $\mathbf{P}_S$. We can now restrict from $\tilde{x}(T)$ to the subset $\tilde{x}(T_S)$ consisting of the elements labeled by diagonals and sides of $\mathbf{P}_S$. The resulting seed $(\tilde{x}(T_S), Q(T_S))$ defines a rank $\ell - 3$ cluster subalgebra (isomorphic to $R_{2,\ell}$) of the cluster algebra $R_{2,n+3}$.

The following properties of seed subpatterns and cluster subalgebras follow immediately from the definitions.

**Lemma 4.2.8.**

- *A seed subpattern of a seed subpattern is again a seed subpattern.*
- *If a cluster algebra has finitely many cluster variables, then so does any of its cluster subalgebras.*
- *A connected component of a quiver gives rise to a seed subpattern.*

## 4.3. Changing the coefficients

In this section we will establish a connection between seed patterns utilizing the same exchange matrices but different coefficient tuples. For a more thorough and comprehensive treatment, see [**14**].

We begin by a brief discussion of a couple of very simple special instances of coefficient change.

**Definition 4.3.1.** Let $(\tilde{\mathbf{x}}, \tilde{B})$ be a seed with an $m$-element extended cluster $\tilde{\mathbf{x}}$, and let $x_i \in \tilde{\mathbf{x}}$ be a coefficient (or frozen) variable. *Trivialization* (at the index $i$; or, of the variable $x_i$) is a transformation of the seed that removes $x_i$ from $\tilde{\mathbf{x}}$, and accordingly removes $i$ from the set of indices, and the $i$th row from the exchange matrix $\tilde{B}$. (As in the case of freezing, cf. Definition 4.2.1, a renumbering may be required if we want to use the indices $1, \dots, m-1$ after the trivialization.) More generally, we can trivialize any subset of coefficient variables; the order of operations does not matter.

In the quiver case, trivialization amounts to a removal of a subset of frozen vertices, together with all arrows incident to them; and the removal of the corresponding coefficient variables.

**Lemma 4.3.2.** *Trivialization of coefficients commutes with seed mutation.*

**Proof.** The key observation is that trivialization of a coefficient variable $x_i$ can be interpreted as setting $x_i = 1$. □

It is also easy to see that trivializing coefficients commutes with taking a seed subpattern.

**Remark 4.3.3.** Here is another simple way of transforming a seed pattern into a new one: introduce (any number of) additional "dummy" coefficient variables that do not appear in any exchange relations. That is, enlarge the extended exchange matrices by adding rows consisting entirely of zeroes; in the quiver case, just add isolated frozen vertices. This transformation changes the associated cluster algebra by tensoring it with the polynomial ring generated by the dummy variables.

We next describe a large class of "rescaling" transformations of seed patterns. Roughly, the idea is to multiply each cluster variable by a Laurent monomial in the coefficient variables, and then rewrite the exchange relations in terms of the "rescaled" variables. (We use quotation marks since we are not simply rescaling our variables, but also sending them to a different ambient field of rational functions.) The key property of this construction, formalized in Theorem 4.3.4 below, is that the resulting "rescaled" seeds are again related by (the same) mutations, yielding a seed pattern.

**Theorem 4.3.4.** *Let $n$, $m$, $\bar{m}$ be positive integers, with $n \le m$ and $n \le \bar{m}$. Let $\mathcal{F}$ and $\bar{\mathcal{F}}$ be the fields of rational functions in the variables $x_1, \dots, x_m$ and $\bar{x}_1, \dots, \bar{x}_{\bar{m}}$, respectively. We denote by $\mathcal{F}_{\rm sf}$ and $\bar{\mathcal{F}}_{\rm sf}$ the corresponding semifields of subtraction-free rational expressions. Let*

$$\varphi : \mathbb{Q}_{\rm sf}(x_1, \dots, x_m) \to \operatorname{Trop}(\bar{x}_{n+1}, \dots, \bar{x}_{\bar{m}})$$

*be a semifield homomorphism (determined by an arbitrary choice of Laurent monomials $\varphi(x_i) \in \operatorname{Trop}(\bar{x}_{n+1}, \dots, \bar{x}_{\bar{m}})$). Define the semifield map*

$$\psi : \mathbb{Q}_{\rm sf}(x_1, \dots, x_m) \to \mathbb{Q}_{\rm sf}(\bar{x}_1, \dots, \bar{x}_{\bar{m}})$$

*by setting*

$$\psi(x_i) = \begin{cases} \bar{x}_i\, \varphi(x_i) & \text{if } i \le n; \\ \varphi(x_i) & \text{if } i > n. \end{cases} \tag{4.2}$$

*Let $(\mathbf{x}(t), \mathbf{y}(t), B(t))_{t \in \mathbb{T}_n}$ be a seed pattern in $\mathcal{F}$ (cf. Definition 3.6.10), with*

$$\begin{aligned} \mathbf{x}(t) &= (x_{1;t}, \dots, x_{n;t}) \in \mathcal{F}^n, \\ \mathbf{y}(t) &= (y_{1;t}, \dots, y_{n;t}) \in \operatorname{Trop}(x_{n+1}, \dots, x_m)^n, \\ B(t) &= (b^t_{ij}), \end{aligned}$$

*with the initial cluster $\mathbf{x}(t_\circ) = (x_1, \dots, x_n)$, and with the frozen variables $x_{n+1}, \dots, x_m$. Define $\bar{\mathbf{x}}(t) = (\bar{x}_{1;t}, \dots, \bar{x}_{n;t})$ and $\bar{\mathbf{y}}(t) = (\bar{y}_{1;t}, \dots, \bar{y}_{n;t})$ by*

$$\bar{x}_{i;t} = \frac{\psi(x_{i;t})}{\varphi(x_{i;t})}, \tag{4.3}$$

$$\bar{y}_{k;t} = \varphi(\hat{y}_{k;t}) = \varphi(y_{k;t}) \prod_{i=1}^{n} \varphi(x_{i;t})^{b^t_{ik}}. \tag{4.4}$$

*Then $(\bar{\mathbf{x}}(t), \bar{\mathbf{y}}(t), B(t))_{t \in \mathbb{T}_n}$ is a seed pattern in $\bar{\mathcal{F}}$, with the same exchange matrices $B(t)$ and with the frozen variables $\bar{x}_{n+1}, \dots, \bar{x}_{\bar{m}}$.*

**Proof.** By Corollary 3.5.4, the elements $\hat{y}_{i;t}$ satisfy the $Y$-pattern recurrences. It follows that their images under the semifield homomorphism $\varphi$, cf. (4.4), satisfy (3.23), the tropical version of these recurrences. It remains to check that the elements $\bar{x}_{i;t}$ and $\bar{y}_{j;t}$ satisfy the exchange relations

$$\bar{x}_{k;t}\, \bar{x}_{k;t'} = \frac{\bar{y}_{k;t}}{\bar{y}_{k;t} \oplus 1} \prod_{b^t_{ik} > 0} \bar{x}_{i;t}^{b^t_{ik}} + \frac{1}{\bar{y}_{k;t} \oplus 1} \prod_{b^t_{ik} < 0} \bar{x}_{i;t}^{-b^t_{ik}}, \tag{4.5}$$

for $t \overset{k}{\text{———}} t'$ (cf. (3.24)). We shall deduce (4.5) from the exchange relation

$$x_{k;t}\, x_{k;t'} = \frac{1}{y_{k;t} \oplus 1}\Big( y_{k;t} \prod_{b^t_{ik}>0} x_{i;t}^{b^t_{ik}} + \prod_{b^t_{ik}<0} x_{i;t}^{-b^t_{ik}} \Big). \tag{4.6}$$

Applying the semifield homomorphism $\psi$ (resp., $\varphi$) to both sides of (4.6), dividing respective images by each other, using the fact that $\psi$ and $\varphi$ agree on the frozen variables $x_{n+1}, \dots, x_m$, and using (4.3) and (4.4), we get:

$$\begin{aligned}
\bar{x}_{k;t}\, \bar{x}_{k;t'} &= \frac{\psi(x_{k;t})\, \psi(x_{k;t'})}{\varphi(x_{k;t})\, \varphi(x_{k;t'})} \\
&= \frac{\varphi(y_{k;t} \oplus 1)}{\psi(y_{k;t} \oplus 1)} \cdot \frac{\psi(y_{k;t}) \prod_{b^t_{ik}>0} \psi(x_{i;t})^{b^t_{ik}} + \prod_{b^t_{ik}<0} \psi(x_{i;t})^{-b^t_{ik}}}{\varphi(y_{k;t}) \prod_{b^t_{ik}>0} \varphi(x_{i;t})^{b^t_{ik}} \oplus \prod_{b^t_{ik}<0} \varphi(x_{i;t})^{-b^t_{ik}}} \\
&= \frac{1}{\bar{y}_{k;t} \oplus 1} \cdot \frac{\varphi(y_{k;t}) \prod_{b^t_{ik}>0} \psi(x_{i;t})^{b^t_{ik}} + \prod_{b^t_{ik}<0} \psi(x_{i;t})^{-b^t_{ik}}}{\prod_{b^t_{ik}<0} \varphi(x_{i;t})^{-b^t_{ik}}} \\
&= \frac{1}{\bar{y}_{k;t} \oplus 1}\Big( \varphi(y_{k;t}) \prod_{b^t_{ik}>0} \psi(x_{i;t})^{b^t_{ik}} \prod_{b^t_{ik}<0} \varphi(x_{i;t})^{b^t_{ik}} + \prod_{b^t_{ik}<0} \bar{x}_{i;t}^{-b^t_{ik}} \Big) \\
&= \frac{1}{\bar{y}_{k;t} \oplus 1}\Big( \bar{y}_{k;t} \prod_{b^t_{ik}>0} \bar{x}_{i;t}^{b^t_{ik}} + \prod_{b^t_{ik}<0} \bar{x}_{i;t}^{-b^t_{ik}} \Big).
\end{aligned}$$

□

We next restate Theorem 4.3.4 in terms of extended exchange matrices.

**Proposition 4.3.5.** *Keep the assumptions and notation of Theorem 4.3.4. Let $\tilde{B}_\circ = \tilde{B}(t_\circ)$ be the initial extended exchange matrix of the original exchange pattern. Then the new initial seed $(\bar{\mathbf{x}}(t_\circ), \bar{\mathbf{y}}(t_\circ), B(t_\circ))$ has the extended exchange matrix $\overline{\tilde{B}}_\circ = \Psi \tilde{B}_\circ$ where $\Psi = (\psi_{ij})$ is the $\bar{m} \times m$ matrix whose entries $\psi_{ij}$ are defined by*

$$\psi(x_j) = \prod_{i=1}^{\bar{m}} \bar{x}_i^{\psi_{ij}}. \tag{4.7}$$

*(Note that (4.2) implies that $\psi(x_j)$ is a Laurent monomial in $\bar{x}_1, \dots, \bar{x}_{\bar{m}}$.)*

**Proof.** We use the notation $\overline{\tilde{B}}_\circ = (\bar{b}_{ik})$ and $\tilde{B}_\circ = (b_{ik})$. Recall that $\bar{b}_{ik} = b_{ik}$ for $i \le n$. We have:

$$\begin{aligned}
\prod_{i \le \bar{m}} \bar{x}_i^{\bar{b}_{ik}} &= \bar{y}_{k;t_\circ} \prod_{j \le n} \bar{x}_j^{\bar{b}_{jk}} = \varphi(y_{k;t_\circ}) \prod_{j \le n} \varphi(x_j)^{b_{jk}} \prod_{j \le n} \bar{x}_j^{b_{jk}} \\
&= \prod_{j \le m} \varphi(x_j)^{b_{jk}} \prod_{j \le n} \bar{x}_j^{b_{jk}} = \prod_{j \le m} \psi(x_j)^{b_{jk}} = \prod_{i \le \bar{m}} \prod_{j \le m} \bar{x}_i^{\psi_{ij} b_{jk}},
\end{aligned}$$

establishing that $\overline{\tilde{B}}_\circ = \Psi \tilde{B}_\circ$. (Here we used (3.22) and (4.7).) □

The following corollary will be particularly useful in the sequel.

**Corollary 4.3.6.** *Consider two seed patterns*

$$(\Sigma(t))_{t\in\mathbb{T}_n} = (\mathbf{x}(t), \mathbf{y}(t), B(t))_{t\in\mathbb{T}_n},$$
$$(\bar\Sigma(t))_{t\in\mathbb{T}_n} = (\bar{\mathbf{x}}(t), \bar{\mathbf{y}}(t), \bar B(t))_{t\in\mathbb{T}_n}$$

*with the same exchange matrices $B(t) = \bar B(t)$, for $t \in \mathbb{T}_n$. Suppose that all rows of the initial extended exchange matrix $\overline{\tilde B}_\circ$ for the second seed pattern lie in the $\mathbb{Z}$-span of the rows of the initial extended exchange matrix $\tilde B_\circ$ for the first seed pattern. If two labeled (resp., unlabeled) seeds $\Sigma(t) = \Sigma(t')$ coincide in the first pattern, then the corresponding seeds $\bar\Sigma(t) = \bar\Sigma(t')$ in the second pattern coincide as well.*

**Proof.** We use the notation of Proposition 4.3.5; thus $\tilde B_\circ$ and $\overline{\tilde B}_\circ$ are $m\times n$ and $\bar m \times n$ matrices whose top $n \times n$ submatrices equal $B(0)$. The $\mathbb{Z}$-span condition means that $\overline{\tilde B}_\circ = \Psi \tilde B_\circ$ where $\Psi = (\psi_{ij})$ is an integer $\bar m \times m$ matrix. Since the top $n \times n$ submatrices of $\tilde B_\circ$ and $\overline{\tilde B}_\circ$ coincide, we may assume that $\Psi$ is a block matrix of the form

$$\Psi = \begin{bmatrix} I & 0 \\ \Psi_1 & \Psi_2 \end{bmatrix}$$

where $I$ is the $n \times n$ identity matrix. We then define the maps $\psi$ and $\varphi$ by

$$\psi(x_j) = \prod_{i=1}^{\bar m} \bar x_i^{\psi_{ij}}, \quad \varphi(x_j) = \prod_{i=n+1}^{\bar m} \bar x_i^{\psi_{ij}},$$

for $1 \le j \le m$ (cf. (4.7)), so as to completely match the initial setup in Theorem 4.3.4/Proposition 4.3.5. Then Theorem 4.3.4 implies that the seed pattern $(\bar\Sigma(t))_{t\in\mathbb{T}_n}$ is given by the formulas (4.3)–(4.4). Consequently, each seed $\bar\Sigma(t)$ is uniquely determined by the corresponding seed $\Sigma(t)$ in the original pattern, and the claim follows. □

**Remark 4.3.7.** The $\mathbb{Z}$-span condition in Corollary 4.3.6 is in particular satisfied if the initial extended exchange matrix $\tilde B_\circ = \tilde B(t_\circ)$ has *full $\mathbb{Z}$-rank*, i.e., the $\mathbb{Z}$-span of its rows is the entire lattice $\mathbb{Z}^n$ of integer row-vectors.

This condition is also satisfied if the second seed pattern has trivial coefficients, i.e., if it has no frozen variables.

**Remark 4.3.8.** In Corollary 4.3.6, the $\mathbb{Z}$-span condition can be replaced by one involving a $\mathbb{Q}$-span. The proof remains essentially the same but requires allowing rational powers of the variables. This can be handled in two alternative ways: algebraically, by working in the semifield of "subtraction-free Puiseux expressions;" or analytically, by always choosing the branch of a fractional power that takes positive values at positive arguments.

## 4.4. Folding

Folding is a procedure that, under certain conditions, produces new seed patterns from existing ones. The basic idea is to exploit symmetries of a quiver to construct a quotient object (a folded extended exchange matrix), then design an equivariant mutation dynamics that would drive the algebraic dynamics of "folded seeds."

In this text, we discuss folding in a somewhat limited generality; see [**6**] for a more elaborate treatment. Our main application of folding will occur in Chapter 5, where we use it to construct cluster algebras of finite types $BCFG$ from those of "simply-laced" types $ADE$. Folding has also been used in [**7**] to classify the cluster algebras of finite mutation type; see Section 8.2.

We begin with a motivating example. Consider the quiver

$$1 \longleftarrow 2 \longrightarrow 3$$

of type $A_3$, with three mutable vertices. We notice the $\mathbb{Z}/2\mathbb{Z}$ symmetry of the quiver, and place only *two* distinct variables at its vertices, as follows:

$$x_0 \longleftarrow x_1 \longrightarrow x_0\,.$$

The exchange relations then become:

$$x_0 x_0' = x_1 + 1,$$
$$x_1 x_1' = x_0^2 + 1.$$

If we want to preserve the symmetry, we can now mutate either at vertex 2, or simultaneously at 1 and 3. Continuing in this fashion, we recover the seed pattern from Example 3.2.5.

**Definition 4.4.1.** Let $Q$ be a labeled quiver, as in Definition 2.7.1. More explicitly, we assume that $Q$ has $m$ vertices labeled $1, \dots, m$; the vertices labeled $1, \dots, n$ are mutable; the vertices labeled $n+1, \dots, m$ are frozen. Let $G$ be a group acting on the vertex set of $Q$, or equivalently on $\{1, \dots, m\}$. (For all practical purposes, it is safe to assume that the group $G$ is finite.) The notation $i \sim i'$ will mean that $i$ and $i'$ lie in the same $G$-orbit. We say that the quiver $Q$ (or the corresponding $m \times n$ extended exchange matrix $\tilde{B} = \tilde{B}(Q) = (b_{ij})$) is *$G$-admissible* if

(1) for any $i \sim i'$, index $i$ is mutable (i.e., $i \le n$) if and only if $i'$ is;

(2) for any indices $i$ and $j$, and any $g \in G$, we have $b_{ij} = b_{g(i),g(j)}$ ;

(3) for mutable indices $i \sim i'$, we have $b_{ii'} = 0$;

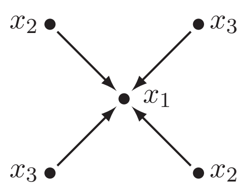
(4) for any $i \sim i'$, and any mutable $j$, we have $b_{ij} b_{i'j} \ge 0$.

Assume that $Q$ is $G$-admissible. We call a $G$-orbit *mutable* (resp., *frozen*) if it consists of mutable (resp., frozen) vertices, cf. condition (1) above. Let $\tilde{B}^G = \tilde{B}(Q)^G = (b^G_{IJ})$ be the matrix whose rows (resp., columns) are labeled by the $G$-orbits (resp., mutable $G$-orbits), and whose entries are given by

$$b^G_{IJ} = \sum_{i\in I} b_{ij}\,, \tag{4.8}$$

where $j$ is an arbitrary index in $J$. (By condition (2), the right-hand side of (4.8) does not depend on the choice of $j$.) We then say that $\tilde{B}^G$ is obtained from $\tilde{B}$ (or from the quiver $Q$) by *folding* with respect to the given $G$-action.

An example of folding is shown in Figure 4.2.




$$\tilde{B}^G = \begin{bmatrix} 0 & -1 & -1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1\,x_1' &= x_2^2\,x_3^2 + 1 \\ x_2\,x_2' &= 1 + x_1 \\ x_3\,x_3' &= 1 + x_1 \end{aligned}$$

**Figure 4.2.** The quiver $Q$ shown on the left is $G$-admissible with respect to the action of the group $G = \mathbb{Z}/2\mathbb{Z}$ wherein the generator of $G$ acts on the vertices of $Q$ by a $180^\circ$ rotation. All 5 vertices are mutable.

**Remark 4.4.2.** Condition (3) can be restated as saying that each $G$-orbit $I$ is *totally disconnected*; that is, there is no arrow between two vertices in $I$. Condition (4) means that there is no oriented path of length 2 through a mutable vertex that connects two vertices belonging to the same $G$-orbit. These conditions are dictated by the following considerations. If $i$ and $i'$ are in the same $G$-orbit $I$, then in the folded seed (to be defined below in this section), the same variable $x_I$ will be associated with both $i$ and $i'$. We do not want $x_I$ to appear on the right-hand side of the exchange relation for $x_I$ (hence (3)), nor do we want $x_I$ to appear in both monomials on the right-hand side of the exchange relation for some variable $x_J$ (hence (4)).

**Lemma 4.4.3.** *Let $Q$ be a $G$-admissible quiver. Then $\tilde{B}(Q)^G$ is an extended skew-symmetrizable matrix.*

**Proof.** As above, we use the notation $\tilde{B} = \tilde{B}(Q) = (b_{ij})$. Formula (4.8) implies that, for $I$ and $J$ mutable,

$$|J|\, b^G_{IJ} = \sum_{i\in I}\sum_{j\in J} b_{ij} = -\sum_{j\in J}\sum_{i\in I} b_{ji} = -|I|\, b^G_{JI}\,, \tag{4.9}$$

so the square matrix $(|J|\, b^G_{IJ})$ is skew-symmetric. □

**Lemma 4.4.4.** *Let $Q$ be a $G$-admissible quiver, with $\tilde{B}(Q) = (b_{ij})$. Let $I$ and $J$ be two $G$-orbits, with $J$ mutable. Then either all entries $b_{ij}$, for $i \in I$ and $j \in J$, are nonnegative, or all are nonpositive. Consequently the following are equivalent:*

- $b^G_{IJ} > 0$;
- *there exist $i \in I$ and $j \in J$ such that $b_{ij} > 0$;*
- *for every $i \in I$, there exists $j \in J$ such that $b_{ij} > 0$.*

*Similar equivalences hold with all inequality signs reversed.*

**Proof.** Let $i, i' \in I$ and $j, j' \in J$. Let $g \in G$ be such that $g(j') = j$. Using conditions (2) and (4) of Definition 4.4.1, we get $b_{ij}b_{i'j'} = b_{ij}b_{g(i'),j} \geq 0$, as desired. The equivalence statements in the lemma follow by (4.8) together with condition (2). □

For $Q$ a $G$-admissible quiver, condition (3) of Definition 4.4.1 ensures (cf. Exercise 2.1.4(3)) that the result of mutating $Q$ at the vertices in a mutable $G$-orbit $K$ does not depend on the order of mutations. We will denote this composition of mutations by

$$\mu_K = \prod_{k \in K} \mu_k \,, \tag{4.10}$$

making sure to only use this notation when it is well defined.

**Lemma 4.4.5.** *Let $Q$ be a $G$-admissible quiver, with $\tilde{B} = \tilde{B}(Q)$. Let $K$ be a mutable $G$-orbit such that $\mu_K(Q)$ is also $G$-admissible. Then*

$$(\mu_K(\tilde{B}))^G = \mu_K(\tilde{B}^G).$$

Note the abuse of notation in the last equality: on the left, $\mu_K$ is a composition of mutations defined by (4.10); on the right, $\mu_K$ is a single mutation at the mutable index $K$ of the folded matrix $\tilde{B}^G$.

**Proof.** Using the definition of matrix mutation (2.1) in combination with Lemma 4.4.4, we obtain:

(4.11)
$$\mu_K(\tilde{B}^G)_{IJ} = \begin{cases} -b^G_{IJ} & \text{if } K \in \{I, J\}; \\ b^G_{IJ} + b^G_{IK} b^G_{KJ} & \text{if } K \notin \{I, J\} \text{ and } b_{ik} > 0 \text{ and } b_{kj} > 0 \\ & \qquad \text{for some } i \in I, j \in J, k \in K; \\ b^G_{IJ} - b^G_{IK} b^G_{KJ} & \text{if } K \notin \{I, J\} \text{ and } b_{ik} < 0 \text{ and } b_{kj} < 0 \\ & \qquad \text{for some } i \in I, j \in J, k \in K; \\ b^G_{IJ} & \text{otherwise.} \end{cases}$$

On the other hand, mutating $\tilde{B}$ at each $k \in K$ (recall that $K$ is totally disconnected), we get:
(4.12)

$$\mu_K(\tilde{B})_{ij} = \begin{cases} -b_{ij} & \text{if } i \in K \text{ or } j \in K \text{ (i.e., } K \in \{I, J\}); \\ b_{ij} + \sum_{k\in K} b_{ik}b_{kj} & \text{if } i \notin K,\ j \notin K \text{ (i.e., } K \notin \{I, J\}), \text{ and} \\ & \qquad b_{ik} > 0 \text{ and } b_{kj} > 0 \text{ for some } k \in K; \\ b_{ij} - \sum_{k\in K} b_{ik}b_{kj} & \text{if } i \notin K,\ j \notin K \text{ (i.e., } K \notin \{I, J\}), \text{ and} \\ & \qquad b_{ik} < 0 \text{ and } b_{kj} < 0 \text{ for some } k \in K; \\ b_{ij} & \text{otherwise.} \end{cases}$$

By (4.8) (recall that $\mu_K(\tilde{B})$ is $G$-admissible), we have, for any $j \in J$:

$$((\mu_K(\tilde{B}))^G)_{IJ} = \sum_{i\in I} (\mu_K(\tilde{B}))_{ij}\,.$$

We note furthermore that all terms in the last sum will fall into the same case in (4.12). It is now easy to check that $(\mu_K(\tilde{B}^G))_{IJ} = ((\mu_K(\tilde{B}))^G)_{IJ}$. For example, in the second case of (4.12), we have:

$$\begin{aligned} ((\mu_K(\tilde{B}))^G)_{IJ} &= \sum_{i\in I}\big(b_{ij} + \sum_{k\in K} b_{ik}b_{kj}\big) \\ &= b^G_{IJ} + \sum_{k\in K} b_{kj} \sum_{i\in I} b_{ik} \\ &= b^G_{IJ} + b^G_{IK} b^G_{KJ} \\ &= \mu_K(\tilde{B}^G)_{IJ}. \end{aligned}$$

□

The “mutation commutes with folding” statement in Lemma 4.4.5 comes with a caveat: it requires admissibility of *both* quivers $Q$ and $\mu_K(Q)$ with respect to the group action at hand. Unfortunately, admissibility does not propagate via mutations: $Q$ may be $G$-admissible while $\mu_K(Q)$ is not.

**Example 4.4.6.** Let $Q$ be an oriented 6-cycle with six mutable vertices labeled 1 through 6 in clockwise order. Let the generator of $G = \mathbb{Z}/2\mathbb{Z}$ act by sending each vertex $i$ to the vertex $(i+3) \bmod 6$. Then $Q$ is $G$-admissible but each quiver $\mu_K(Q)$ is not.

We next proceed to the folding of seeds. This will require, in addition to an action of a group $G$, a choice of a semifield homomorphism that “bundles together” the variables associated with the vertices in the same $G$-orbit.

**Definition 4.4.7.** Let $G$ be a group acting on the set of indices $\{1, \dots, m\}$ so that every $g \in G$ maps the subset $\{1, \dots, n\}$ to itself. Let $m^G$ denote the number of orbits of this action. Let $\mathcal{F}$ (resp., $\mathcal{F}^G$) be a field isomorphic to the field of rational functions in $m$ (resp., $m^G$) independent variables, and let $\mathcal{F}_{\text{sf}}$

(resp., $\mathcal{F}_{\rm sf}^G$) denote the corresponding semifields of subtraction-free rational expressions. Let $\psi : \mathcal{F}_{\rm sf} \to \mathcal{F}_{\rm sf}^G$ be a surjective semifield homomorphism.

Let $Q$ be a quiver as above. A seed $\Sigma = (\tilde{\mathbf{x}}, \tilde{B}(Q))$ in $\mathcal{F}$, with the extended cluster $\tilde{\mathbf{x}} = (x_i)$, is called $(G,\psi)$-*admissible* if

- $Q$ is a $G$-admissible quiver;
- for any $i \sim i'$, we have $\psi(x_i) = \psi(x_{i'})$.

In this situation, we define a new "folded" seed $\Sigma^G = (\tilde{\mathbf{x}}^G, \tilde{B}^G)$ in $\mathcal{F}_{\rm sf}^G \subset \mathcal{F}^G$ whose extended exchange matrix $\tilde{B}^G$ is given by (4.8), and whose extended cluster $\tilde{\mathbf{x}}^G = (x_I)$ has $m^G$ elements $x_I$ indexed by the $G$-orbits and defined by $x_I = \psi(x_i)$, for $i \in I$. Note that since $\psi$ is a surjective homomorphism, the elements $x_I$ generate $\mathcal{F}^G$, hence are algebraically independent.

We can now extend Lemma 4.4.5 to the folding of seeds.

**Lemma 4.4.8.** *Let $\Sigma = (\tilde{\mathbf{x}}, \tilde{B}(Q))$ be a $(G,\psi)$-admissible seed as above. Let $K$ be a mutable $G$-orbit. If the quiver $\mu_K(Q)$ is $G$-admissible, then the seed $\mu_K(\Sigma)$ is $(G,\psi)$-admissible, and moreover $(\mu_K(\Sigma))^G = \mu_K(\Sigma^G)$.*

**Proof.** Let $\mathcal{O}$ denote the set of $G$-orbits. As above, we use the notation

$$\begin{aligned} \Sigma &= (\tilde{\mathbf{x}}, \tilde{B}), & \tilde{\mathbf{x}} &= (x_i), & \tilde{B} &= \tilde{B}(Q) = (b_{ij}), \\ \Sigma^G &= (\tilde{\mathbf{x}}^G, \tilde{B}^G), & \tilde{\mathbf{x}}^G &= (x_I)_{I \in \mathcal{O}}, & \tilde{B}^G &= (b_{IJ}^G) \end{aligned}$$

By Lemma 4.4.5, all we need to show is that the extended clusters in $\mu_K(\Sigma^G)$ and in $(\mu_K(\Sigma))^G$ are the same. The extended cluster in $\mu_K(\Sigma^G)$ is obtained from $\tilde{\mathbf{x}}^G$ by replacing $x_K$ by the element $x'_K$ defined by

$$x_K x'_K = \prod_{b_{IK}^G > 0} x_I^{b_{IK}^G} + \prod_{b_{IK}^G < 0} x_I^{-b_{IK}^G}, \tag{4.13}$$

for any $k$ in the orbit $K$.

The extended cluster in $\mu_K(\Sigma)$ is obtained from $\tilde{\mathbf{x}}$ by replacing each $x_k$, for $k \in K$, by the element $x'_k$ defined by

$$x_k x'_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}. \tag{4.14}$$

The extended cluster in $(\mu_K(\Sigma))^G$ is then obtained by applying the homomorphism $\psi$; as we know, $\psi$ sends each $x_i$ to $x_{[i]}$ where $[i]$ denotes the $G$-orbit containing $i$. As a result, we get the variables $x_I$ for $I \in \mathcal{O} - \{K\}$, together with $\psi(x'_k)$; here $k$ is an arbitrary element of $K$. We need to show that $\psi(x'_k) = x'_K$ where $x'_K$ is defined by (4.13).

Applying $\psi$ to (4.14), we get

$$x_K \psi(x'_k) = \prod_{b_{ik}>0} x_{[i]}^{b_{ik}} + \prod_{b_{ik}<0} x_{[i]}^{-b_{ik}}. \tag{4.15}$$

Note that in the first monomial on the right-hand side of (4.15), the exponent of a given $x_I$ will be nonzero if and only if there exists $i \in I$ such that $b_{ik} > 0$, in which case the exponent will be $\sum_{i\in I} b_{ik} = b^G_{IK}$. But by Lemma 4.4.4, the existence of such an index $i \in I$ is equivalent to the inequality $b^G_{IK} > 0$. It follows that the first monomials in the right-hand sides of (4.15) and (4.13) agree. A similar argument shows that the second monomials agree, and we are done. □

**Definition 4.4.9.** Let $G$ be a group acting on the vertex set of a quiver $Q$. We say that $Q$ is *globally foldable* with respect to $G$ if $Q$ is $G$-admissible and moreover for any sequence of mutable $G$-orbits $J_1, \dots, J_k$, the quiver $(\mu_{J_k} \circ \cdots \circ \mu_{J_1})(Q)$ is $G$-admissible.

**Exercise 4.4.10.** For the quiver $Q$ and the action of the group $G = \mathbb{Z}/2\mathbb{Z}$ described in Figure 4.2, show that $Q$ is globally foldable with respect to $G$.

Lemma 4.4.8 implies that if $Q$ is globally foldable, then we can fold all the seeds in the corresponding seed pattern.

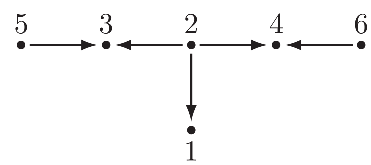
**Corollary 4.4.11.** *Let $Q$ be a quiver which is globally foldable with respect to a group $G$ acting on the set of its vertices. Let $\Sigma = (\tilde{\mathbf{x}}, \tilde{B}(Q))$ be a seed in the field $\mathcal{F}$ of rational functions freely generated by an extended cluster $\tilde{\mathbf{x}} = (x_i)$. Let $\tilde{\mathbf{x}}^G = (x_I)$ be a collection of formal variables labeled by the $G$-orbits $I$, and let $\mathcal{F}^G$ denote the field of rational functions in these variables. Define the surjective homomorphism*

$$\begin{aligned} \psi : \mathcal{F}_{\mathrm{sf}} &\to \mathcal{F}^G_{\mathrm{sf}} \\ x_i &\mapsto x_I \quad (i \in I) \end{aligned}$$

*of the corresponding semifields of subtraction-free rational expressions, so that $\Sigma$ is a $(G, \psi)$-admissible seed. Then for any mutable $G$-orbits $J_1, \dots, J_k$, the seed $(\mu_{J_k} \circ \cdots \circ \mu_{J_1})(\Sigma)$ is $(G, \psi)$-admissible, and moreover the folded seeds $((\mu_{J_k} \circ \cdots \circ \mu_{J_1})(\Sigma))^G$ form a seed pattern in $\mathcal{F}^G$, with the initial extended exchange matrix $(\tilde{B}(Q))^G$.*

In general, it may be very difficult to determine whether a quiver is globally foldable. Fortunately, the cases that we will need in Chapter 5 for the purposes of finite type classification will turn out to be easy to handle. One of these cases is discussed in Exercise 4.4.12 below. We will revisit this exercise in Section 5.7.

**Exercise 4.4.12.** Let $Q$ be the quiver at the left of Figure 4.3, with all its vertices mutable. Let the generator of the group $G = \mathbb{Z}/2\mathbb{Z}$ act on the vertices of $Q$ by fixing the vertices 1 and 2, exchanging the vertices 3 and 4, and exchanging the vertices 5 and 6. Then $Q$ is $G$-admissible; the folded skew-symmetrizable matrix $B(Q)^G$ is shown at the right in Figure 4.3. (The rows and columns of this matrix are labeled by the orbits $\{1\}, \{2\}, \{3,4\}, \{5,6\}$.) Show that $Q$ is globally foldable, by cataloguing all quivers (up to $G$-equivariant isomorphism) which can be obtained from $Q$ by iterating the transformations $\mu_J$ associated with $G$-orbits $J$.




$$\tilde{B}^G = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

**Figure 4.3.** Folding a type $E_6$ quiver.

**Remark 4.4.13.** There exist skew-symmetrizable square matrices which cannot be obtained by folding a globally foldable quiver, see [**22**, Section 14]. Consequently the technique of folding (in the current state of the art) is not powerful enough to reduce the study of general seed patterns, and the associated cluster algebras, to the quiver case.

Chapter 5

# Finite type classification

A seed pattern (or the corresponding cluster algebra) is said to be of *finite type* if it has finitely many distinct seeds. To rephrase, a seed gives rise to a pattern (or cluster algebra) of finite type if the process of iterated mutation produces finitely many distinct seeds.

If a seed pattern has finite type, then it obviously has finitely many distinct cluster variables. In fact the converse is also true, see Corollary 5.1.6.

The main result of this chapter is the classification, originally obtained in [**11**], of seed patterns (equivalently cluster algebras) of finite type. It turns out that the property of a cluster algebra with an initial seed $(\mathbf{x}, \mathbf{y}, B)$ to be of finite type depends only on (the mutation class of) the exchange matrix $B$ but not on the choice of a coefficient tuple $\mathbf{y}$. Even more remarkably, such mutation classes are in one-to-one correspondence with Cartan matrices of finite type, or equivalently with finite crystallographic root systems.

To show that a cluster algebra of finite type must come from a Cartan matrix of finite type, we follow the approach of [**11**]. In particular we show that a cluster algebra has finite type if and only if all its exchange matrices $B = (b_{ij})$ have the property that $|b_{ij}b_{ji}| \leq 3$ for any pair of indices $i, j$.

The fact that cluster algebras coming from finite-type Cartan matrices are of finite type is established by a string of case-by-case arguments. For each type, we construct a particular seed pattern that has finitely many seeds, and whose initial extended exchange matrix has full $\mathbb{Z}$-rank. A generalization to arbitrary coefficients is then obtained via Remark 4.3.7. We note that the original proof [**11**] of this direction of the classification theorem used a different (root-theoretic) strategy relying on some rather delicate properties of *generalized associahedra*, cf. Section 6.2.

## 5.1. Finite type classification in rank 2

Any seed pattern of rank 1 has two seeds, so is of finite type. In this section, we determine which seed patterns of rank 2 are of finite type.

Recall that the seeds in any seed pattern of rank 2 can be labeled by integers $t \in \mathbb{Z}$. Without loss of generality, we may assume that the exchange matrices $B(t)$ are given by

$$B(t) = (-1)^t \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix} \tag{5.1}$$

where the integers $b$ and $c$ are either both positive, or both equal to 0.

**Theorem 5.1.1.** *A seed pattern of rank 2, with exchange matrices given by (5.1), is of finite type if and only if $bc \leq 3$.*

**Remark 5.1.2.** The statement of Theorem 5.1.1 is very similar to the classification of finite crystallographic reflection groups of rank 2, which we review below. (The proof of Theorem 5.1.1 will not depend on this.) Consider the subgroup $W \subset \mathrm{GL}_2$ generated by the reflections

$$s_1 = \begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 1 & 0 \\ c & -1 \end{bmatrix}. \tag{5.2}$$

Since $s_1^2 = s_2^2 = 1$, the group $W$ is finite if and only if the element

$$s_1 s_2 = \begin{bmatrix} bc-1 & -b \\ c & -1 \end{bmatrix}$$

is of finite order. An eigenvalue $\lambda$ of $s_1 s_2$ satisfies the characteristic equation

$$\lambda^2 - (bc-2)\lambda + 1 = 0. \tag{5.3}$$

If $bc \in \{1, 2, 3\}$, then the roots of (5.3) are two distinct roots of unity, so $s_1 s_2$ has finite order (specifically, order 3, 4, or 6, respectively). If $b = c = 0$, then $(s_1 s_2)^2 = 1$ by inspection. If $bc > 4$, then the eigenvalues are real and not equal to $\pm 1$, so $W$ is infinite. For $bc = 4$, one checks that

$$(s_1 s_2)^k = \begin{bmatrix} 2k+1 & -kb \\ kc & -2k+1 \end{bmatrix},$$

implying that $W$ is infinite in this case as well.

**Proof of Theorem 5.1.1.** The case $b = c = 0$ is trivial. The cases $bc \in \{1, 2, 3\}$ are handled by the calculation in Exercise 5.1.3 below. An alternative (more conceptual) explanation of why the rank 2 cluster algebras with $bc \leq 3$ are of finite type will be given later in this chapter.

**Exercise 5.1.3.** Use Corollary 4.3.6 to show that any seed pattern with exchange matrices

$$B(t) = (-1)^t \begin{bmatrix} 0 & 1 \\ -c & 0 \end{bmatrix}, \tag{5.4}$$

with $c \in \{1, 2, 3\}$, has finitely many distinct seeds. To this end, consider the seed with the initial extended exchange matrix

$$\tilde{B}_\circ = \begin{bmatrix} 0 & 1 \\ -c & 0 \\ 1 & 0 \end{bmatrix}.$$

Note that the rows of $\tilde{B}_\circ$ span $\mathbb{Z}^2$ over $\mathbb{Z}$, so by Remark 4.3.7, it suffices to check that the seed pattern defined by $\tilde{B}_\circ$ has finitely many seeds. (In fact, it has five seeds for $c = 1$, six seeds for $c = 2$, and eight seeds for $c = 3$.)

Now suppose that $bc \geq 4$. Let $(\mathbf{x}(t), \mathbf{y}(t), B(t))_{t \in \mathbb{Z}}$ be a seed pattern in an ambient field $\mathcal{F}$ whose exchange matrices are given by (5.1). We denote the cluster variables in this pattern by $z_t$ $(t \in \mathbb{Z})$, so that

$$\ldots, \mathbf{x}(0) = (z_1, z_2), \quad \mathbf{x}(1) = (z_3, z_2), \quad \mathbf{x}(2) = (z_3, z_4), \quad \mathbf{x}(3) = (z_5, z_4), \ldots$$

Our goal is to show that the set $\{z_t : t \in \mathbb{Z}\} \subset \mathcal{F}$ is infinite. (In fact, all cluster variables $z_t$ turn out to be distinct.)

Let $u$ be a formal variable, and consider the semifield $U = \{u^r : r \in \mathbb{R}\}$ of formal monomials in $u$ with real exponents, with the operations defined by

$$u^r \oplus u^s = u^{\max(r,s)},$$
$$u^r \cdot u^s = u^{r+s}.$$

We shall prove that the set $\{z_t\}$ is infinite by constructing a semifield homomorphism $\psi : \mathcal{F} \to U$ such that the image $\{\psi(z_t) : t \in \mathbb{Z}\} \subset U$ is infinite. As in Remark 5.1.2, we consider the cases $bc > 4$ and $bc = 4$ separately.

**Case 1:** $bc > 4$. In this case, there is a real number $\lambda > 1$ satisfying (5.3). Let $\psi$ be the map uniquely defined by setting the image of every frozen variable to be 1, and setting

$$\psi(z_1) = u^c, \quad \psi(z_2) = u^{\lambda+1}. \tag{5.5}$$

The exchange relations imply that the images $\psi(z_t)$ satisfy

$$\psi(z_{t-1})\,\psi(z_{t+1}) = \begin{cases} \psi(z_t)^c \oplus 1 & \text{if } t \text{ is even;} \\ \psi(z_t)^b \oplus 1 & \text{if } t \text{ is odd} \end{cases} \tag{5.6}$$

(cf. (3.3)). We claim that, for $k = 0, 1, 2, \ldots$, one has

$$\psi(z_{2k+1}) = u^{\lambda^k c}, \quad \psi(z_{2k+2}) = u^{\lambda^k(\lambda+1)}. \tag{5.7}$$

The base case $k = 0$ holds in view of (5.5). Induction step:

$$\psi(z_{2k+3}) = \frac{\psi(z_{2k+2})^c \oplus 1}{\psi(z_{2k+1})} = u^{\lambda^k(\lambda+1)c-\lambda^k c} = u^{\lambda^{k+1}c},$$
$$\psi(z_{2k+4}) = \frac{\psi(z_{2k+3})^b \oplus 1}{\psi(z_{2k+2})} = u^{\lambda^{k+1}cb-\lambda^k(\lambda+1)} = u^{\lambda^k(\lambda cb-\lambda-1)} = u^{\lambda^{k+1}(\lambda+1)},$$

where the last equality relies on (5.3). Since $\lambda > 1$, formulas (5.7) imply that the image set $\{\psi(z_t)\} \subset U$ is infinite.

**Case 2:** $bc = 4$. While the general logic of the argument remains the same, it has to be adjusted since in this case, the only root of the equation (5.3) is $\lambda = 1$. So we replace (5.5) by

$$\psi(z_1) = u, \quad \psi(z_2) = u^b,$$

and verify by induction that $\psi(z_{2k-1}) = u^{2k-1}$ and $\psi(z_{2k}) = u^{kb}$, implying the claim. Details are left to the reader. □

Theorem 5.1.1 suggests the following definition.

**Definition 5.1.4.** A skew-symmetrizable matrix $B = (b_{ij})$ is called 2-*finite* if and only if for any matrix $B'$ mutation equivalent to $B$ and any indices $i$ and $j$, we have $|b'_{ij}b'_{ji}| \le 3$.

**Corollary 5.1.5.** *In a seed pattern of finite type, every exchange matrix is* 2-*finite.*

**Proof.** This is an immediate consequence of Theorem 5.1.1. If an exchange matrix $B$ is mutation equivalent to $B' = (b'_{ij})$ such that $|b'_{ij}b'_{ji}| \ge 4$ for some $i$ and $j$, then alternately applying mutations $\mu_i$ and $\mu_j$ to the corresponding seed, we obtain infinitely many distinct cluster variables, and infinitely many distinct seeds. □

**Corollary 5.1.6.** *A seed pattern is of finite type if and only if it has finitely many cluster variables.*

**Proof.** One direction is obvious: if a seed pattern has finitely many distinct seeds, then it has finitely many cluster variables. Conversely, if a seed pattern has finitely many cluster variables, then the only way it could possibly have infinitely many distinct seeds is if there were infinitely many distinct exchange matrices. But then the entries of the exchange matrices could not be bounded, and in particular there would exist an exchange matrix which is not 2-finite, leading to a sequence of infinitely many distinct cluster variables. □

## 5.2. Cartan matrices and Dynkin diagrams

The classification of cluster algebras of finite type turns out to be completely parallel to the famous Cartan-Killing classification of semisimple Lie algebras, finite crystallographic root systems, etc. The latter classification can be found in many books, see, e.g., [**15, 17**]. In this section, we quickly review it, using the language of *Cartan matrices* and *Dynkin diagrams*. We then explain the connection between (symmetrizable) Cartan matrices and (skew-symmetrizable) exchange matrices.

**Definition 5.2.1.** A square integer matrix $A = (a_{ij})$ is called a *symmetrizable generalized Cartan matrix* if it satisfies the following conditions:

- all diagonal entries of $A$ are equal to 2;
- all off-diagonal entries of $A$ are non-positive;
- there exists a diagonal matrix $D$ with positive diagonal entries such that the matrix $DA$ is symmetric.

We call $A$ *positive* if $DA$ is positive definite; this is equivalent to the positivity of all principal minors $\Delta_{I,I}(A)$. In particular, any such matrix satisfies

$$\Delta_{\{i,j\},\{i,j\}}(A) = \det \begin{pmatrix} 2 & a_{ij} \\ a_{ji} & 2 \end{pmatrix} = 4 - a_{ij}a_{ji} > 0,$$

or equivalently

$$a_{ij}a_{ji} \leq 3 \text{ for } i \neq j. \tag{5.8}$$

Positive symmetrizable generalized Cartan matrices are often referred to simply as *Cartan matrices*, or *Cartan matrices of finite type*.

**Example 5.2.2.** In view of (5.8), a $2 \times 2$ matrix

$$A = \begin{bmatrix} 2 & -b \\ -c & 2 \end{bmatrix}$$

is a Cartan matrix of finite type if and only if one of the following holds:

- $b = c = 0$;
- $b = c = 1$;
- $b = 1$, $c = 2$ or $b = 2$, $c = 1$;
- $b = 1$, $c = 3$ or $b = 3$, $c = 1$.

Note that this matches the classifications in Theorem 5.1.1 and Remark 5.1.2. The latter match has a well-known explanation: there is a canonical correspondence between Cartan matrices of finite type and finite Weyl groups (or finite crystallographic root systems). The relationship between these objects and cluster algebras of finite type is much more subtle.

**Remark 5.2.3.** A Cartan matrix encodes essential information about the geometry of a root system (or the corresponding Weyl group). The classification of finite crystallographic root systems (resp., associated reflection groups) can be reduced to classifying Cartan matrices of finite type. This standard material can be found in many books, see, e.g., [**16**].

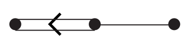
**Definition 5.2.4.** The *Coxeter graph* of an $n \times n$ Cartan matrix $A$ is a simple graph with vertices $1, \dots, n$ in which vertices $i$ and $j \neq i$ are joined by an edge whenever $a_{ij} \neq 0$. If $a_{ij} \in \{0, -1\}$ for all $i \neq j$, then $A$ is uniquely determined by its Coxeter graph. (Such matrices are called *simply-laced*.) If $A$ is not simply-laced but of finite type then, in view of (5.8), one needs a little additional information to specify $A$. This is done by replacing the Coxeter graph of $A$ with its *Dynkin diagram* in which, instead of a single edge, each pair of vertices $i$ and $j$ with $a_{ij}a_{ji} > 1$ is shown as follows:

$$i \bullet\!\!\Rightarrow\!\!\bullet\, j \quad \text{if } a_{ij} = -1 \text{ and } a_{ji} = -2;$$

$$i \bullet\!\!\Rrightarrow\!\!\bullet\, j \quad \text{if } a_{ij} = -1 \text{ and } a_{ji} = -3.$$

Note that our usage of the terms *Coxeter graph* and *Dynkin diagram* is a bit non-standard: Coxeter graphs are usually defined as labeled graphs, and Dynkin diagrams are often assumed to be connected. (We make no such requirement.) Also, as in [**11**], we use the conventions of [**17**] (as opposed to those in [**2**]) in going between Dynkin diagrams and Cartan matrices.

A couple of examples are shown in Figure 5.1. The notation $B_3$ and $C_3$ is explained in Figure 5.2.

$B_3$  $\begin{bmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

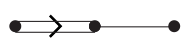
$C_3$  $\begin{bmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

**Figure 5.1.** Dynkin diagrams and Cartan matrices of types $B_3$ and $C_3$.

**Remark 5.2.5.** It is important to stress that the meaning of double and triple arrows in a Dynkin diagram is very different from the meaning of multiple arrows in a quiver. A double arrow $1 \bullet\!\!\Rightarrow\!\!\bullet\, 2$ in a Dynkin diagram corresponds to the submatrix $\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$ of the associated Cartan matrix. Meanwhile, a double arrow $1 \rightrightarrows 2$ in a quiver corresponds to the submatrix $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ of the associated exchange matrix.

A Cartan matrix is called *indecomposable* if its Dynkin diagram is connected. By a simultaneous permutation of rows and columns, any Cartan matrix $A$ can be transformed into a block-diagonal matrix with indecomposable blocks. This corresponds to decomposing the Dynkin diagram of $A$ into connected components. The *type* of $A$ (i.e., its equivalence class with respect to simultaneous permutations of rows and columns) is determined by specifying the multiplicity of each type of connected Dynkin diagram in this decomposition. Thus to classify Cartan matrices, one needs to produce the list of all connected Dynkin diagrams. The celebrated *Cartan-Killing classification* asserts that this list is given as follows.

**Theorem 5.2.6.** *Figure 5.2 gives a complete list of connected Dynkin diagrams corresponding to indecomposable Cartan matrices of finite type.*

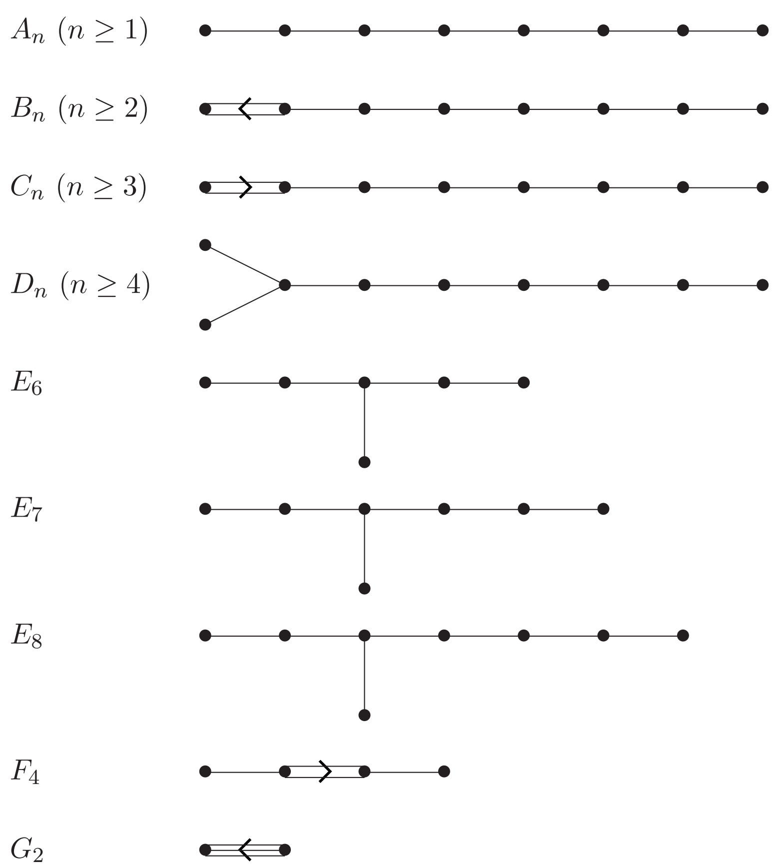



**Figure 5.2.** Dynkin diagrams of indecomposable Cartan matrices. The subscript $n$ indicates the number of nodes in the diagram.

We do not include the proof of Theorem 5.2.6 in this book, nor do we rely on this theorem anywhere in our proofs. We will, however, make extensive use of the standard nomenclature of Dynkin diagrams presented in Figure 5.2. The notation $X_n \sqcup Y_{n'}$ will denote the disjoint union of two Dynkin diagrams $X_n$ and $Y_{n'}$.

The relationship between Cartan matrices and skew-symmetrizable matrices is based on the following definition [**11**].

**Definition 5.2.7.** Let $B = (b_{ij})$ be a skew-symmetrizable integer matrix. Its *Cartan counterpart* is the symmetrizable generalized Cartan matrix

$$A = A(B) = (a_{ij}) \tag{5.9}$$

of the same size defined by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j; \\ -|b_{ij}| & \text{if } i \neq j. \end{cases} \tag{5.10}$$

The main result of this chapter is the following classification of cluster algebras of finite type [**11**].

**Theorem 5.2.8.** *A seed pattern (or the corresponding cluster algebra) is of finite type if and only if it contains an exchange matrix $B$ such that $A(B)$ (see Equation* (5.9) *and Equation* (5.10)*) is a Cartan matrix of finite type.*

The proof of Theorem 5.2.8 spans Sections 5.3–5.10. An overview of the proof is given below in this section.

One important feature of Theorem 5.2.8 is that the finite type property depends solely on the exchange matrix $B$ but not on the coefficient tuple $\mathbf{y}$. In other words, the top $n \times n$ submatrix $B$ of an extended exchange matrix $\tilde{B}$ determines whether the seed pattern at hand has finitely many seeds. The bottom part of $\tilde{B}$ has no effect on this property.

**Definition 5.2.9.** Let $X_n$ be a Dynkin diagram on $n$ vertices. A seed pattern of rank $n$ (or the corresponding cluster algebra) is said to be *of type $X_n$* if one of its exchange matrices $B$ has Cartan counterpart of type $X_n$.

**Example 5.2.10.** The matrices

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}$$

define seed patterns (and cluster algebras) of types $A_1 \sqcup A_1$, $A_2$, $B_2$, and $G_2$, respectively.

**Remark 5.2.11.** Suppose $X_n$ is simply laced, i.e., is one of the types $A_n$, $D_n$, $E_6$, $E_7$, $E_8$. Then a seed pattern is of type $X_n$ if one of its exchange matrices $B$ corresponds to a quiver which is an orientation of a Dynkin diagram of type $X_n$. We note that by Exercise 2.6.3, all orientations of a tree are mutation equivalent to each other, so if one of them is present in the pattern, then all of them are.

A priori, Definition 5.2.9 allows for the possibility that a given seed pattern is simultaneously of two different types. However, the following companion to Theorem 5.2.8 shows that this cannot happen.

**Theorem 5.2.12.** *Let $B'$ and $B''$ be skew-symmetrizable square matrices whose Cartan counterparts $A(B')$ and $A(B'')$ are Cartan matrices of finite type. Then the following are equivalent:*

(1) *the Cartan matrices $A(B')$ and $A(B'')$ have the same type;*
(2) *$B'$ and $B''$ are mutation equivalent.*

This theorem will be proved in Section 5.9.

By Theorem 5.2.8, a seed pattern of finite type must contain exchange matrices whose Cartan counterparts are Cartan matrices of finite type. By Theorem 5.2.12, all these matrices have the same type. Consequently the (Cartan-Killing) type of a seed pattern (resp., cluster algebra) of finite type is unambiguously defined.

We conclude this section by providing an overview of the remainder of Chapter 5.

Sections 5.3–5.8 are dedicated to showing that any seed pattern that has an exchange matrix whose Cartan counterpart is of one of the types $A_n, B_n, \ldots, G_2$ has finitely many seeds. This is done case by case. The idea is to explicitly construct, for each (indecomposable) type, a particular seed pattern whose exchange matrices have full $\mathbb{Z}$-rank, and show that this pattern has finitely many seeds. Then an argument based on Corollary 4.3.6 and Remark 4.3.7 will imply the same for *all* cluster algebras of the corresponding type.

In Sections 5.3–5.4 we exhibit seed patterns of types $A_n$ and $D_n$ possessing the requisite properties. In type $A_n$, we utilize the construction involving the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}_{2,n+3}$, cf. Section 1.2. A modification of this construction (admittedly much more technical than the type $A_n$ case) is then used for the type $D_n$. In Section 5.5, we handle the types $B_n$ and $C_n$ using the technique of folding introduced in Section 4.4.

The exceptional types are treated in a different way. In Section 5.6, we use a computer check to verify that the cluster algebras of type $E_8$ are of finite type. This implies the same for the types $E_6$ and $E_7$. Types $F_4$ and $G_2$ are then handled in Section 5.7, via folding of $E_6$ and $D_4$, respectively.

In Section 5.8, we discuss decomposable types, and complete the first part of the proof of Theorem 5.2.8.

We note that Sections 5.3–5.6 contain many incidental examples of cluster algebras of finite type, in addition to those used in the proof of the classification theorem.

Theorem 5.2.12 is proved in Section 5.9. Also in this section, we complete and summarize the enumeration of the seeds (equivalently, clusters) and the cluster variables for all finite types.

The proof of Theorem 5.2.8 is completed in Section 5.10, by demonstrating that any seed pattern of finite type comes from a Cartan matrix of finite type. This is done by exploiting the fact that every exchange matrix appearing in such a seed pattern is 2-finite, see Corollary 5.1.5.

Theorem 5.2.8 provides a characterization of seed patterns of finite type which, while conceptually satisfying, is not particularly useful in practice. In Section 5.11, we discuss an alternative criterion for recognizing whether a seed pattern is of finite type. This criterion is formulated directly in terms of the given exchange matrix $B$ (as opposed to its mutation class).

## 5.3. Seed patterns of type $A_n$

The main result of this chapter is Theorem 5.3.2, which says that seed patterns of type $A_n$ are of finite type. To prove this result, we use the fact that seed patterns of type $A_n$ are governed by the combinatorics of triangulated polygons. A case in point is the cluster algebra structure in the Plücker ring $R_{2,n+3}$ (the homogeneous coordinate ring of $\mathrm{Gr}_{2,n+3}$), discussed in Section 1.2. Working with this explicitly-defined ring is very helpful because it allows us to verify that a cluster variable indexed by a diagonal in a triangulation only depends on the diagonal, and not on the triangulation, or the sequence of mutations we took to get to that cluster. After verifying that the cluster structure in $R_{2,n+3}$ is of finite type, we check that one of its exchanges matrices has full $\mathbb{Z}$-rank, which completes the proof of Theorem 5.3.2 using an argument based on Corollary 4.3.6 and Remark 4.3.7.

Let $T$ be a triangulation of a convex $(n+3)$-gon $\mathbf{P}_{n+3}$ by $n$ noncrossing diagonals labeled $1, \dots, n$. We define the skew-symmetric $n \times n$ matrix $B(T) = (b_{ij}(T))$ as follows:

$$b_{ij}(T) = \begin{cases} 1 & \text{if } i \text{ and } j \text{ label two sides of a triangle in } T, \\ & \quad \text{with } j \text{ following } i \text{ in the clockwise order;} \\ -1 & \text{if } i \text{ and } j \text{ label two sides of a triangle in } T, \\ & \quad \text{with } i \text{ following } j \text{ in the clockwise order;} \\ 0 & \text{if } i \text{ and } j \text{ do not belong to the same triangle in } T. \end{cases} \tag{5.11}$$

Thus the matrix $B(T)$ corresponds to the mutable part of the quiver $Q(T)$ described in Definition 2.2.1 and Figure 2.2.

The following easy lemma provides an alternative definition of the notion of a seed pattern of type $A_n$.

**Lemma 5.3.1.** *A seed pattern has type $A_n$ if and only if one (equivalently, any) of its exchange matrices can be identified with the exchange matrix $B(T)$ corresponding to a triangulation $T$ of a convex $(n+3)$-gon, cf.* (5.11).

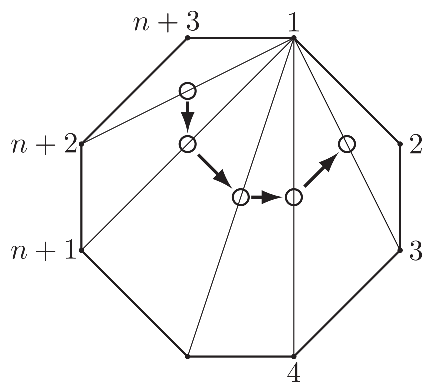



**Figure 5.3.** A triangulation $T_\circ$ of the polygon $\mathbf{P}_{n+3}$ (here $n = 5$). The mutable part of the quiver $Q(T_\circ)$ (see Definition 2.2.1) is the *equioriented* Dynkin quiver of type $A_n$.

**Theorem 5.3.2.** *Seed patterns of type $A_n$ are of finite type.*

**Remark 5.3.3.** Definition 5.2.9 imposes no restrictions on the bottom part of the extended exchange matrices, nor on the number of frozen variables. In light of Lemma 5.3.1, Theorem 5.3.2 asserts that as long as the mutable part of a quiver comes from a triangulated polygon, the total number of seeds is finite.

**Proof.** We start by showing that a particular seed pattern of type $A_n$, the one associated with the Plücker ring $R_{2,n+3}$, is of finite type.

As in Section 1.2, we label the vertices of the polygon $\mathbf{P}_{n+3}$ clockwise by the numbers $1, \dots, n+3$. For a triangulation $T$ of $\mathbf{P}_{n+3}$ as in Definition 2.2.1, we use the labels $1, \dots, n$ for the diagonals of $T$ (in arbitrary fashion), and use the labels $n+1, \dots, 2n+3$ for the sides of $\mathbf{P}_{n+3}$, as follows:

- the side with vertices $\ell$ and $\ell+1$ is labeled $n+\ell$, for $\ell = 1, \dots, n+2$;
- the side with vertices 1 and $n+3$ is labeled $2n+3$.

We define the matrix $\tilde{B}(T) = (b_{ij}(T))$ by the formula (5.11), this time with $i \in \{1, \dots, 2n+3\}$ and $j \in \{1, \dots, n\}$. Thus $\tilde{B}(T)$ is the extended exchange matrix for the quiver $Q(T)$ from Definition 2.2.1. By Exercise 2.2.2, flips of triangulations translate into mutations of associated quivers.

We now reformulate the construction described in Section 1.2. Let $V = \mathbb{C}^2$ be a 2-dimensional complex vector space, and let $\langle \cdot, \cdot \rangle$ denote the standard skew-symmetric nondegenerate bilinear form on $V$. Simply put, $\langle u, v \rangle$ is the determinant of the $2 \times 2$ matrix with columns $u, v \in V$. Let $\mathcal{K}$ be the field of rational functions on $V^{n+3}$ written in terms of $2n+6$ variables, the coordinates of $n+3$ vectors $v_1, \dots, v_{n+3}$. The special linear group naturally acts on $V$, hence on $V^{n+3}$ and on $\mathcal{K}$. Let $\mathcal{F} = \mathcal{K}^{\mathrm{SL}_2}$ be the subfield of $\mathrm{SL}_2$-invariant rational functions. All the action will take place in $\mathcal{F}$.

We associate the Plücker coordinate $P_{ij} = \langle v_i, v_j \rangle \in \mathcal{F}$ with the line segment connecting vertices $i$ and $j$. For each triangulation $T$ of $\mathbf{P}_{n+3}$, we let $\tilde{\mathbf{x}}(T)$ be the collection of the $2n+3$ Plücker coordinates $P_{ij}$ associated with the sides and diagonals of $T$, as in Section 1.2. By Lemma 5.3.4 below, the elements of $\tilde{\mathbf{x}}(T)$ are algebraically independent. We view the Plücker coordinates associated with the sides of $\mathbf{P}_{n+3}$ as frozen variables. As observed earlier, the Grassmann-Plücker relations (1.3) satisfied by the elements $P_{ij}$ can be interpreted as exchange relations encoded by the matrices $\tilde{B}(T)$.

We denote $\Sigma(T) = (\tilde{\mathbf{x}}(T), \tilde{B}(T))$. As discussed above, flips of triangulations correspond to mutations of seeds. Consequently, the seeds form a seed pattern of type $A_n$. (Strictly speaking, the labeled seeds in a seed pattern are labeled by the vertices of the $n$-regular tree $\mathbb{T}_n$. In our situation, the unlabeled seed $\Sigma(T)$ obtained after a sequence of mutations from some initial seed $\Sigma(T_\circ)$ will only depend on $T$ but not on that sequence.) The number of distinct seeds in this pattern is obviously finite.

We complete the proof of Theorem 5.3.2 using an argument based on Corollary 4.3.6 and Remark 4.3.7. All we need to do is to check that for some triangulation $T_\circ$, the matrix $\tilde{B}(T_\circ)$ has full $\mathbb{Z}$-rank. Taking $T_\circ$ as in Figure 5.3, we obtain the matrix

$$\tilde{B}(T_\circ) = \left[\begin{array}{cccccc} 0 & -1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \hline -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{array}\right],$$

where the line is drawn under the $n$th row. The matrix is easily seen to have full $\mathbb{Z}$-rank. □

**Lemma 5.3.4.** *For any triangulation $T$ of the polygon $\mathbf{P}_{n+3}$, the elements of $\tilde{\mathbf{x}}(T)$ are algebraically independent. Thus $(\tilde{\mathbf{x}}(T), \widetilde{B}(T))$ is a seed in $\mathcal{F}$.*

**Proof.** One way to establish this is to observe that $\tilde{\mathbf{x}}(T)$ generates the field $\mathcal{F}$ (since each Plücker coordinate $P_{ij}$ is a rational function in $\tilde{\mathbf{x}}(T)$), and combine this with the fact that the transcendence degree of $\mathcal{F}$ over $\mathbb{C}$ (equivalently, the dimension of the affine cone over $\mathrm{Gr}_{2,n+3}$) is $2n+3$. □

**Remark 5.3.5.** An alternative proof of Theorem 5.3.2 can be based on the description of the fundamental group of the graph whose vertices are the triangulations of the polygon $\mathbf{P}_{n+3}$, and whose edges correspond to the flips. (We view this graph as a 1-dimensional simplicial complex, the 1-skeleton of the $n$-dimensional associahedron.) The fundamental group of this graph is generated by 4-cycles and 5-cycles (the boundaries of 2-dimensional faces of the associahedron) pinned down to a basepoint. For each of these cycles, the corresponding sequence of 4 or 5 mutations in a seed pattern of type $A_n$ brings us back to the original seed; this follows from the analysis of the type $A_2$ case in Section 5.1. Consequently, the seeds in such a pattern can be labeled by the triangulations of $\mathbf{P}_{n+3}$, implying the claim of finite type.

**Corollary 5.3.6.** *Cluster variables in a seed pattern of type $A_n$ can be labeled by the diagonals of a convex $(n+3)$-gon $\mathbf{P}_{n+3}$ so that*

- *clusters correspond to triangulations of the polygon $\mathbf{P}_{n+3}$ by non-crossing diagonals,*
- *mutations correspond to flips, and*
- *exchange matrices are given by* (5.11).

*Cluster variables labeled by different diagonals are distinct, so there are altogether $\frac{n(n+3)}{2}$ cluster variables and $\frac{1}{n+2}\binom{2n+2}{n+1}$ seeds (and as many clusters).*

**Proof.** It is well known that the number of triangulations a convex $(n+3)$-gon has is equal to the Catalan number $\frac{1}{n+2}\binom{2n+2}{n+1}$, see e.g., [**26**, Exercise 6.19a]. So the only claim remaining to be proved is that all these cluster variables are distinct in any seed pattern of type $A_n$. Let $x$ and $x'$ be two cluster variables labeled by distinct diagonals $d$ and $d'$. If $d$ and $d'$ do not cross each other, then there is a cluster containing $x$ and $x'$, so $x$ and $x'$ are algebraically independent and therefore distinct. If $d$ and $d'$ do cross, then there is an exchange relation of the form $x\,x' = M_1 + M_2$ where $M_1$ and $M_2$ are monomials in the elements of some extended cluster $\tilde{\mathbf{x}}$ containing $x$. Now the equality $x = x'$ would imply $x^2 = M_1 + M_2$, contradicting the condition that the elements of $\tilde{\mathbf{x}}$ are algebraically independent. □

In the rest of this section, we examine several seed patterns (or cluster algebras) of type $A_n$ which naturally arise in various mathematical contexts.

**Exercise 5.3.7.** A *frieze pattern* [**4, 5**] is a table of the form

$$
n+2\text{ rows}\left\{\begin{array}{ccccccccccccccccc}
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & & * & & * & & * & & * & & * & & * & & * & & \cdots \\
\cdots & * & & * & & * & & * & & * & & * & & * & & * & \cdots \\
\cdots & & * & & * & & * & & * & & * & & * & & * & & \cdots \\
\cdots & * & & * & & * & & * & & * & & * & & * & & * & \cdots \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots
\end{array}\right.
$$

with (say) positive integer entries such that every quadruple

$$
\begin{array}{ccc}
 & B & \\
A & & C \\
 & D &
\end{array}
$$

satisfies $AC - BD = 1$. Identify the entries in a frieze pattern with cluster variables in a seed pattern of type $A_n$. How many distinct entries does a frieze pattern have? What kind(s) of periodicity does it possess?

**Example 5.3.8.** Let us discuss, somewhat informally, the example of a seed pattern associated with the basic affine space for $\mathrm{SL}_4$. We choose the initial seed for this pattern as shown in Figure 5.4. (This seed has already appeared in Figure 2.5; it corresponds to a particular choice of a wiring diagram.) The variables $P_2$, $P_3$, and $P_{23}$ are mutable; the remaining six variables are frozen. The mutable part of the initial quiver is an oriented 3-cycle; as such, it is easily identified as the mutable part of a quiver associated with a particular triangulation of a hexagon. Thus we are dealing here with a seed of type $A_3$.

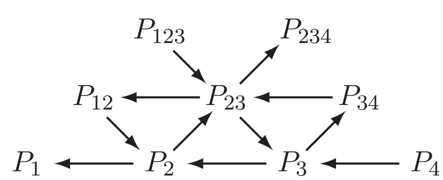



**Figure 5.4.** A seed of flag minors in $\mathbb{C}[\mathrm{SL}_4]^U$.

A matrix in $\mathrm{SL}_4$ has $2^4 - 2 = 14$ nontrivial flag minors: the 6 frozen variables

$$P_1,\ P_{12},\ P_{123},\ P_4,\ P_{34},\ P_{234}$$

(recall that they correspond to the unbounded chambers in a wiring diagram) and 8 additional flag minors

$$P_2,\ P_3,\ P_{13},\ P_{14},\ P_{23},\ P_{24},\ P_{124},\ P_{134},$$

all of which can be obtained by the mutation process from our initial seed. Note however that a seed pattern of type $A_3$ should have 9 cluster variables—so one of them is still missing!

Examining the initial seed shown in Figure 5.4, we see that it can be mutated in three possible ways. The corresponding exchange relations are:

$$\begin{aligned} P_2\,P_{13} &= P_{12}\,P_3 + P_1\,P_{23}\,, \\ P_3\,P_{24} &= P_4\,P_{23} + P_{34}\,P_2\,, \\ P_{23}\,\Omega &= P_{123}\,P_{34}\,P_2 + P_{12}\,P_{234}\,P_3\,. \end{aligned}$$

The first two relations correspond to the two braid moves that can be applied to the initial wiring diagram $D_\circ$. The third relation is different in nature: it produces a seed that does not correspond to any wiring diagram, as the new cluster variable $\Omega$ is not a flag minor. Ostensibly, $\Omega$ is a rational expression (indeed, a Laurent polynomial) in the flag minors. One can check that in fact

$$\Omega = \frac{P_{123}\,P_{34}\,P_2 + P_{12}\,P_{234}\,P_3}{P_{23}} = -P_1P_{234} + P_2P_{134} \tag{5.12}$$

—so $\Omega$ is not merely a rational function on the basic affine space but a *regular* function. It follows that the corresponding cluster algebra is precisely the invariant ring $\mathbb{C}[\mathrm{SL}_4]^U$ (recall that the latter is generated by the flag minors). It turns out that this phenomenon holds for any special linear group $\mathrm{SL}_k$, resulting in a cluster algebra structure in $\mathbb{C}[\mathrm{SL}_k]^U$.

In the case under consideration, we get 14 distinct extended clusters, in agreement with Corollary 5.3.6. See Figure 5.5.

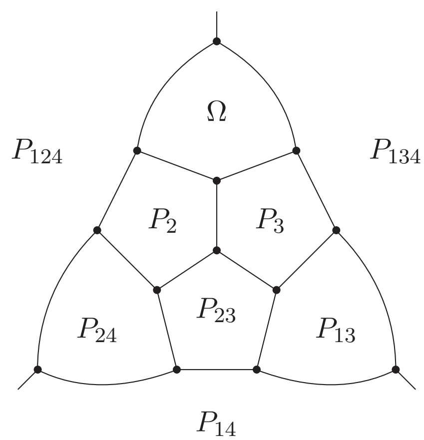



**Figure 5.5.** Clusters in $\mathbb{C}[\mathrm{SL}_4]^U$. The 14 clusters for this seed pattern are shown as the vertices of a graph; the edges of the graph correspond to seed mutations. Note that there is one additional vertex at infinity, so the graph should be viewed as drawn on a sphere rather than a plane. The regions are labeled by cluster variables. Each cluster consists of the three elements labeling the regions adjacent to the corresponding vertex. The 6 frozen variables are not shown. This graph is isomorphic to the 1-skeleton of the three-dimensional associahedron, shown in Figure 1.4.

**Example 5.3.9.** We conclude this section by presenting a family of seed patterns of type $A_n$ introduced in [**28**]. They correspond to cluster structures in particular *double Bruhat cells* for the special linear groups $\mathrm{SL}_{n+1}(\mathbb{C})$, more specifically to the cells associated with *pairs of Coxeter elements* in the associated symmetric group $\mathcal{S}_{n+1}$. This construction can be extended to arbitrary simply connected semisimple complex Lie groups, see [**28**].

Let $L_n \subset \mathrm{SL}_{n+1}(\mathbb{C})$ be the subvariety of tridiagonal matrices

$$z = \begin{bmatrix} v_1 & q_1 & 0 & \cdots & 0 \\ 1 & v_2 & q_2 & \ddots & \vdots \\ 0 & 1 & v_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & q_n \\ 0 & \cdots & 0 & 1 & v_{n+1} \end{bmatrix} \tag{5.13}$$

of determinant 1. For $i, j \in \{1, \ldots, n+3\}$ satisfying $i + 2 \le j$, consider the *solid principal minor* (cf. Exercise 1.4.2)

$$U_{ij} = \Delta_{[i,j-2],[i,j-2]} \in \mathbb{C}[L_n],$$

the determinant of the submatrix with rows and columns $i, i+1, \ldots, j-2$. For example, $U_{i,i+2} = v_i$ and $U_{1,n+3} = \det(z) = 1$. By convention, $U_{i,i-1} = 1$.

**Exercise 5.3.10.** Prove that these functions satisfy the relations

$$U_{ik}\, U_{j\ell} = q_{j-1} q_j \cdots q_{k-2}\, U_{ij}\, U_{k\ell} + U_{i\ell}\, U_{jk}\,, \tag{5.14}$$

for $1 \le i < j < k < l \le n+3$. Then show that these relations are the exchange relations in a particular seed pattern of type $A_n$. More precisely, show that there is a seed pattern of type $A_n$, with the frozen variables $q_1, \ldots, q_n$, in which the cluster variables $U_{ij}$ associated to the diagonals $[i, j]$ of the convex polygon $\mathbf{P}_{n+3}$ satisfy the exchange relations (5.14).

What is required to establish the latter claim? Let us extend each $n \times n$ exchange matrix $B(T)$ associated to a triangulation $T$ of $\mathbf{P}_{n+3}$ (see (5.11)) to the $2n \times n$ matrix $\tilde{B}(T)$ whose columns encode the relations (5.14) corresponding to the $n$ possible flips from $T$. One then needs to verify that whenever triangulations $T$ and $T'$ are related by a flip, the associated matrices $\tilde{B}(T)$ and $\tilde{B}(T')$ are related by the corresponding mutation.

One of the clusters in this seed pattern consists of the $n$ leading principal minors $U_{13}\,, \ldots, U_{1,n+2}\,$. The exchange relations from this cluster are the relations (5.14) with $(i, j, k, \ell) = (1, k-1, k, k+1)$. They can be rewritten as follows:

$$U_{1,k+1} = v_{k-1}\, U_{1,k} - q_{k-2}\, U_{1,k-1} \quad (k = 3, \ldots, n+2). \tag{5.15}$$

These relations play important roles in the classical theory of orthogonal polynomials in one variable [**27**], in the study of a generalized Toda lattice [**21**], and in other mathematical contexts.

## 5.4. Seed patterns of type $D_n$

In this section we show that seed patterns of type $D_n$ are of finite type. The proof of this theorem, while substantially more technical than the proof of Theorem 5.3.2, follows the same general plan. It relies on two main ingredients. The first ingredient is a combinatorial construction ("tagged arcs" in a punctured disk) that enables us to explicitly describe the combinatorics of mutations in type $D_n$ and introduce the relevant nomenclature. The second ingredient is an algebraic construction of a particular seed pattern of type $D_n$. In this pattern, each cluster variable associated with a tagged arc has an intrinsic definition independent of the mutation path; this will imply that the number of seeds is finite. The "full $\mathbb{Z}$-rank" argument will then allow us to generalize the finiteness statement to arbitrary coefficients.

While type $D_n$ Dynkin diagrams are usually defined for $n \geq 4$, in this section we will allow for the possibility of $n = 3$ (in which case one recovers type $A_3$).

**Exercise 5.4.1.** Show that a seed pattern is of type $D_n$ if and only if one of its exchange matrices corresponds to a quiver which is an oriented $n$-cycle.

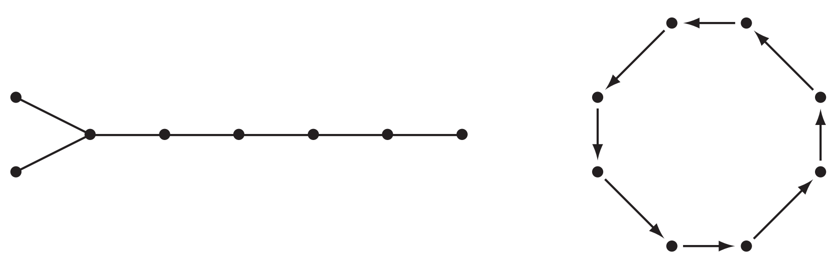


**Figure 5.6.** Dynkin diagram of type $D_n$ and an oriented $n$-cycle.

**Theorem 5.4.2.** *Seed patterns of type $D_n$ are of finite type.*

As in Remark 5.3.3, the key point of Theorem 5.4.2 is that a pattern of type $D_n$ has finitely many seeds regardless of the number of frozen variables and of the entries in the bottom parts of extended exchange matrices.

**Definition 5.4.3.** Let $\mathbf{P}_n^\bullet$ be a convex $n$-gon ($n \geq 3$) with a distinguished point $p$ (a *puncture*) in its interior. We label the vertices of $\mathbf{P}_n^\bullet$ clockwise from 1 to $n$. These vertices and the puncture $p$ are the *marked points* of $\mathbf{P}_n^\bullet$.

An *arc* in $\mathbf{P}_n^\bullet$ is a non-selfintersecting curve $\gamma$ in $\mathbf{P}_n^\bullet$ such that

- the endpoints of $\gamma$ are two different marked points;
- except for its endpoints, $\gamma$ is disjoint from the boundary of $\mathbf{P}_n^\bullet$ and from the puncture $p$;
- $\gamma$ does not cut out an unpunctured digon.

We consider each arc up to isotopy within the class of such curves.

The arcs incident to the puncture are called *radii*.

**Definition 5.4.4.** A *tagged arc* in $\mathbf{P}_n^\bullet$ is either an ordinary non-radius arc, or a radius that has been labeled ("tagged") in one of two possible ways, *plain* or *notched*. Two tagged arcs are called *compatible* with one another if their untagged versions do not cross each other, with the following modification: the plain and notched versions of the same radius are compatible, but the the plain and notched versions of two different radii are not.

A *tagged triangulation* is a maximal (by inclusion) collection of pairwise compatible tagged arcs. See Figure 5.7.

**Lemma 5.4.5.** *Any tagged triangulation $T$ of $\mathbf{P}_n^\bullet$ consists of $n$ tagged arcs. Any tagged arc in a tagged triangulation $T$ can be replaced in a unique way by a tagged arc not belonging to $T$, to form a new tagged triangulation $T'$.*

**Proof.** It is easy to see that tagged triangulations come in three flavors:

(1) a triangulation in the usual sense, with every radius plain;

(2) a triangulation in the usual sense, with every radius notched;

(3) the plain and notched versions of the same radius inside a punctured digon. Outside of the digon, it is an ordinary triangulation.

In each of these cases, the total number of tagged arcs is $n$. □

In the situation described in Lemma 5.4.5, we say that the tagged triangulations $T$ and $T'$ are related by a *flip*.

**Exercise 5.4.6.** Verify that any two tagged triangulations of $\mathbf{P}_n^\bullet$ are connected via a sequence of flips.

We next define an extended exchange matrix $\tilde{B}(T)$ associated with a tagged triangulation $T$ of the punctured polygon $\mathbf{P}_n^\bullet$. The construction is similar to the one in type $A_n$. In the cases (1) and (2) above, the rule (5.11) is used; in the case (3), some adjustments are needed around the radii. Figure 5.8 illustrates the recipe used to define the matrix $\tilde{B}(T)$, or equivalently the corresponding quiver. The vertices of the quiver corresponding to boundary segments (i.e., the sides of the polygon) are frozen; the ones corresponding to tagged arcs are mutable. Details are left to the reader.

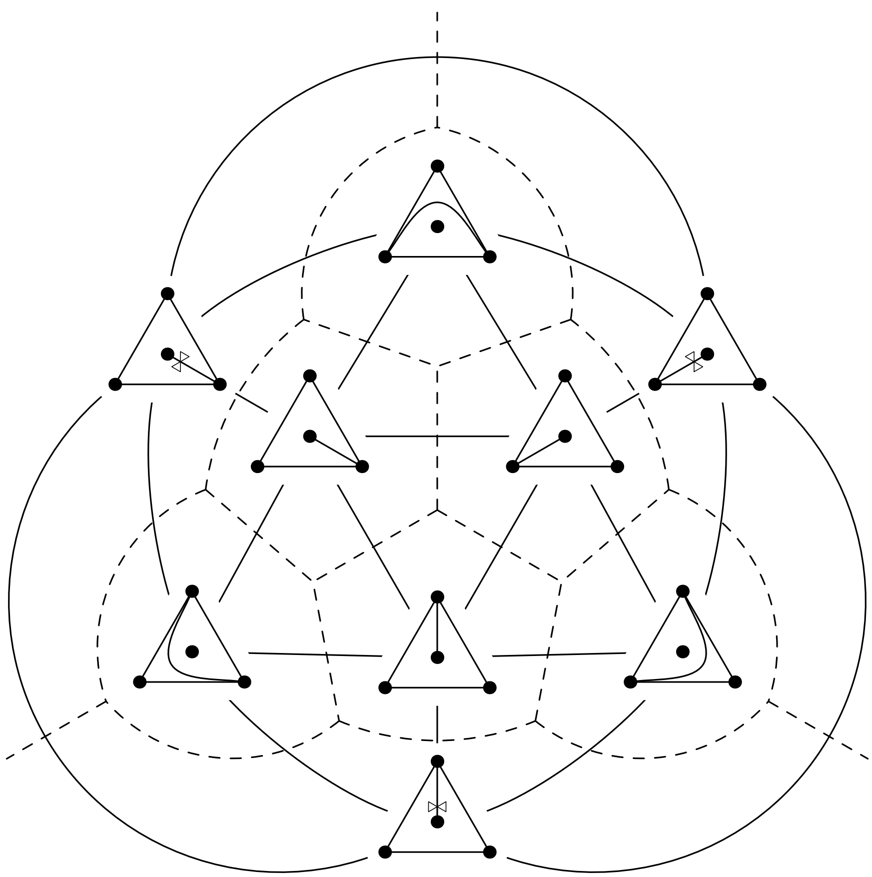


**Figure 5.7.** Tagged arcs in a once-punctured triangle $\mathbf{P}_3^\bullet$. Solid lines indicate which arcs are compatible. The vertices of the dashed graph correspond to tagged triangulations; its edges correspond to flips.

**Exercise 5.4.7.** Verify that if two tagged triangulations $T$ and $T'$ are related by a flip, then the extended skew-symmetric matrices $\tilde{B}(T)$ and $\tilde{B}(T')$ are related by the corresponding matrix mutation.

Unfortunately the matrices $\tilde{B}(T)$ have rank $< n$, so exhibiting a seed pattern with these matrices and finitely many seeds would not yield a proof of Theorem 5.4.2, cf. Remark 4.3.7. We will remove this obstacle by introducing additional frozen variables besides those labeled by boundary segments.

We now prepare the algebraic ingredients for the proof of Theorem 5.4.2. Similarly to the type $A_n$ case, the idea is to interpret tagged arcs in a once-punctured polygon as a family of rational functions. These rational functions satisfy the type $D_n$ exchange relations, which in turn are associated to flips of tagged arcs. The astute reader will notice that the algebraic construction we

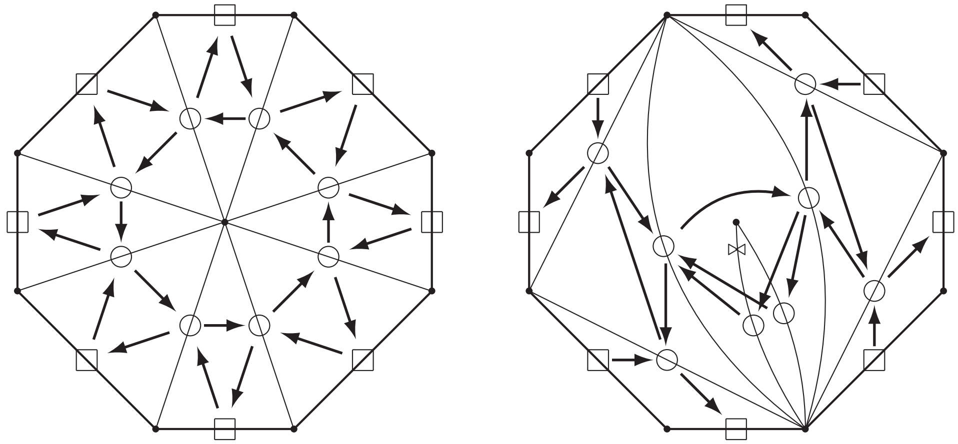



**Figure 5.8.** Quivers associated with tagged triangulations of a once-punctured convex polygon $\mathbf{P}_n^\bullet$.

associate to the type $D_n$ case contains the algebraic construction associated to the type $A_{n-1}$ case (Plücker coordinates of an $(n+2)$-tuple of vectors in $\mathbb{C}^2$), reflective of the fact that the type $A_{n-1}$ Dynkin diagram is contained inside the type $D_n$ Dynkin diagram.

As in Section 5.3, we set $V = \mathbb{C}^2$, and let $\langle u, v\rangle$ be the determinant of the $2\times 2$ matrix with columns $u, v \in V$. Let $\mathcal{K}$ be the field of rational functions on $V^n \times V \times V \times \mathbb{C}^2 \cong \mathbb{C}^{2n+6}$, written in terms of $2n+6$ variables: the coordinates of $n+2$ vectors

$$v_1, \dots, v_n, a, \overline{a} \in V,$$

plus two additional variables $\lambda$ and $\overline{\lambda}$.

Let $A \in \mathrm{End}(V)$ denote the linear operator defined by

$$Av = \frac{\overline{\lambda}\langle v, a\rangle\, \overline{a} - \lambda\langle v, \overline{a}\rangle\, a}{\langle \overline{a}, a\rangle}, \tag{5.16}$$

so that $a$ (resp. $\overline{a}$) is an eigenvector for $A$ with eigenvalue $\lambda$ (resp. $\overline{\lambda}$). Let

$$a^{\bowtie} = \frac{\overline{\lambda} - \lambda}{\langle \overline{a}, a\rangle}\,\overline{a};$$

thus $a^{\bowtie}$ is also an eigenvector for $A$, with eigenvalue $\overline{\lambda}$. We choose this normalization because of the following property, which is immediate from the definitions.

**Lemma 5.4.8.** *For $v \in V$, we have $\langle v, Av\rangle = \langle v, a\rangle\langle v, a^{\bowtie}\rangle$.*

We next describe a seed pattern inside $\mathcal{K}$, and in fact inside its subfield $\mathcal{F}$ of $\mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^*$-invariant rational functions. (The group $\mathrm{SL}_2$ acts in the standard way on each of the vectors $v_1, \dots, v_n, a, \overline{a}$, and fixes $\lambda$ and $\overline{\lambda}$. The group $\mathbb{C}^*$ acts by rescaling the vector $\overline{a}$.) Informally speaking, we are going to think of the vectors $v_1, \dots, v_n$ as located at the corresponding vertices of $\mathbf{P}_n^\bullet$, and we shall associate both eigenvectors $a$ and $a^{\bowtie}$ with the puncture $p$. We make a *cut* running from the puncture $p$ to the boundary segment (the side of the polygon) that connects the vertices 1 and $n$. We will think of crossing the cut as picking up an application of $A$.

**Definition 5.4.9.** We associate an element $P_\gamma \in \mathcal{K}$ to each tagged arc or boundary segment $\gamma$ in $\mathbf{P}_n^\bullet$, as follows (see Figure 5.9):

$$P_\gamma = \begin{cases} \langle v_i, v_j \rangle & \text{if } \gamma \text{ doesn't cross the cut, and has endpoints } i \text{ and } j > i; \\ \langle v_j, Av_i \rangle & \text{if } \gamma \text{ crosses the cut, and has endpoints } i \text{ and } j > i; \\ \langle v_i, a \rangle & \text{if } \gamma \text{ is a plain radius with endpoints } p \text{ and } i; \\ \langle v_i, a^{\bowtie} \rangle & \text{if } \gamma \text{ is a notched radius with endpoints } p \text{ and } i. \end{cases}$$

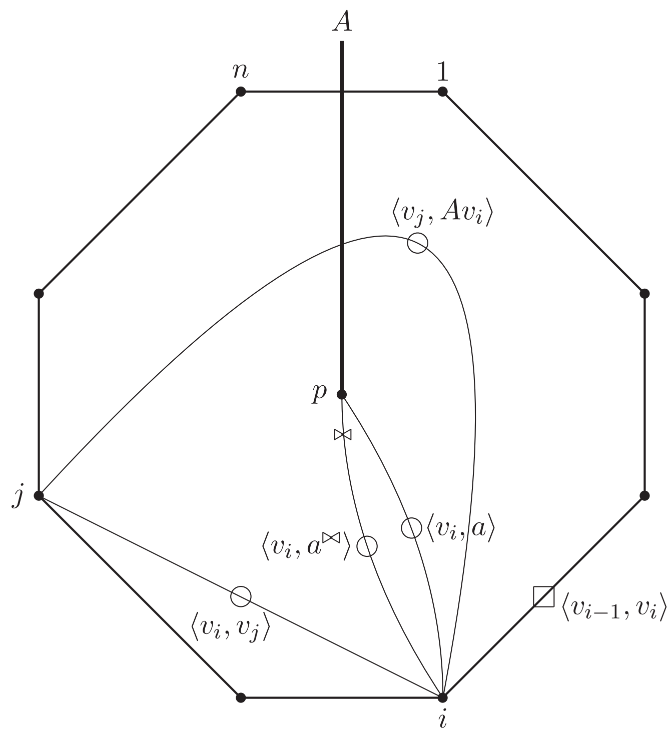



**Figure 5.9.** Elements of the field $\mathcal{K}$ associated with tagged arcs in $\mathbf{P}_n^\bullet$. The boundary segment crossed by the cut corresponds to $\langle v_n, Av_1 \rangle$.

We can now verify that the $P_\gamma$ satisfy a family of relations that topologists know as "skein relations" for tagged arcs.

For example, Figure 5.10 illustrates the equation

$$\langle v_i, v_l \rangle \langle v_k, Av_i \rangle = \langle v_i, v_k \rangle \langle v_l, Av_i \rangle + \langle v_k, v_l \rangle \langle v_i, a \rangle \langle v_i, a^{\bowtie} \rangle, \tag{5.17}$$

which follows from the Grassmann-Plücker relation (1.3) combined with Lemma 5.4.8.

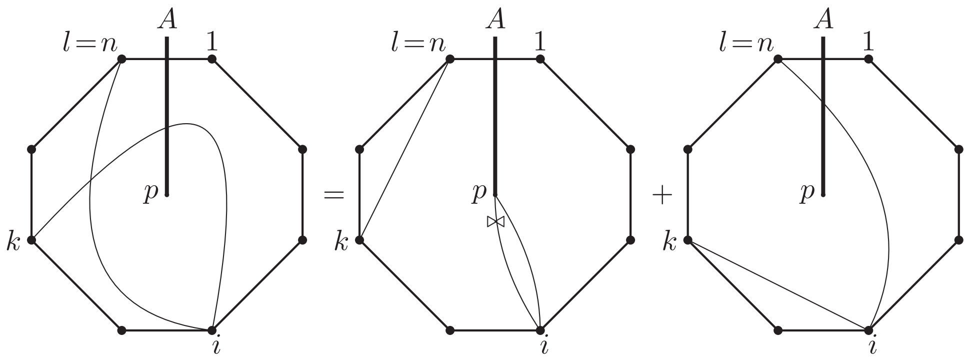


**Figure 5.10.** Pictorial representation of the relation (5.17) (or (5.22)).

More generally, we have the following relations. The reader may want to draw a figure corresponding to each relation to get some intuition about the form of the relations.

**Proposition 5.4.10.** *The elements $P_\gamma$ described in Definition 5.4.9 satisfy the following relations:*

$$\langle v_i, v_k\rangle\langle v_j, v_l\rangle = \langle v_i, v_j\rangle\langle v_k, v_l\rangle + \langle v_i, v_l\rangle\langle v_j, v_k\rangle, \tag{5.18}$$

$$\langle v_j, v_l\rangle\langle v_k, Av_i\rangle = \langle v_j, v_k\rangle\langle v_l, Av_i\rangle + \langle v_j, Av_i\rangle\langle v_k, v_l\rangle, \tag{5.19}$$

$$\langle v_k, Av_i\rangle\langle v_l, Av_j\rangle = \lambda\overline{\lambda}\langle v_i, v_j\rangle\langle v_k, v_l\rangle + \langle v_k, Av_j\rangle\langle v_l, Av_i\rangle, \tag{5.20}$$

$$\langle v_i, v_k\rangle\langle v_l, Av_j\rangle = \langle v_l, Av_i\rangle\langle v_j, v_k\rangle + \langle v_l, Av_k\rangle\langle v_i, v_j\rangle, \tag{5.21}$$

$$\langle v_i, v_l\rangle\langle v_k, Av_i\rangle = \langle v_i, v_k\rangle\langle v_l, Av_i\rangle + \langle v_k, v_l\rangle\langle v_i, a\rangle\langle v_i, a^{\bowtie}\rangle, \tag{5.22}$$

$$\langle v_j, Av_i\rangle\langle v_l, Av_j\rangle = \lambda\overline{\lambda}\langle v_i, v_j\rangle\langle v_j, v_l\rangle + \langle v_j, a^{\bowtie}\rangle\langle v_j, a\rangle\langle v_l, Av_i\rangle, \tag{5.23}$$

$$\langle v_i, v_l\rangle\langle v_l, Av_j\rangle = \langle v_l, Av_i\rangle\langle v_j, v_l\rangle + \langle v_l, a^{\bowtie}\rangle\langle v_l, a\rangle\langle v_i, v_j\rangle \tag{5.24}$$

$$\langle v_i, v_k\rangle\langle v_j, a\rangle = \langle v_i, v_j\rangle\langle v_k, a\rangle + \langle v_i, a\rangle\langle v_j, v_k\rangle, \tag{5.25}$$

$$\langle v_j, Av_i\rangle\langle v_k, a\rangle = \overline{\lambda}\langle v_j, v_k\rangle\langle v_i, a\rangle + \langle v_j, a\rangle\langle v_k, Av_i\rangle, \tag{5.26}$$

$$\langle v_k, Av_j\rangle\langle v_i, a\rangle = \langle v_k, Av_i\rangle\langle v_j, a\rangle + \lambda\langle v_k, a\rangle\langle v_i, v_j\rangle, \tag{5.27}$$

$$\langle v_i, a^{\bowtie}\rangle\langle v_j, a\rangle = \overline{\lambda}\langle v_i, v_j\rangle + \langle v_j, Av_i\rangle. \tag{5.28}$$

*where $1 \le i < j < k < \ell \le n$. In addition, they satisfy the relations obtained from* (5.25)–(5.28) *by interchanging $\lambda$ with $\overline{\lambda}$ and $a$ with $a^{\bowtie}$ throughout.*

**Proof.** Each of these relations follows from a suitable instance of the Grassmann-Plücker relation (1.3), using Lemma 5.4.8 and the identities

$$\begin{aligned}\langle Av, Av'\rangle &= \det(A)\,\langle v, v'\rangle = \lambda\overline{\lambda}\langle v, v'\rangle,\\ \langle Av, a\rangle &= \overline{\lambda}\langle v, a\rangle,\\ \langle Av, a^{\bowtie}\rangle &= \lambda\langle v, a^{\bowtie}\rangle.\end{aligned}$$

For example, (5.27) can be obtained from the identity

$$\langle v_k, Av_j\rangle\langle Av_i, a\rangle = \langle v_k, Av_i\rangle\langle Av_j, a\rangle + \langle v_k, a\rangle\langle Av_i, Av_j\rangle,$$

while (5.28) follows from

$$\langle v_i, Av_i\rangle\langle v_j, a\rangle = \langle v_i, v_j\rangle\langle Av_i, a\rangle + \langle v_i, a\rangle\langle v_j, Av_i\rangle. \quad \square$$

For a tagged triangulation $T$ of the punctured polygon $\mathbf{P}_n^\bullet$, let $\tilde{\mathbf{x}}(T)$ be the $(2n+2)$-tuple consisting of the elements $P_\gamma$ labeled by the tagged arcs and boundary segments in $T$, together with $\lambda$ and $\overline{\lambda}$. We view the elements $P_\gamma \in \tilde{\mathbf{x}}(T)$ labeled by tagged arcs as cluster variables, and those labeled by the boundary segments as frozen variables; $\lambda$ and $\overline{\lambda}$ are frozen variables as well.

Let $\gamma$ be a tagged arc in a tagged triangulation $T$, and let $\gamma'$ be the tagged arc that replaces $\gamma$ when the latter is flipped. One can check that in every such instance, exactly one of the relations in Proposition 5.4.10 has the product $P_\gamma\, P_{\gamma'}$ on the left-hand side; the corresponding right-hand side is always a sum of two monomials in the elements of $\tilde{\mathbf{x}}(T)$. We let $\tilde{B}^\bullet(T)$ denote the matrix encoding these relations for all tagged arcs in $T$. The matrix $\tilde{B}^\bullet(T)$ can be seen to be an extension of the matrix $\tilde{B}(T)$ by two extra rows corresponding to $\lambda$ and $\overline{\lambda}$. (Put differently, setting $\lambda = \overline{\lambda} = 1$ produces relations encoded by $\tilde{B}(T)$.)

**Example 5.4.11.** Figure 5.11 shows a triangulation $T_\circ$ of a punctured pentagon, and the associated matrix $\tilde{B}^\bullet(T_\circ)$. Columns 1 and 5 of $\tilde{B}^\bullet(T_\circ)$ encode the exchange relations

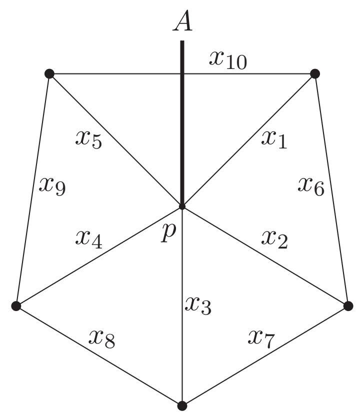
$$\langle v_5, Av_2\rangle\langle v_1, a\rangle = \langle v_5, Av_1\rangle\langle v_2, a\rangle + \lambda\langle v_5, a\rangle\langle v_1, v_2\rangle = x_{10}x_2 + \lambda x_5 x_6$$

$$\langle v_4, Av_1\rangle\langle v_5, a\rangle = \overline{\lambda}\langle v_4, v_5\rangle\langle v_1, a\rangle + \langle v_4, a\rangle\langle v_5, Av_1\rangle = \overline{\lambda}x_9 x_1 + x_4 x_{10}.$$

The matrix $\tilde{B}^\bullet(T_\circ)$ has full $\mathbb{Z}$-rank. However, the submatrix of $\tilde{B}(T_\circ)$ consisting of the first ten rows does not have full rank (each row sum is 0).

This example can be straightforwardly generalized to $n \neq 5$.

**Proposition 5.4.12.** *For any tagged triangulation $T$ of $\mathbf{P}_n^\bullet$, the elements of $\tilde{\mathbf{x}}(T)$ are algebraically independent, so $(\tilde{\mathbf{x}}(T), \tilde{B}^\bullet(T))$ is a seed in the field generated by $\tilde{\mathbf{x}}(T)$. The seeds associated to tagged triangulations related by a flip are related to each other by the corresponding mutation.*




$$\tilde{B}^\bullet(T_\circ) = \left[\begin{array}{ccccc} 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 \\ \hline -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ \hline -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

**Figure 5.11.** The (tagged) triangulation $T_\circ$ of a punctured pentagon ($n = 5$), and the corresponding matrix $\tilde{B}^\bullet(T_\circ)$. The columns of $\tilde{B}^\bullet(T_\circ)$ correspond to $x_1, \ldots, x_5$. The rows correspond to $x_1, \ldots, x_{10}, \lambda, \overline{\lambda}$, in this order.

**Proof.** It is a straightforward but tedious exercise to verify that the matrix $\tilde{B}^\bullet(T)$ undergoes a mutation when a tagged arc in $T$ is flipped to produce a new tagged triangulation $T'$. We already know that the elements of $\tilde{\mathbf{x}}(T)$ and $\tilde{\mathbf{x}}(T')$ satisfy the corresponding exchange relation. It remains to show algebraic independence. Since each exchange is a birational transformation, it suffices to prove algebraic independence for one particular choice of $T$.

Consider the triangulation $T_\circ$ made up of $n$ plain radii, cf. Figure 5.11 and Example 5.4.11. Then $\lambda$, $\overline{\lambda}$, and $\langle v_n, Av_1\rangle$ are the only elements of $\tilde{\mathbf{x}}(T_\circ)$ which involve $\lambda$, $\overline{\lambda}$, or $\overline{a}$. The remaining $2n-1$ elements are $2\times 2$ minors of the matrix with columns $v_1, v_2, \ldots, v_n, a$; they form an extended cluster in the corresponding seed pattern of type $A_{n-2}$. Hence they are algebraically independent, and the claim follows. □

**Proof of Theorem 5.4.2**. By Proposition 5.4.12, the seeds $(\tilde{\mathbf{x}}(T), \tilde{B}^\bullet(T))$ form a seed pattern. The number of seeds in this pattern is finite. As verified in Example 5.4.11, the extended exchange matrix $\tilde{B}^\bullet(T_\circ)$ has full $\mathbb{Z}$-rank. The theorem follows by Corollary 4.3.6. □

**Corollary 5.4.13.** *Cluster variables in a seed pattern of type $D_n$ can be labeled by the tagged arcs in a once-punctured convex $n$-gon $\mathbf{P}_n^\bullet$ so that clusters correspond to tagged triangulations, and mutations correspond to flips. Cluster variables labeled by different tagged arcs are distinct. There are altogether $n^2$ cluster variables and $\frac{3n-2}{n}\binom{2n-2}{n-1}$ seeds (or clusters).*

Most of the work in the proof of the corollary concerns the enumeration of seeds. Let $a_n$ (resp., $d_n$) denote the number of seeds in a pattern of type $A_n$ (resp., $D_n$), including $a_0 = 1$, $d_2 = 4$, and $d_3 = 14$ by convention. We know from Corollary 5.3.6 that $a_n = \frac{1}{n+2}\binom{2n+2}{n+1}$.

**Lemma 5.4.14.** *The number of tagged triangulations $T$ containing a given radius $\gamma$ is equal to $a_{n-1}$.*

**Proof.** Assume without loss of generality that $\gamma$ is a plain radius connecting the puncture $p$ to a boundary point $i$. If $T$ contains the notched counterpart of $\gamma$, then the remaining $n-1$ arcs of $T$ form a triangulation of the $(n+1)$-gon obtained from $\mathbf{P}_n^\bullet$ by removing a loop based at $i$ which goes around the puncture. The number of such triangulations is $a_{n-2}$. If $T$ does not contain the notched counterpart of $\gamma$, then it is an ordinary triangulation of $\mathbf{P}_n^\bullet$ cut open along $\gamma$, in which we are not allowed to use the aforementioned loop. The number of such triangulations is $a_{n-1} - a_{n-2}$. □

**Lemma 5.4.15.** *The numbers $d_n$ satisfy the recurrence*

$$d_n = \sum_{k=0}^{n-3} a_k\, d_{n-1-k} + 2a_{n-1}\,.$$

**Proof.** Consider two ways of counting the triples $(T, \gamma, i)$ in which $T$ is a tagged triangulation of $\mathbf{P}_n^\bullet$, $\gamma$ is a tagged arc in $T$, and $i$ is an endpoint of $\gamma$. Selecting $T$ first, then $\gamma$ and then $i$, we see that the number of such triples is equal to $d_n \cdot n \cdot 2$. Selecting $i$ first, then $\gamma$ and $T$ together, treating the cases $i \neq p$ and $i = p$ separately, and using Lemma 5.4.14, we obtain:

$$2nd_n = n\Big(2\sum_{k=0}^{n-3} a_k\, d_{n-1-k} + 2a_{n-1}\Big) + 2na_{n-1}\,,$$

as desired. (The factor of 2 before the sum accounts for the two ways of cutting up the polygon: leaving the puncture to the left or to the right of $\gamma$, as we move away from $i$.) □

**Proof of Corollary 5.4.13.** The statements in the first sentence of the corollary have already been established. The claim of distinctness can be verified in the special case $n = 4$ by direct calculation; the general case then follows by restriction.

The cluster variables are labeled by $\frac{n^2-3n+2}{2}$ ordinary arcs not crossing the cut, $\frac{n^2-n-2}{2}$ ordinary arcs crossing the cut, and $n$ radii of each of the two flavors, bringing the total to $n^2$.

The formula $d_n = \frac{3n-2}{n}\binom{2n-2}{n-1}$ can now be proved by induction using the recurrence in Lemma 5.4.15. We leave this step to the reader. □

We conclude this section by examining a couple of examples of cluster algebras of type $D_n$ that occur in the settings discussed in Chapter 1.

**Example 5.4.16.** The ring of polynomials in 9 variables $z_{ij}$ $(i, j \in \{1,2,3\})$, viewed as matrix entries of a $3 \times 3$ matrix

$$z = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} \in \mathrm{Mat}_{3,3}(\mathbb{C}) \cong \mathbb{C}^9,$$

carries a natural cluster algebra structure of type $D_4$, see, e.g., [**12**]. The seed pattern giving rise to this cluster structure contains, among others, the seeds associated with double wiring diagrams with 3 wires of each kind.

Let us use the initial seed associated with the diagram $D$ in Figure 2.6. Thus the initial cluster is

$$\mathbf{x} = (x_1, x_2, x_3, x_4) = (\Delta_{1,2}, \Delta_{3,2}, \Delta_{13,12}, \Delta_{13,23}), \tag{5.29}$$

the 5 coefficient variables are

$$(x_5, \ldots, x_9) = (\Delta_{123,123}, \Delta_{12,23}, \Delta_{1,3}, \Delta_{3,1}, \Delta_{23,12}), \tag{5.30}$$

and the exchange relations (encoded by the quiver shown in Figure 2.6) are:

$$\begin{aligned} \Delta_{1,2}\, \Delta_{356} &= \Delta_{1,3}\, \Delta_{3,2} + \Delta_{13,23}\,, \\ \Delta_{3,2}\, \Delta_{145} &= \Delta_{13,12} + \Delta_{3,1}\, \Delta_{1,2}\,, \\ \Delta_{13,12}\, \Delta_{236} &= \Delta_{23,12}\, \Delta_{13,23} + \Delta_{123,123}\, \Delta_{3,2}\,, \\ \Delta_{13,23}\, \Delta_{124} &= \Delta_{123,123}\, \Delta_{1,2} + \Delta_{12,23}\, \Delta_{13,12}\,. \end{aligned}$$

The mutable part of $Q(D)$ is an oriented 4-cycle, so we are indeed dealing with a pattern of type $D_4$. By Corollary 5.4.13, it has 50 seeds, which include 34 seeds associated with double wiring diagrams, see Figure 1.12. There are 16 cluster variables, labeled by the 16 tagged arcs in a once-punctured quadrilateral. They include the 14 minors of $z$ not listed in (5.30). As suggested by Exercise 1.4.4, the remaining two cluster variables are the polynomials $K(z)$ and $L(z)$ given by (1.6)–(1.7).

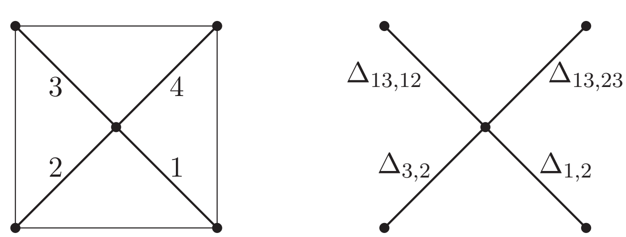



**Figure 5.12.** The triangulation representing the initial cluster (5.29).

**Example 5.4.17.** The coordinate ring $\mathbb{C}[\mathrm{SL}_5]^U$ of the basic affine space for $\mathrm{SL}_5$ has a natural cluster algebra structure of type $D_6$, to be discussed in detail in Example 7.5.4. This cluster algebra has 36 cluster variables and 8 frozen variables. This set of 44 generators includes $2^5-2=30$ flag minors plus 14 non-minor elements. The total number of clusters is 672.

**Remark 5.4.18.** For any irreducible representation of a semisimple algebraic group $G$, Lusztig [**23**] introduced the concept of a canonical (resp., dual canonical) basis; these correspond to the lower and upper global bases of Kashiwara [**19**]. While we will not define the dual canonical bases here, we note that they are strongly connected to the theory of cluster algebras. In particular, it was conjectured in [**10**, p. 498] that every cluster monomial in $\mathbb{C}[\mathrm{SL}_k]^U$ (i.e., a monomial in the elements of some extended cluster) belongs to the dual canonical basis; this conjecture was proved in [**18**]. For $k \le 5$, cluster monomials make up the entire dual canonical basis in $\mathbb{C}[\mathrm{SL}_k]^U$.

**Example 5.4.19.** The homogeneous coordinate ring of a Schubert divisor in the Grassmannian $\mathrm{Gr}_{2,n+2}$ has a structure of a cluster algebra of type $D_n$; see Example 7.3.4 for details.

## 5.5. Seed patterns of types $B_n$ and $C_n$

By Definition 5.2.9, a seed pattern of rank $n \ge 2$ (or the associated cluster algebra) is of type $B_n$ if one of its exchange matrices is

$$B = \begin{bmatrix} 0 & -2 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{5.31}$$

(up to simultaneous permutation of rows and columns).

Simiarly, a seed pattern (or cluster algebra) of rank $n \ge 3$ is of type $C_n$ if one of its exchange matrices has the form

$$B = \begin{bmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{5.32}$$

(We continue to use the conventions of [**17**], cf. Definition 5.2.4.)

**Theorem 5.5.1.** *Seed patterns of type $B_n$ are of finite type.*

**Theorem 5.5.2.** *Seed patterns of type $C_n$ are of finite type.*

Note that type $B_2$ is already covered by Theorem 5.1.1, with $bc = 2$.

We will prove Theorems 5.5.1–5.5.2 using the technique of folding introduced in Section 4.4. Specifically, we will obtain seed patterns of type $C_n$ by folding seed patterns of type $A_{2n-1}$, and we will get type $B_n$ from type $D_{n+1}$.

The following statement follows easily from the definitions.

**Lemma 5.5.3.** *Let $Q$ be a quiver globally foldable with respect to an action of a group $G$. Let $\overline{Q}$ be a quiver constructed from $Q$ by introducing new frozen vertices together with some arrows connecting them to the mutable vertices in $Q$. Extend the action of $G$ from $Q$ to $\overline{Q}$ by making $G$ fix every newly added vertex. Then the quiver $\overline{Q}$ is globally foldable with respect to $G$.*

**Corollary 5.5.4.** *Let $Q$ be a quiver without frozen vertices. Suppose that $Q$ is globally foldable with respect to an action of a group $G$. If any seed pattern with the initial exchange matrix $B(Q)$ is of finite type (regardless of the choice of an initial extended exchange matrix $\tilde{B}$ containing $B(Q)$), then any seed pattern with the initial exchange matrix $B(Q)^G$ is of finite type.*

**Proof.** Combine Corollary 4.4.11 and Lemma 5.5.3. □

**Proof of Theorem 5.5.2.** Our proof strategy is as follows. We will construct a type $A_{2n-1}$ quiver $Q_0$ with a group $G$ acting on its vertices, so that $Q_0$ is globally foldable with respect to $G$, and $B(Q_0)^G$ is the $n \times n$ exchange matrix of type $C_n$ from Equation (5.32). Theorem 5.5.2 will then follow from Theorem 5.3.2 and Corollary 5.5.4.

The combinatorial model for a seed pattern of type $A_{2n-1}$ presented in Section 5.3 uses triangulations of a convex $(2n+2)$-gon $\mathbf{P}_{2n+2}$, with vertices numbered $1, 2, \ldots, 2n+2$ in clockwise order. Consider the centrally symmetric triangulation $T_0$ (see Figure 5.13) consisting of the following diagonals:

- a "diameter" $d_1$ connecting vertices 1 and $n+2$;
- diagonals $d_2, d_3, \ldots, d_n$ connecting $n+2$ with $2, 3, \ldots, n$;
- diagonals $d_{2'}, d_{3'}, \ldots, d_{n'}$ connecting 1 with $n+3$, $n+4, \ldots, 2n+1$.

The quiver $Q_0 = Q(T_0)$ (see Section 2.2) is an orientation of the type $A_{2n-1}$ Dynkin diagram with vertices labeled $n', \ldots, 3', 2', 1, 2, 3, \ldots, n$ in order, and arrows directed towards the central vertex 1, see Figure 5.13. The group $G = \mathbb{Z}/2\mathbb{Z}$ acts on the vertices of $Q_0$ by exchanging $i'$ and $i$ for $2 \le i \le n$, and fixing the vertex 1. It is easy to see that $Q_0$ is $G$-admissible, and moreover $B(Q_0)^G$ is the exchange matrix of type $C_n$ from Equation (5.32).

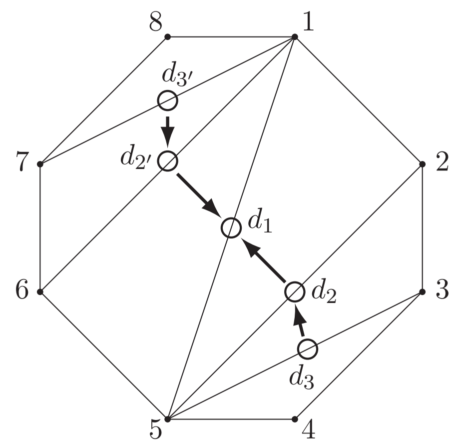



**Figure 5.13.** A centrally symmetric triangulation $T_0$ and its quiver $Q_0 = Q(T_0)$. The folded matrix $B(Q(T_0))^G$ has type $C_n$; here $n = 3$.

It remains to show that $Q_0$ is globally foldable. Besides acting on the set $\{n', \dots, 3', 2', 1, 2, 3, \dots, n\}$, the group $G = \mathbb{Z}/2\mathbb{Z}$ naturally acts—by central symmetry—on the set of diagonals of the polygon $\mathbf{P}_{2n+2}$. This action has two kinds of orbits: (1) the "diameters" of $\mathbf{P}_{2n+2}$ fixed by the $G$-action, and (2) pairs of centrally symmetric diagonals.

A transformation $\mu_{J_k} \circ \cdots \circ \mu_{J_1}$ associated with a sequence $J_1, \dots, J_k$ of $G$-orbits in $\{n', \dots, 3', 2', 1, 2, 3, \dots, n\}$ corresponds to a sequence of flips associated with diameters or pairs of centrally symmetric diagonals. Such a sequence transforms $T_0$ into another centrally symmetric triangulation $T$ in which the diameter retains the label 1 and each pair of centrally symmetric diagonals have labels $i$ and $i'$, for some $i$. It is easy to see that the quiver associated to $T$ is $G$-admissible. Thus $Q_0$ is globally foldable. □

**Proof of Theorem 5.5.1.** We will follow the strategy used in the proof of Theorem 5.5.2. Namely, we will construct a type $D_{n+1}$ quiver $Q_0$ with a group $G$ acting on its vertices, so that $Q_0$ is globally foldable with respect to $G$, and $B(Q_0)^G$ is the exchange matrix of type $B_n$ from Definition 5.31. Theorem 5.5.1 will then follow from Theorem 5.4.2 and Corollary 5.5.4.

The combinatorial model for a seed pattern of type $D_{n+1}$ ($n \geq 3$) uses tagged triangulations of a convex $(n+1)$-gon $\mathbf{P}^{\bullet}_{n+1}$ with a puncture $p$ in its interior. The vertices of $\mathbf{P}^{\bullet}_{n+1}$ are numbered $1, 2, \dots, n+1$ in clockwise order. Consider the tagged triangulation $T_0$ formed by:

- two radii $d_1$ and $d_2$ (tagged plain and notched) connecting the vertex 1 with the puncture $p$;
- plain arcs $d_3, d_4, \dots, d_{n+1}$ connecting the vertex 1 with vertices $2, 3, \dots, n$, respectively, as shown in Figure 5.14.

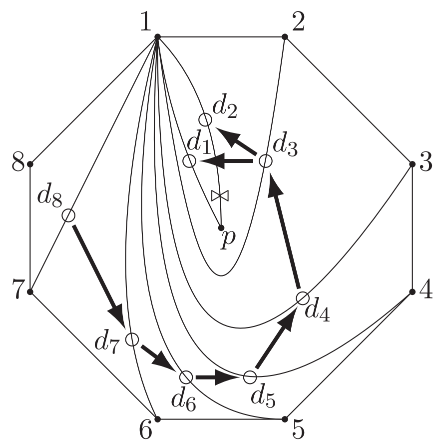



**Figure 5.14.** A tagged triangulation $T_0$ with quiver $Q_0$ such that $B(Q_0)^G$ is the type $B_n$ exchange matrix; here $n = 7$.

The corresponding quiver $Q_0 = Q(T_0)$ is an orientation of the type $D_{n+1}$ Dynkin diagram. The group $G = \mathbb{Z}/2\mathbb{Z}$ acts on the vertices of $Q_0$ by exchanging the vertices 1 and 2, and fixing all the other vertices. It is easy to see that $Q_0$ is $G$-admissible, and moreover $B(Q_0)^G$ is the exchange matrix of type $B_n$ from Equation (5.31).

It remains to show that $Q_0$ is globally foldable. In addition to the action on the set $\{1, 2, \ldots, n+1\}$ described above, the group $G = \mathbb{Z}/2\mathbb{Z}$ acts on the set of tagged arcs in the punctured polygon $\mathbf{P}^\bullet_{n+1}$ by fixing every non-radius arc, and toggling the tags of the radii. A transformation $\mu_{J_k} \circ \cdots \circ \mu_{J_1}$ associated with a sequence $J_1, \ldots, J_k$ of $G$-orbits in $\{1, 2, \ldots, n+1\}$ corresponds to a sequence of flips associated with non-radius arcs or pairs of "parallel" radii with different tagging. (The latter step replaces a pair of parallel radii inside a punctured digon by another such pair of radii.) Such a sequence transforms $T_0$ into another $G$-invariant tagged triangulation $T$ of the punctured polygon $\mathbf{P}^\bullet_{n+1}$ in which the labels 1 and 2 are assigned to a pair of parallel radii. It is easy to see that the quiver associated to $T$ is $G$-admissible (cf., e.g., the right picture in Figure 5.8), and so $Q_0$ is globally foldable. □

**Lemma 5.5.5.** *The cluster variables labeled by different $G$-orbits in a cluster algebra of type $B_n$ (respectively, $C_n$) are not equal to each other.*

**Proof.** In type $C$, consider two pairs of centrally symmetric diagonals, or a diameter and a pair of centrally symmetric diagonals, or two diameters. In all three cases, there is an octagon that contains the two $G$-orbits. Freezing all cluster variables except those corresponding to the diagonals of this octagon yields a seed subpattern of type $C_3$. In type $C_3$, one can check directly that each of 12 distinct $G$-orbits corresponds to a different cluster variable, say by verifying that the Laurent expansions expressing these cluster variables in terms of a particular initial cluster have different denominators.

In type $B$, consider two pairs of radii in $\mathbf{P}^{\bullet}_{n+1}$, or a pair of radii and an arc which is not a radii, or two arcs which are not radii. In all three cases, the two $G$-orbits lie inside a certain punctured quadrilateral, reducing the problem to the treatment of a seed subpattern of type $B_3$. In type $B_3$, there are 12 different $G$-orbits; using the same method as in type $C_3$, we can verify that they correspond to 12 distinct cluster variables. □

**Exercise 5.5.6.** By enumerating $G$-orbits, verify that a cluster algebra of type $B_n$ or $C_n$ has $n^2 + n$ cluster variables. By enumerating $G$-invariant tagged triangulations and centrally symmetric triangulations, verify that a cluster algebra of type $B_n$ or $C_n$ has $\binom{2n}{n}$ clusters (or seeds).

**Proposition 5.5.7.** *The same seed pattern cannot be simultaneously of type $B_n$ and of type $C_n$ for $n \geq 3$.*

**Proof.** By Lemma 2.7.13, if the diagram of an exchange matrix is connected, then its skew-symmetrizing vector is unique up to rescaling. By Exercise 2.7.7, the skew-symmetrizing vector is preserved under mutation. It remains to note that the skew-symmetrizing vectors for the exchange matrices of types $B_n$ and $C_n$ are $(1, 2, 2, \ldots, 2)$ and $(2, 1, 1, \ldots, 1)$, respectively (up to rescaling and permuting the entries). □

Examples of coordinate rings having natural cluster algebra structures of types $B_n$ and $C_n$ will be given in Section 7.3.

## 5.6. Seed patterns of types $E_6$, $E_7$, $E_8$

In this section, we describe a computer-assisted proof of the statement that the cluster algebras (or seed patterns) of exceptional types $E_6$, $E_7$, $E_8$ are of finite type. The proof utilizes one of several software packages for computing with cluster algebras, freely available online. Our personal favorites are the `Java` applet [**20**] and the `Sage` package [**24**], cf. the links at [**8**]. Among other things, both the applet and the `Sage` package allow one to compute seeds and Laurent expansions of cluster variables obtained by applying a sequence of mutations to a given initial seed.

**Theorem 5.6.1.** *Seed patterns of types $E_6$, $E_7$, and $E_8$ are of finite type.*

**Proof.** It suffices to verify that a seed pattern of type $E_8$ is of finite type. A pattern of type $E_6$ or $E_7$ can be viewed as a subpattern of a pattern of type $E_8$, so if the latter has finitely many seeds, then so does the former.

The main part of the proof is a verification that a cluster algebra of type $E_8$ *with trivial coefficients* has finitely many seeds. (The case of general coefficients will follow easily, see below.) With the `Sage` package [**24**], this is done as follows. The `Sage` command

```
S24 = ClusterSeed([[0,1],[1,2],[2,3],[4,5],[5,6],[6,7],
        [0,4],[1,5],[2,6],[3,7],[5,0],[6,1],[7,2]]);
```

defines a seed `S24` with the quiver shown in Figure 5.15. Recall that by Exercise 2.6.6, this quiver is mutation equivalent to any orientation of a Dynkin diagram of type $E_8$. Next, the `Sage` command

```
VC = S24.variable_class(ignore_bipartite_belt=True);
```

performs an exhaustive depth-first search to find all seeds that can be obtained from `S24` using $\le N$ mutations, for $N = 0, 1, 2, 3, \dots$; the cluster variables appearing in these seeds are recorded in the list `VC`. As soon as the calculation stops, the finiteness of the seed pattern is thereby established. Then the command

```
len(VC);
```

produces the output

```
128
```

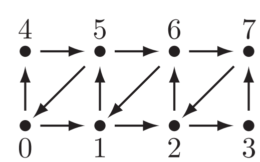
which is the total number of cluster variables in the pattern. These 128 cluster variables, or more precisely their Laurent expansions in terms of the chosen initial seed, can be displayed by executing the commands

```
for k in range(128):  print(VC[k]); print("...");
```

To get a better idea of what happens in the course of the depth-first search, one can run the command

```
SC=S24.mutation_class(show_depth=True,return_paths=True);
```

its output will show how many seeds have been obtained after each stage. These data are recorded in Figure 5.15. We see that no new seeds are found for $N = 14$. The total number of seeds is 25080.




| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | $\ge 13$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 9 | 50 | 196 | 614 | 1582 | 3525 | 6863 | 11626 | 17098 | 21706 | 24220 | 24974 | 25080 |

**Figure 5.15.** Top: the triangulated grid quiver of type $E_8$. Bottom: the table showing, for each $N \ge 0$, the number of distinct seeds that can be obtained from the seed with this quiver using $\le N$ mutations.

The proof of Theorem 5.6.1 can now be completed using a standard argument based on Corollary 4.3.6 and Remark 4.3.7. One only needs to check that the $8 \times 8$ exchange matrix

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

associated to the quiver in Figure 5.15 has full $\mathbb{Z}$-rank. □

**Remark 5.6.2.** The reader may be wondering: why we chose as the initial quiver the triangulated grid quiver in Figure 5.15, rather than an orientation of a Dynkin diagram of type $E_8$? The answer is that a straightforward implementation of the latter strategy appears to be computationally infeasible. A seed pattern of type $E_8$ has 128 cluster variables. The formulas expressing them as Laurent polynomials in the 8 initial cluster variables are recursively computed in the process of the depth-first search, and then used to compare the seeds to each other. When the triangulated grid quiver is chosen as the initial one, these Laurent polynomials turn out to be quite manageable. As a result, the entire calculation took less than one hour on a MacBook Pro laptop computer (manufactured in 2013), with a 2.6 GHz processor and 8 GB RAM. By comparison, a similar calculation using, as the initial quiver, the Dynkin diagram of type $E_8$ with an alternating orientation (i.e., each vertex is either a source or a sink) did not terminate within a few days. The explanation likely lies in the fact that the Laurent polynomials expressing some of the 128 cluster variables in terms of the initial ones are extremely cumbersome in this case. To get an idea of the size of these Laurent polynomials (which are known to have positive coefficients), one can specialize the 8 initial variables to 1, and compute the remaining 120 cluster variables recursively. When the initial quiver is the alternating Dynkin quiver of type $E_8$, the largest of these specializations turns out to be equal to 2820839; for the triangulated grid quiver, the corresponding value is 107.

**Remark 5.6.3.** One naturally arising cluster algebra of type $E_8$ is the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}_{3,8}$ of 3 planes in 8-space. Another closely related example is the coordinate ring of the affine space of $3 \times 5$ matrices. (See Chapter 7 and Chapter 9 for more details.) Each of these constructions can in principle be used, with or without a computer, to verify that seed patterns of type $E_8$ are of finite type.

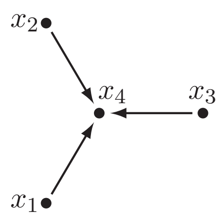
## 5.7. Seed patterns of types $F_4$ and $G_2$

We will now use folding to take care of types $F_4$ and $G_2$.

By Exercise 4.4.12, one can realize a type $F_4$ exchange matrix as $B(Q)^G$, where $Q$ is an orientation of the type $E_6$ Dynkin diagram, $G = \mathbb{Z}/2\mathbb{Z}$, and $Q$ is globally foldable. Now Corollary 5.5.4, together with the fact that type $E_6$ cluster algebras are of finite type, implies that the same is true in type $F_4$.

An alternative approach is to use a computer to check directly that a cluster algebra of type $F_4$ (with no frozen variables) has finitely many seeds (there are 105 of them). We can then use our standard argument based on Remark 4.3.7, together with the fact that the matrix $\tilde{B}^G$ from Figure 4.3 has full $\mathbb{Z}$-rank, to complete the proof.

We now turn to cluster algebras of type $G_2$. While it follows from the results of Section 5.1 that cluster algebras of type $G_2$ are of finite type, this result can also be obtained via folding of the type $D_4$ quiver shown in Figure 5.16. It is easy to see that this quiver is $G$-foldable, with respect to the natural action of $G = \mathbb{Z}/3\mathbb{Z}$. Since every seed pattern of type $D_4$ is of finite type, Corollary 5.5.4 implies the same for the type $G_2$.




$$\tilde{B}^G = \begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}$$

**Figure 5.16.** The generator of the group $G = \mathbb{Z}/3\mathbb{Z}$ acts on the vertices of the quiver shown on the left by sending $1 \mapsto 2 \mapsto 3 \mapsto 1$ and $4 \mapsto 4$. All vertices are mutable. The rows and columns of the matrix $\tilde{B}^G$ are indexed by the $G$-orbits $\{1, 2, 3\}$ and $\{4\}$.

## 5.8. Decomposable types

We refer to the disjoint union of two Dynkin diagrams of types $X_n$ and $Y_{n'}$ as a Dynkin diagram of type $X_n \sqcup Y_{n'}$; and similarly for disjoint unions of three or more Dynkin diagrams.

We have now defined cluster algebras of types $A_n$, $B_n$, $C_n$, $D_n$, $E_6$, $E_7$, $E_8$, $F_4$, and $G_2$, corresponding to the indecomposable Cartan matrices of finite type, or equivalently, the connected Dynkin diagrams. A seed pattern (or cluster algebra) of rank $n + n'$ is said to be of type $X_n \sqcup Y_{n'}$ if one of its exchange matrices is (up to a simultaneous permutation of rows and columns) a block-diagonal matrix with blocks whose Cartan counterparts are of types $X_n$ and $Y_{n'}$. We have the following simple lemma.

**Lemma 5.8.1.** *If $\mathcal{A}$ is a cluster algebra of decomposable type $X_n \sqcup Y_{n'}$, then the number of cluster variables is the sum of those of the cluster algebras of types $X_n$ and $Y_{n'}$, and the number of clusters is the product.*

**Proof.** Suppose that $\mathcal{A}$ is of type $X_n \sqcup Y_{n'}$. Then (after a simultaneous permutation of rows and columns) one of its exchange matrices $B$ is block-diagonal with blocks whose Cartan counterparts are of types $X_n$ and $Y_{n'}$. Then the labeled seeds of $\mathcal{A}$ are all pairs of the form $(\mathbf{x}^1 \sqcup \mathbf{x}^2, B^1 \sqcup B^2)$, where each $(\mathbf{x}^i, B^i)$ is a labeled seed of the cluster algebra associated to the $i$th block of $B$, $\mathbf{x}^1 \sqcup \mathbf{x}^2$ denotes the concatenation of the clusters $\mathbf{x}^1$ and $\mathbf{x}^2$, and $B^1 \sqcup B^2$ denotes the block-diagonal matrix with blocks $B^1$ and $B^2$. The statement of the lemma follows. □

We are now ready to complete the proof of the "if" direction of Theorem 5.2.8, restated below for the convenience of the reader.

**Corollary 5.8.2.** *If the Cartan counterpart of an exchange matrix of a seed pattern (or a cluster algebra) is a Cartan matrix of finite type, then the seed pattern is of finite type.*

**Proof.** Putting together Theorems 5.3.2, 5.4.2, 5.5.1, 5.5.2, and the results of Sections 5.6–5.7, we conclude that if the Cartan counterpart $A = A(B)$ of one of the exchange matrices $B$ associated to a seed pattern is an indecomposable Cartan matrix of finite type, then the seed pattern has finite type. In the general (decomposable) case, the same conclusion follows from Lemma 5.8.1. □

## 5.9. Enumeration of clusters and cluster variables

In this section we present formulas for the number of cluster variables and clusters in each cluster algebra of finite type. These enumerative invariants provide a way to distinguish cluster algebras of different types, leading to a proof of Theorem 5.2.12.

One important property of finite type cluster algebras is that the underlying combinatorics does not depend on the choice of coefficients. (Conjecturally this holds for arbitrary cluster algebras, see Section 6.1.) We have seen that in cluster algebras of type $A_n$, seeds are in bijection with triangulations of an $(n+3)$-gon, regardless of the choice of coefficients. Similarly, in cluster algebras of type $D_n$, seeds are in bijection with tagged triangulations of a punctured $n$-gon, for any choice of coefficients. Seeds of cluster algebras of types $C_n$ and $B_n$ are in bijection with folded triangulations and tagged triangulations, respectively. For cluster algebras of exceptional types other than $E_7$, the exchange matrices all have full $\mathbb{Z}$-rank. It then follows from Corollary 4.3.6 and Remark 4.3.7 that the combinatorics of seeds is independent of the choice of coefficients: if an exchange matrix $B(t)$ has full $\mathbb{Z}$-rank, and $\tilde{B}(t)$ is obtained from $B(t)$ by appending some additional rows, then the rows of $\tilde{B}(t)$ lie in the $\mathbb{Z}$-span of the rows of $B(t)$, and also the rows of $B(t)$ lie in the $\mathbb{Z}$-span of the rows of $\tilde{B}(t)$. Therefore Corollary 4.3.6 implies that the seeds of the seed patterns associated with $B(t)$ and with $\tilde{B}(t)$ are in bijection with each other.

In type $E_7$, one needs to add one additional row to the exchange matrix $B$ in order to obtain an exchange matrix $\tilde{B}$ of full $\mathbb{Z}$-rank. One can then check (for example by computer) that the seeds of the seed patterns associated with $B$ and $\tilde{B}$, respectively, are in bijection. The same argument as before implies that the underlying combinatorics of any cluster algebra of type $E_7$ does not depend on the choice of coefficients.

**Proposition 5.9.1.** *Let $X_n$ be a connected Dynkin diagram. The numbers of seeds and cluster variables in a seed pattern of type $X_n$ are given by the values in the corresponding column of the table in Figure 5.17. Alternatively, let $\Phi$ be an irreducible finite crystallographic root system of type $X_n$. Then*

$$\#\text{seeds}(X_n) = \prod_{i=1}^{n} \frac{e_i + h + 1}{e_i + 1}, \tag{5.33}$$

$$\#\text{clvar}(X_n) = \frac{n(h+2)}{2}, \tag{5.34}$$

*where $e_1, \ldots, e_n$ are the* exponents *of $\Phi$, and $h$ is the corresponding* Coxeter number*.*

| $X_n$ | $A_n$ | $B_n, C_n$ | $D_n$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $G_2$ |
|---|---|---|---|---|---|---|---|---|
| $\#\text{seeds}(X_n)$ | $\frac{1}{n+2}\binom{2n+2}{n+1}$ | $\binom{2n}{n}$ | $\frac{3n-2}{n}\binom{2n-2}{n-1}$ | 833 | 4160 | 25080 | 105 | 8 |
| $\#\text{clvar}(X_n)$ | $\frac{n(n+3)}{2}$ | $n(n+1)$ | $n^2$ | 42 | 70 | 128 | 28 | 8 |

**Figure 5.17.** Enumeration of seeds and cluster variables

**Proof.** The values in Figure 5.17 can be verified case by case. The types $A_n$, $B_n/C_n$, and $D_n$ were worked out in Corollary 5.3.6, Exercise 5.5.6, and Corollary 5.4.13, respectively. Exceptional types can be handled using the software packages discussed in Section 5.6. It is then straightforward to check that the formulas (5.33)–(5.34) match the values shown in Figure 5.17. (See, e.g., [**2, 9**] for the values of the exponents and Coxeter numbers for all types.) □

**Remark 5.9.2.** Since the number $\frac{1}{n+2}\binom{2n+2}{n+1}$ of seeds in type $A_n$ is a Catalan number, the numbers in (5.33) can be regarded as generalizations of the Catalan numbers to arbitrary Dynkin diagrams.

**Remark 5.9.3.** The number of cluster variables is alternatively given by

$$\#\text{clvar}(X_n) = \tfrac{\#\,\text{roots}(X_n)}{2} + n,$$

where $\#\,\text{roots}(X_n)$ denotes the number of roots in the root system $\Phi$ of type $X_n$. Thus cluster variables are equinumerous to the roots which are either positive or negative simple (i.e., the negatives of simple roots). A natural labeling of the cluster variables by these "almost positive" roots was described and studied in [**13**].

The reader is referred to [**9**] for a detailed discussion of cluster combinatorics of finite type, and further references.

Recall from Lemma 5.8.1 that if $\mathcal{A}$ is a cluster algebra of decomposable type $X_n \sqcup Y_{n'}$, then the number of cluster variables is the sum of those for the cluster algebras of types $X_n$ and $Y_{n'}$, and the number of seeds is the product. Therefore Proposition 5.9.1 allows us to compute the number of cluster variables and seeds for any cluster algebra of finite type.

**Proof of Theorem 5.2.12.** The implication (1)⇒(2) is easy to establish. Suppose that the Cartan counterparts $A(B')$ and $A(B'')$ are Cartan matrices of the same finite type. In the case of simply laced types $ADE$, this means that the corresponding quivers are (possibly different) orientations of isomorphic Dynkin diagrams. By Exercise 2.6.3, these two quivers are related to each other by a sequence of mutations at sources and sinks, and consequently $B'$ and $B''$ are mutation equivalent. The remaining cases $BCFG$ are treated in a similar fashion, using an appropriate analogue of Exercise 2.6.3.

Let us prove the implication (2)⇒(1). We first observe that it suffices to establish this result in the indecomposable case, since mutations transform the connected components of a quiver (or their analogues for skew-symmetrizable matrices) independently of each other.

Let $B'$ and $B''$ be mutation equivalent exchange matrices of types $X'$ and $X''$, respectively, where $X'$ and $X''$ are connected Dynkin diagrams. We need to show that $X'$ and $X''$ are of the same type. This is done as follows. By Proposition 5.9.1, the number of cluster variables in a seed pattern associated with such an exchange matrix is uniquely determined by the type of the corresponding Dynkin diagram (i.e., it does not depend on the choice of coefficients). Moreover no two connected Dynkin types of the same rank produce the same number of cluster variables—with the exception of the pairs $(B_n, C_n)$. For $B_n$ versus $C_n$, the claim follows from Proposition 5.5.7. □

## 5.10. 2-finite exchange matrices

In this section, we complete the proof of Theorem 5.2.8, closely following [**11**]. The notion of a 2-finite (skew-symmetrizable) matrix introduced in Definition 5.1.4 plays a key role.

We establish Theorem 5.2.8 by including it in the following statement.

**Theorem 5.10.1.** *For a seed $\Sigma = (\mathbf{x}, \mathbf{y}, B)$, the following are equivalent:*

(1) *there exists a matrix $B'$ mutation equivalent to $B$ such that its Cartan counterpart $A(B')$ is a Cartan matrix of finite type;*
(2) *the seed pattern (or the cluster algebra) defined by $\Sigma$ is of finite type;*
(3) *the exchange matrix $B$ is 2-finite.*

The implication $\boxed{(1) \Rightarrow (2)}$ in Theorem 5.10.1 is nothing but Corollary 5.8.2. To the best of our knowledge, the only known proof of the reverse implication $(2) \Rightarrow (1)$ goes through property (3).

The implication $\boxed{(2) \Rightarrow (3)}$ is precisely Corollary 5.1.5.

All that remains in order to complete the proof of Theorem 5.10.1 (hence Theorem 5.2.8) is to prove the implication $\boxed{(3) \Rightarrow (1)}$. We reformulate the latter below as a standalone statement.

**Proposition 5.10.2.** *Let $B = (b_{ij})$ be a 2-finite skew-symmetrizable integer matrix. Then there exists a matrix $B'$ mutation equivalent to $B$ such that its Cartan counterpart $A(B')$ is a Cartan matrix of finite type.*

In the rest of this section, we outline the proof of Proposition 5.10.2 given in [**11**, Sections 7–8]. The proof is purely combinatorial and rather technical. The missing details (all of them relatively minor) can be found in *loc. cit.*

The proof of Proposition 5.10.2 makes heavy use of the notion of diagram from Definition 2.7.10. Recall from Proposition 2.7.11 that mutation is well-defined for diagrams, and we write $\Gamma \sim \Gamma'$ to denote that diagrams $\Gamma$ and $\Gamma'$ are mutation equivalent.

A diagram $\Gamma$ is called 2-*finite* if any diagram $\Gamma' \sim \Gamma$ has all edge weights equal to $1, 2$ or $3$; otherwise we refer to $\Gamma$ as 2-*infinite.* Thus a matrix $B$ is 2-finite if and only if its diagram $\Gamma(B)$ is 2-finite. Note that a diagram is 2-finite if and only if so are all its connected components.

We now restate Proposition 5.10.2 in the language of diagrams.

**Proposition 5.10.3.** *Any 2-finite diagram is mutation equivalent to an orientation of a Dynkin diagram (where the weight of each edge is understood as its multiplicity in the corresponding Dynkin diagram).*

We note that all orientations of a given Dynkin diagram are mutation equivalent to each other, as they are related by source-or-sink mutations as in the proof of Theorem 5.2.12. This is true more generally for any diagram whose underlying graph is a tree.

A *subdiagram* of a diagram $\Gamma$ is a diagram $\Gamma' \subset \Gamma$ obtained by taking an induced directed subgraph of $\Gamma$ and keeping all its edge weights intact.

The proof of Proposition 5.10.3 repeatedly makes use of the following obvious property: any subdiagram of a 2-finite diagram is 2-finite. Equivalently, any diagram that has a 2-infinite subdiagram is 2-infinite. Thus, in order to show that a given diagram is 2-infinite, it suffices to exhibit a sequence of mutations that creates an edge of weight 3 or larger, or a subdiagram which is already known to be 2-infinite. The strategy is to catalogue enough 2-infinite subdiagrams to be able to show that any diagram avoiding them has to be mutation equivalent to an orientation of a Dynkin diagram.

We first examine two special classes of diagrams, those whose underlying graphs are trees or cycles, respectively. We refer to them as *tree diagrams* and *cycle diagrams*, respectively.

**Proposition 5.10.4.** *Any* 2*-finite tree diagram is an orientation of a connected Dynkin diagram.*

For the proof we consider a class of diagrams defined as follows. A diagram $\Gamma$ is called an *extended Dynkin tree diagram* if

- $\Gamma$ is a tree diagram with edge weights $\leq 3$;
- $\Gamma$ is not on the Dynkin diagram list;
- any proper subdiagram of $\Gamma$ is a Dynkin diagram (possibly disconnected).

To prove the proposition, it is enough to show that any orientation of any extended Dynkin tree diagram is 2-infinite. Direct inspection shows that Figure 5.18 provides a complete list of such diagrams (as discussed above, the choice of an orientation for a tree diagram is immaterial). We note that all these diagrams are associated with untwisted affine Lie algebras and can be found in the tables in [**2**] or in [**17**, Chapter 4, Table Aff 1]. The only diagram from those tables that is missing in Figure 5.18 is $A_n^{(1)}$, which is an $(n+1)$-cycle; it will appear later in our discussion of cycle diagrams.

In showing that an extended Dynkin tree diagram is 2-infinite, we can arbitrarily choose its orientation. We start with the three infinite series $B_n^{(1)}, C_n^{(1)}$, and $D_n^{(1)}$, each time orienting all the edges left to right. Let us denote the diagram in question by $X_n^{(1)}$, and let $n_\circ$ be the minimal value of $n$. So if $X = D$ (resp., $B$, $C$), then $n_\circ$ equals 4 (resp., 3, 2). If $n > n_\circ$, then mutating at the second vertex from the left, and subsequently removing

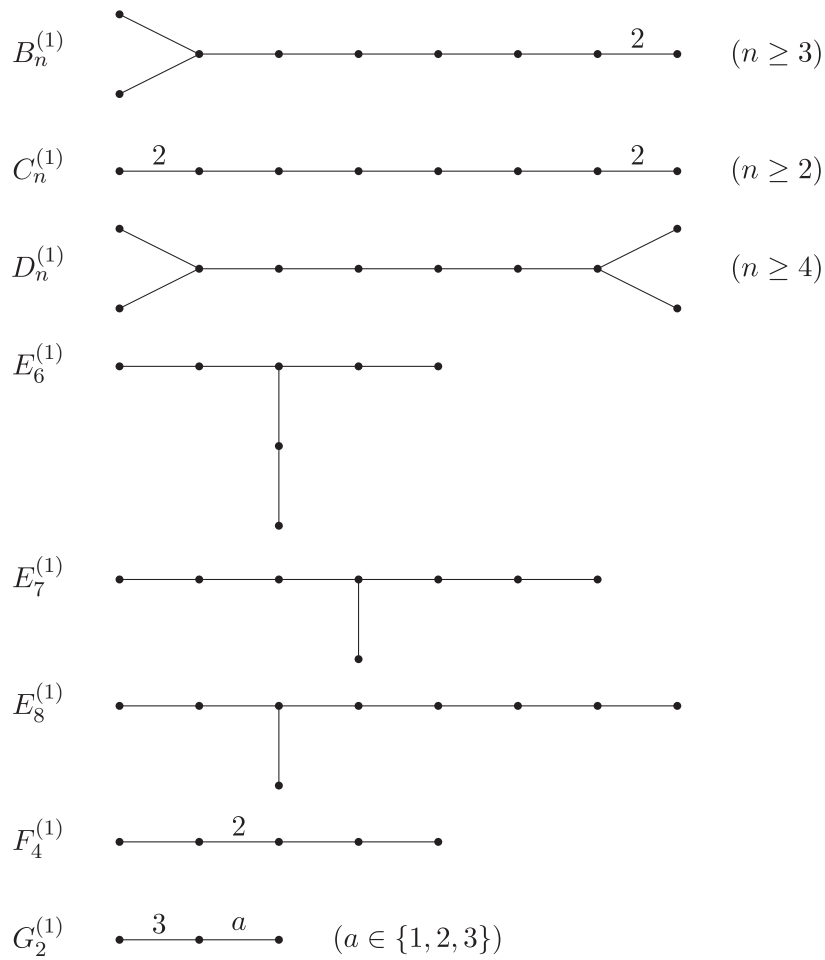



**Figure 5.18.** Extended Dynkin tree diagrams. Each tree $X_n^{(1)}$ has $n+1$ vertices. All unspecified edge weights are equal to 1.

this vertex (together with all incident edges) leaves us with a subdiagram of type $X_{n-1}^{(1)}$. Using induction on $n$, it suffices to check the base cases $D_4^{(1)}, B_3^{(1)}$ and $C_2^{(1)}$. It is not hard to check that each of these three diagrams is 2-infinite. The same applies to extended Dynkin trees of types $F_4^{(1)}$ and $G_2^{(1)}$.

The remaining three cases $E_6^{(1)}, E_7^{(1)}$ and $E_8^{(1)}$ can be treated in a similar manner (with or without a computer) but we prefer another approach. To describe it, we will need to introduce some notation.

**Definition 5.10.5.** For $p, q, r \in \mathbb{Z}_{\geq 0}$, we denote by $T_{p,q,r}$ the tree diagram (with all edge weights equal to 1) on $p + q + r + 1$ vertices obtained by connecting an endpoint of each of the three chains $A_p$, $A_q$ and $A_r$ to a single extra vertex (see Figure 5.19).

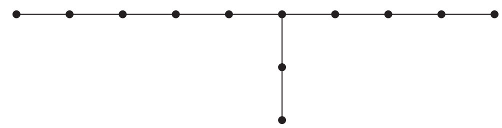


**Figure 5.19.** The tree diagram $T_{5,4,2}$ .

**Definition 5.10.6.** For $p, q, r \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}_{\geq 0}$, let $S^s_{p,q,r}$ denote the diagram (with all edge weights equal to 1) on $p+q+r+s$ vertices obtained by attaching three branches $A_{p-1}$, $A_{q-1}$, and $A_{r-1}$ to three consecutive vertices on a *cyclically oriented* $(s+3)$-cycle (see Figure 5.20).

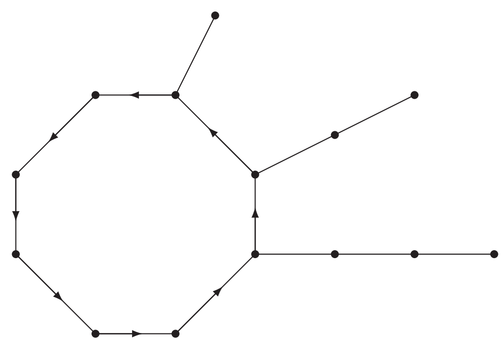


**Figure 5.20.** The diagram $S^5_{4,3,2}$ .

Note that in both definitions the choice of orientations for the edges where they are not shown, is immaterial: different choices lead to mutation-equivalent diagrams. For $T_{p,q,r}$ , this follows from Theorem 5.2.12; for $S^s_{p,q,r}$, one needs a slight generalization of this result, see [**11**, Proposition 9.2].

**Exercise 5.10.7.** Show that the diagram $S^s_{p,q,r}$ is mutation equivalent to $T_{p+r-1,q,s}$.

With the help of Exercise 5.10.7, the proof of Proposition 5.10.4 can now be completed, using the observations that

$$\begin{aligned} E_6^{(1)} &= T_{2,2,2} \sim S^2_{2,2,1} \supset D_5^{(1)}\,; \\ E_7^{(1)} &= T_{3,1,3} \sim S^3_{3,1,1} \supset E_6^{(1)}\,; \\ E_8^{(1)} &= T_{2,1,5} \sim S^5_{2,1,1} \supset E_7^{(1)}\,. \end{aligned}$$

Turning to the cycle diagrams, we have the following classification.

**Exercise 5.10.8.** Let $\Gamma$ be a 2-finite diagram whose underlying graph is an $n$-cycle (with some orientation of edges). Show that $\Gamma$ is cyclically oriented, and moreover it must be one of the following (see Figure 5.21):

(a) an $n$-cycle with all weights equal to 1 (in this case, $\Gamma \sim D_n$);

(b) a 3-cycle with edge weights $2, 2, 1$ (in this case, $\Gamma \sim B_3$);

(c) a 4-cycle with edge weights $2, 1, 2, 1$ (in this case, $\Gamma \sim F_4$).

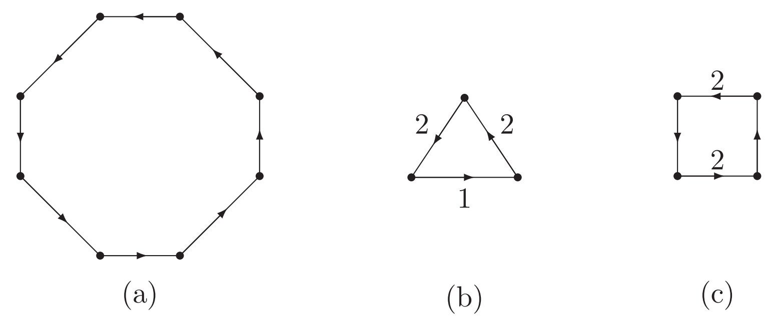



**Figure 5.21.** 2-finite cycles.

**Proof of Proposition 5.10.3.** We proceed by induction on $n$, the number of vertices in $\Gamma$. If $n \leq 3$, then $\Gamma$ is either a tree or a cycle, and the theorem follows by Proposition 5.10.4 and Exercise 5.10.8. So let us assume that the statement is already known for some $n \geq 3$; we need to show that it holds for a diagram $\Gamma$ on $n+1$ vertices. Pick a vertex $v \in \Gamma$ such that the subdiagram $\Gamma' = \Gamma - \{v\}$ is connected. Since $\Gamma'$ is 2-finite, it is mutation equivalent to some Dynkin diagram $X_n$. Furthermore, we may assume that $\Gamma'$ is (isomorphic to) our favorite representative of the mutation equivalence class of $X_n$. For each $X_n$, we will choose a representative that is most convenient for the purposes of this proof, and use the classifications of 2-finite tree and cycle diagrams obtained above to achieve the desired goal.

**Case 1:** $\Gamma'$ *is an orientation of a Dynkin diagram with no branching point, i.e., is of one of the types* $A_n$*,* $B_n$*,* $C_n$*,* $F_4$*, or* $G_2$*.* Let us orient the edges of $\Gamma'$ so that they all point in the same direction. If $v$ is adjacent to exactly one vertex of $\Gamma'$, then $\Gamma$ is a tree, and we are done by Proposition 5.10.4. If $v$ is adjacent to more than two vertices of $\Gamma'$, then $\Gamma$ has a cycle subdiagram whose edges are not cyclically oriented, contradicting Exercise 5.10.8. Thus we may assume that $v$ is adjacent to precisely two vertices $v_1$ and $v_2$ of $\Gamma'$, see Figure 5.22. Then $\Gamma$ has precisely one cycle $\mathcal{C}$, which must be of one of the types (a)–(c) shown in Figure 5.21.

Suppose that $\mathcal{C}$ is an oriented cycle with unit edge weights. If $\Gamma$ has an edge of weight $\geq 2$, then it contains a subdiagram of type $B_m^{(1)}$ or $G_2^{(1)}$, unless $\mathcal{C}$ is a 3-cycle, in which case $\mu_v(\Gamma) \sim B_{n+1}$. If all edges in $\Gamma$ are of weight 1, then it is one of the diagrams $S_{p,q,r}^s$ in Exercise 5.10.7 (with $q = 0$). Hence $\Gamma$ is mutation equivalent to a tree, and we are done by Proposition 5.10.4.

Suppose that $\mathcal{C}$ is as in Figure 5.21(b). If one of the edges $(v, v_1)$ and $(v, v_2)$ has weight 1, then $\mu_v$ removes the edge $(v_1, v_2)$, resulting in a tree, and we are done again. So assume that both $(v, v_1)$ and $(v, v_2)$ have weight 2.

If at least one edge outside $\mathcal{C}$ has weight $\geq 2$, then $\Gamma \supset C_m^{(1)}$ or $\Gamma \supset G_2^{(1)}$. It remains to consider the case shown in Figure 5.22. A direct check shows that $\mu_l \circ \cdots \circ \mu_2 \circ \mu_1 \circ \mu_{v_2} \circ \mu_v(\Gamma) = B_{n+1}$, and we are done.

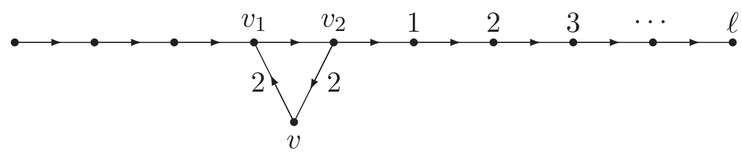



**Figure 5.22.** Second subcase in Case 1.

Suppose that $\mathcal{C}$ is as in Figure 5.21(c). It suffices to show that any diagram $\mathcal{C}'$ obtained from $\mathcal{C}$ by adjoining a single vertex adjacent to one of its vertices is 2-infinite. Indeed, if this extra edge has weight 1 (resp., 2, 3), then $\mathcal{C}'$ has a 2-infinite subdiagram of type $B_3^{(1)}$ (resp., $C_2^{(1)}$, $G_2^{(1)}$).

**Case 2:** $\Gamma' \sim D_n$ *($n \geq 4$).* We may assume that $\Gamma'$ is an oriented $n$-cycle with unit edge weights.

If $v$ is adjacent to two non-adjacent vertices of $\Gamma'$ (and possibly others), then $\Gamma$ contains an improperly oriented cycle, contradicting Exercise 5.10.8.

Suppose $v$ is adjacent to a single vertex $v_1 \in \Gamma'$. If the edge $(v, v_1)$ has weight $\geq 2$, then $\Gamma$ has a subdiagram $B_3^{(1)}$ or $G_2^{(1)}$. If $(v, v_1)$ has weight 1, then by Exercise 5.10.7, $\Gamma$ is mutation equivalent to a tree, and we are done.

Suppose that $v$ is adjacent to exactly two vertices $v_1$ and $v_2$ which are adjacent to each other. Then the triangle $(v, v_1, v_2)$ is either an oriented 3-cycle with unit edge weights or the diagram in Figure 5.21(b). In the former case, $\mu_v(\Gamma)$ is an oriented $(n+1)$-cycle, so $\Gamma \sim D_{n+1}$. In the latter case, $\mu_v(\Gamma)$ contains an improperly oriented (hence 2-infinite) cycle.

**Case 3:** $\Gamma' \sim E_n = T_{1,2,n-4}$, *for* $n \in \{6, 7, 8\}$. By Exercise 5.10.7, we may assume that $\Gamma' = S_{1,2,1}^{n-4}$, i.e., $\Gamma'$ consists of an oriented $(n-1)$-cycle $\mathcal{C}$ with unit edge weights, and an extra edge of weight 1 connecting a vertex in $\mathcal{C}$ to a vertex $v_1 \notin \mathcal{C}$.

There are several subcases to examine, depending on how $v$ connects to $\mathcal{C}$. It is routine (if tedious) to check that in each of these subcases, $\Gamma$ must be equivalent to an orientation of a Dynkin diagram (e.g. because it is equivalent to a tree, or to one of the diagrams treated in Cases 1 and 2 above), or else $\Gamma$ is not 2-finite. Details can be found in [**11**].

This concludes the proof of Proposition 5.10.3. As a consequence, we obtain Proposition 5.10.2, Theorem 5.10.1, and Theorem 5.2.8. □

**Remark 5.10.9.** The property of being 2-finite is clearly hereditary. Therefore the other two equivalent properties of exchange matrices appearing in Theorem 5.10.1 are hereditary as well.

## 5.11. Quasi-Cartan companions

An unpleasant feature of the finite type classification (see Theorem 5.10.1) is that it does not provide an effective way to verify whether a given exchange matrix $B$ defines a seed pattern of finite type: both condition (3) (2-finiteness) and condition (1) (being mutation equivalent to a skew-symmetrizable version of a Cartan matrix) impose a restriction on *all* matrices in the mutation class of $B$. An alternative criterion formulated directly in terms of the matrix $B$ (rather than its mutation class) was given in [**1**]. We reproduce this result below while omitting the technical part of the proof.

**Remark 5.11.1.** A different finite type recognition criterion was given in [**25**], by explicitly listing all minimal obstructions to finite type. More precisely, [**25**] provides a list of all (up to isomorphism) *minimal* 2*-infinite diagrams*, i.e., all diagrams which are not 2-finite but whose proper subdiagrams are all 2-finite. Then $B$ is of finite type if and only if $\Gamma(B)$ does not contain a subdiagram on this list. Unfortunately, the list is rather long: it includes 10 infinite series and a large number of exceptional diagrams of size $\leq 9$.

**Definition 5.11.2.** A *quasi-Cartan matrix* is a symmetrizable (square) matrix $A = (a_{ij})$ with integer entries such that $a_{ii} = 2$ for all $i$. Note that in such a matrix, the entries $a_{ij}$ and $a_{ji}$ always have the same sign. Unlike for generalized Cartan matrices, they are allowed to be positive.

A quasi-Cartan matrix $A$ is *positive* if the symmetrized matrix is positive definite, or equivalently if the principal minors of $A$ are all positive.

A quasi-Cartan matrix $A$ is called a *quasi-Cartan companion* of a skew-symmetrizable integer matrix $B$ if $|a_{ij}| = |b_{ij}|$ for all $i \neq j$. Thus $B$ can have several quasi-Cartan companions one of which is the Cartan counterpart of $B$ given by Definition 5.2.7. (To be precise, the number of quasi-Cartan companions of $B$ is $2^e$ where $e$ is the number of edges in the diagram of $B$.)

A *chordless cycle* in the diagram $\Gamma(B)$ is an induced subgraph isomorphic to a cycle (with arbitrary orientation).

**Theorem 5.11.3.** *For a skew-symmetrizable integer matrix $B$, each of the conditions* (2)–(1) *in Theorem 5.10.1 is equivalent to*

(4) *every chordless cycle in $\Gamma(B)$ is cyclically oriented, and $B$ has a positive quasi-Cartan companion.*

The key ingredient in the proof of Theorem 5.11.3 given in [**1**] is the following lemma, whose proof we omit.

**Lemma 5.11.4** ([**1**, Lemma 4.1])**.** *Property* (4) *in Theorem 5.11.3 is preserved under mutations of skew-symmetrizable integer matrices.*

**Proof of Theorem 5.11.3 modulo Lemma 5.11.4.** We deduce the implications (1) $\Rightarrow$ (4) $\Rightarrow$ (3) from Lemma 5.11.4. To prove (1) $\Rightarrow$ (4), it is enough to observe that if $\Gamma(B)$ is a Dynkin diagram, then $B$ satisfies (4). (Indeed, the Cartan counterpart of $B$ is positive, and $\Gamma(B)$ has no cycles.) To prove (4) $\Rightarrow$ (3), note that any positive quasi-Cartan matrix $A = (a_{ij})$ satisfies $|a_{ij}a_{ji}| \leq 3$ for all $i \neq j$ because of the positivity of the principal $2 \times 2$ minor of $A$ occupying the rows and columns $i$ and $j$. □

**Remark 5.11.5.** As explained above, one can use Lemma 5.11.4 to establish the implications (1) $\Rightarrow$ (4) $\Rightarrow$ (3). In combination with the arguments given in Section 5.10, this yields a self-contained combinatorial proof of the equivalence (3) $\Leftrightarrow$ (1).

A skew-symmetrizable integer matrix $B$ can have many quasi-Cartan companions $A$, corresponding to different choices for the signs of its off-diagonal matrix entries. Note that the positivity property of $A$ is preserved by simultaneous sign changes in rows and columns. It turns out that for the purposes of checking (4) for a given matrix $B$, there is a *unique*, up to these transformations, sign pattern for $A$ that needs to be checked for positivity. More precisely, we have the following result, cf. [**1**, Propositions 1.4–1.5].

**Proposition 5.11.6.** *Let $B$ be a skew-symmetrizable integer matrix such that each chordless cycle in $\Gamma(B)$ is cyclically oriented. Then $B$ has a quasi-Cartan companion $A = (a_{ij})$ such that the sign condition*

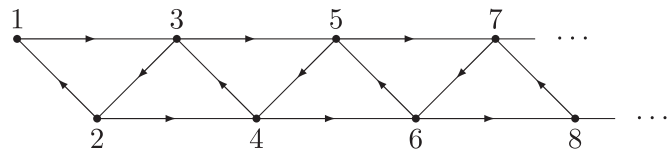
$$\prod_{\{i,j\}\in Z} (-a_{ij}) < 0 \tag{5.35}$$

*(product over all edges $\{i,j\}$ in $Z$) is satisfied for every chordless cycle $Z$. In fact, $A$ is unique up to simultaneous sign changes in rows and columns. Moreover $B$ satisfies* (2)–(1) *if and only if $A$ is positive.*

**Remark 5.11.7.** Several characterizations of positive quasi-Cartan matrices have been given in [**1**, Proposition 2.9]. In particular, these matrices are, up to certain equivalence, also classified by Cartan-Killing types. More precisely, any positive quasi-Cartan matrix corresponding to a root system $\Phi$ has the entries $a_{ij} = \langle \beta_i^\vee, \beta_j \rangle$, where $\{\beta_1, \dots, \beta_n\} \subset \Phi$ is a $\mathbb{Z}$-basis of the root lattice generated by $\Phi$, and $\beta^\vee$ is the coroot dual to a root $\beta \in \Phi$.

We conclude this section by an example illustrating the use of Proposition 5.11.6 for checking whether a particular skew-symmetric matrix is an exchange matrix of a seed pattern of finite type.

**Example 5.11.8.** Let $Q(n)$ be the following quiver with vertices $1, 2, \ldots, n$:



The quiver $Q(n)$ is the diagram of its $n \times n$ exchange matrix

$$B(n) = B(Q(n)) = \begin{bmatrix} 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -1 & 1 & \cdots & 0 & 0 \\ -1 & 1 & 0 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

(that is, $\Gamma(B(n)) = Q(n)$.) This quiver has $n-2$ chordless cycles, the 3-cycles with vertices $\{i, i+1, i+2\}$, for $i = 1, \ldots, n-2$. All of them are cyclically oriented. Now let $A(n)$ be the quasi-Cartan companion of $B(n)$ such that $a_{ij} = b_{ij}$ for $i < j$. One immediately checks that $A(n)$ satisfies the sign condition (5.35). Let $\delta_n = \det(A(n))$. By Sylvester's criterion, $A(n)$ is positive if and only if all the numbers $\delta_1, \ldots, \delta_n$ are positive.

It is not hard to compute the generating function of the sequence $(\delta_n)$, with the convention $\delta_0 = 1$:

$$\sum_{n \geq 0} \delta_n x^n = \frac{(1+x)(1+x+x^2)(1+x^2)(1+x^3)}{1-x^{12}}. \tag{5.36}$$

We see that $\delta_{n+12} = \delta_n$ for $n \geq 0$. Since the numerator in (5.36) is a polynomial of degree 8, we conclude that $\delta_9 = \delta_{10} = \delta_{11} = 0$. The fact that $\delta_9 = 0$ implies that $A(n)$ is not positive (hence $B(n)$ is not 2-finite) for $n \geq 9$.

The values of $\delta_n$ for $1 \leq n \leq 8$ are given in Figure 5.23; cf. Exercise 2.6.6. As all of them are positive, we conclude that $A(n)$ is positive (and so $B(n)$ is 2-finite) if and only if $n \leq 8$. The corresponding Cartan-Killing types are shown in Figure 5.23; we leave the verification to the reader.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\delta_n = \det(A(n))$ | 2 | 3 | 4 | 4 | 4 | 3 | 2 | 1 |
| Cartan-Killing type | $A_1$ | $A_2$ | $A_3$ | $D_4$ | $D_5$ | $E_6$ | $E_7$ | $E_8$ |

**Figure 5.23.** Determinants and Cartan-Killing types of the matrices $A(n)$.

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