Introduction to Cluster Algebras
Chapters 1–3

(preliminary version)

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Preface

This is a preliminary draft of Chapters 1–3 of our forthcoming textbook *Introduction to cluster algebras*, joint with Andrei Zelevinsky (1953–2013). We expect to post additional chapters in the not so distant future.

This book grew from the ten lectures given by Andrei at the NSF CBMS conference on Cluster Algebras and Applications at North Carolina State University in June 2006. The material of his lectures is much expanded but we still follow the original plan aimed at giving an accessible introduction to the subject for a general mathematical audience.

Since its inception in [**23**], the theory of cluster algebras has been actively developed in many directions. We do not attempt to give a comprehensive treatment of the many connections and applications of this young theory. Our choice of topics reflects our personal taste; much of the book is based on the work done by Andrei and ourselves.

Comments and suggestions are welcome.

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Contents

Chapter 1

Total positivity

Total positivity, along with G. Lusztig's theory of canonical bases, was one of the main motivations for the development of cluster algebras. In this chapter, we present the basic notions of total positivity, focusing on three important examples (to be re-examined again in the future): square matrices, Grassmannians, and basic affine spaces. As our main goal here is to provide motivation rather than develop a rigorous theory, the exposition is somewhat informal. Additional details and references can be found in [**18**].

1.1. Totally positive matrices

An $n \times n$ matrix with real entries is called *totally positive* (TP for short) if all its minors—that is, determinants of square submatrices—are positive. A real matrix is called *totally nonnegative* (or TNN) if all its minors are nonnegative. The first systematic study of these classes of matrices was conducted in the 1930s by F. Gantmacher and M. Krein [**28**], following the pioneering work of I. Schoenberg [**45**]. In particular, they showed that the eigenvalues of an $n\times n$ totally positive matrix are real, positive, and distinct.

Total positivity is a remarkably widespread phenomenon: TP and TNN matrices play an important role, *inter alia*, in classical mechanics, probability, discrete potential theory, asymptotic representation theory, algebraic and enumerative combinatorics, and the theory of integrable systems.

The *Binet-Cauchy Theorem* implies that TP (resp., TNN) matrices in $G = \mathrm{SL}_n$ form a multiplicative semigroup, denoted by $G_{>0}$ (resp., $G_{\geq 0}$). The following "splitting lemma" due to C. Cryer [**12, 13**] shows that the study of $G_{\geq 0}$ can be reduced to the investigation of its subsemigroup of upper-triangular unipotent TNN matrices:

Lemma 1.1.1. *A matrix $z \in \mathrm{SL}_n$ is totally nonnegative if and only if it has a Gaussian decomposition*

$$z = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & 1 & 0 & \cdots & 0 \\ * & * & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & 1 \end{bmatrix} \begin{bmatrix} * & 0 & 0 & \cdots & 0 \\ 0 & * & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{bmatrix} \begin{bmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & * & \cdots & * \\ 0 & 0 & 1 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

in which all three factors (lower-triangular unipotent, diagonal, and upper-triangular unipotent) are totally nonnegative.

There is also a counterpart of this statement for totally positive matrices.

The *Loewner-Whitney Theorem* [**36, 51**] identifies the infinitesimal generators of $G_{\geq 0}$ as the *Chevalley generators* of the corresponding Lie algebra. In pedestrian terms, each $n \times n$ TNN matrix can be written as a product of matrices of the form $x_i(t)$, $y_i(t)$, and $z_i(t)$, where each parameter t is positive, the matrices $x_i(t)$ are defined by

$$x_i(t) = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & t & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix},$$

$y_i(t)$ is the transpose of $x_i(t)$, and $z_i(t)$ is the diagonal matrix with diagonal entries $(1, \ldots, 1, t, t^{-1}, 1, \ldots, 1)$ where t and t^{-1} are in positions i and $i+1$. This led G. Lusztig [**37**] to the idea of extending the notion of total positivity to other semisimple groups G, by defining the set $G_{\geq 0}$ of TNN elements in G as the semigroup generated by the Chevalley generators. Lusztig proved that $G_{\geq 0}$ is a semialgebraic subset of G, and described it by inequalities of the form $\Delta(x) \geq 0$ where Δ lies in the appropriate *dual canonical basis*; see [**38**, Section 5]. A simpler description in terms of *generalized minors* [**20**] was given in [**22**].

A yet more general (if informal) concept is one of a *totally positive* (or *totally nonnegative*) (sub)variety of a given complex algebraic variety Z. Vaguely, the idea is this: suppose that Z comes equipped with a family $\mathbf{\Delta}$ of "important" regular functions on Z. The corresponding TP (resp., TNN) variety $Z_{>0}$ (resp., $Z_{\geq 0}$) is the set of points at which all of these functions take positive (resp., nonnegative) values:

$$Z_{>0} = \{z \in Z : \Delta(z) > 0 \text{ for all } \Delta \in \mathbf{\Delta}\}.$$

If Z is the affine space of $n \times n$ matrices (or $Z = \mathrm{GL}_n(\mathbb{C})$, or $Z = \mathrm{SL}_n(\mathbb{C})$), and $\mathbf{\Delta}$ is the set of all minors, then we recover the classical notions. One can restrict this construction to matrices lying in a given stratum of a Bruhat decomposition, or in a given *double Bruhat cell* [**20, 37**]. Another important example is the *totally positive (resp., nonnegative) Grassmannian* consisting of the points in a usual Grassmann manifold where all Plücker coordinates can be chosen to be positive (resp., nonnegative). This construction can be extended to arbitrary partial flag manifolds, and more generally to homogeneous spaces G/P associated to semisimple complex Lie groups.

We note that in each of the examples alluded to above, the notion of positivity depends on a particular choice of a coordinate system: a basis in a vector space allows us to view linear transformations as matrices, determines a system of Plücker coordinates, etc.

Among many questions which one may ask about totally positive/nonnegative varieties $Z_{>0}$ and $Z_{\geq 0}$, let us restrict our attention to the problem of *efficient TP testing*: how many inequalities (and which ones) does one need to check in order to ascertain that a given point in Z is totally positive? In particular, are there efficient ways for testing a very large matrix for total positivity? (It is not hard to see that an $n \times n$ matrix has altogether $\binom{2n}{n} - 1$ minors, a number which grows exponentially in n.) Examples 1.1.2 and 1.1.3 provide a glimpse into the tricks used to construct efficient TP criteria.

Example 1.1.2. A 2×2 matrix $z = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has five minors: the matrix entries a, b, c, d and the determinant $\Delta = \det(z) = ad - bc$. Now the identity

$$ad = \Delta + bc \tag{1.1}$$

shows that we do not have to check all five minors: if a, b, c, and Δ are positive, then so is $d = (\Delta + bc)/a$. (Alternatively, test the minors d, b, c, Δ.)

Example 1.1.3. Now let $n = 3$. To keep it simple (cf. also Lemma 1.1.1), let us consider the subgroup of unipotent upper triangular matrices

$$z = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{SL}_3 \,.$$

Since some of the entries of z are equal to 0, we modify the definition of total positivity by requiring that $\Delta(z) > 0$ for each minor Δ which does not identically vanish on the subgroup. This leaves us with four minors to check for positivity: the matrix entries a, b, c, and the 2×2 minor $P = ac - b$. Again we can reduce the number of needed checks from 4 to 3 using the identity

$$ac = P + b \,. \tag{1.2}$$

Thus each of the sets $\{a, b, P\}$ and $\{b, c, P\}$ provides an efficient TP test.

We note that in each of the above examples, the number of checks involved in each positivity test was equal to the dimension of the variety at hand. It seems implausible that one could do better.

1.2. The Grassmannian of 2-planes in m-space

Before developing efficient total positivity tests for square matrices, we shall discuss the somewhat simpler case of Grassmannians of 2-planes.

Recall that the complex *Grassmann manifold* (the *Grassmannian* for short), denoted $\mathrm{Gr}_{k,m} = \mathrm{Gr}_{k,m}(\mathbb{C})$, is the variety of all k-dimensional subspaces in an m-dimensional complex vector space. Let us fix a basis in this space, thereby identifying it with $\mathbb{C}^m$. Now any $k \times m$ matrix z of rank k defines a point $[z] \in \mathrm{Gr}_{k,m}$, the *row span* of z.

Given a k-element subset $J \subset \{1, \dots, m\}$, the *Plücker coordinate* $P_J(z)$ (evaluated at a matrix z as above) is, by definition, the $k \times k$ minor of z determined by the column set J. The collection $(P_J(z))_{|J|=k}$ only depends on the row span $[z]$ (up to common rescaling), and in fact provides an embedding of $\mathrm{Gr}_{k,m}$ into the complex projective space of dimension $\binom{m}{k} - 1$, called the *Plücker embedding*.

The Plücker coordinates P_J generate the *Plücker ring* $R_{k,m}$, the homogeneous coordinate ring of $\mathrm{Gr}_{k,m}$ with respect to the Plücker embedding. The ideal of relations that they satisfy is generated by the quadratic *Grassmann-Plücker relations*.

The Plücker coordinates are used to define the totally positive points of the Grassmannian, as follows.

Definition 1.2.1. The *totally positive Grassmannian* $\mathrm{Gr}^+_{k,m}$ is the subset of $\mathrm{Gr}_{k,m}$ consisting of points whose Plücker coordinates can be chosen so that all of them are positive real numbers. (Recall that Plücker coordinates are defined up to a common rescaling.)

In simple terms, an element $[z] \in \mathrm{Gr}_{k,m}$ defined by a full-rank $k \times m$ matrix z (without loss of generality, z can be assumed to have real entries) is TP if all the maximal (i.e., $k \times k$) minors $P_J(z)$ are of the same sign. We can begin by checking one particular value $P_J(z)$, and if it happens to be negative, replace the top row of z by its negative. Thus the problem at hand can be restated as follows: find an efficient method for checking whether all maximal minors of a given $k \times m$ matrix are positive. A brute force test requires $\binom{m}{k}$ checks. Can this number be reduced?

In this section, we systematically study the case $k = 2$ (Grassmannians of 2-planes). Chapter 9 will generalize the treatment below to arbitrary Grassmannians $\mathrm{Gr}_{k,m}$.

In the case of the Grassmannian $\mathrm{Gr}_{2,m}$, there are $\binom{m}{2}$ Plücker coordinates $P_{ij} = P_{\{i,j\}}$ labeled by pairs of integers $1 \le i < j \le m$. It turns out however that in order to verify that all the 2×2 minors $P_{ij}(z)$ of a given $2\times m$ matrix z are positive, it suffices to check the positivity of only $2m-3$ special minors. (Note that $2m-3$ is the dimension of the affine cone over $\mathrm{Gr}_{2,m}$.)

Exercise 1.2.2. Show that the 2×2 minors of a $2\times m$ matrix (equivalently, the Plücker coordinates P_{ij}) satisfy the *three-term Grassmann-Plücker relations*

$$P_{ik}\,P_{jl} = P_{ij}\,P_{kl} + P_{il}\,P_{jk} \qquad (1 \le i < j < k < l \le m). \tag{1.3}$$

We are going to construct a family of "optimal" tests for total positivity in $\mathrm{Gr}_{2,m}$ using triangulations of an m-gon. Consider a convex m-gon $\mathbf{P}_m$ with its vertices labeled clockwise. We associate the Plücker coordinate P_{ij} with the chord (i.e., a side or a diagonal of $\mathbf{P}_m$) whose endpoints are i and j.

Now, let T be a *triangulation* of $\mathbf{P}_m$ by pairwise noncrossing diagonals. (Common endpoints are allowed.) We view T as a maximal collection of noncrossing chords; as such, it consists of m sides and $m-3$ diagonals, giving rise to a collection $\tilde{\mathbf{x}}(T)$ of $2m-3$ Plücker coordinates, which we call an *extended cluster*. The Plücker coordinates corresponding to the sides of $\mathbf{P}_m$ are called *frozen variables*. They are present in every extended cluster $\tilde{\mathbf{x}}(T)$, hence the term "frozen;" an alternative terminology is *coefficient variables*. The remaining $m-3$ Plücker coordinates corresponding to diagonals of $\mathbf{P}_m$ are called *cluster variables*; they form a *cluster*. Thus each extended cluster consists of $m-3$ cluster variables and m frozen variables. We note that these $2m-3$ quantities are algebraically independent. See Figure 1.1.

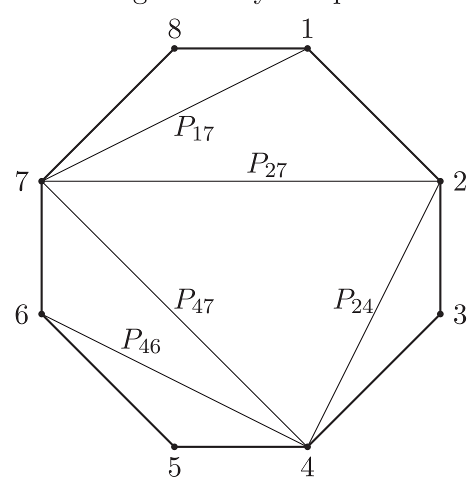


Figure 1.1. A triangulation T of an octagon $\mathbf{P}_8$. The extended cluster $\tilde{\mathbf{x}}(T)$ consists of the cluster variables $P_{17}, P_{24}, P_{27}, P_{46}, P_{47}$ and the frozen variables $P_{12}, P_{23}, \ldots, P_{78}, P_{18}$.

Theorem 1.2.3. *Each Plücker coordinate P_{ij} can be written as a subtraction-free rational expression in the elements of a given extended cluster $\tilde{\mathbf{x}}(T)$. Thus, if the $2m-3$ Plücker coordinates $P_{ij} \in \tilde{\mathbf{x}}(T)$ evaluate positively at a given $2 \times m$ matrix z, then all 2×2 minors of z are positive.*

To clarify, a *subtraction-free* expression is a formula involving variables and positive integers that uses only the operations of addition, multiplication, and division.

Proof of Theorem 1.2.3. Let us visualize the three-term relations (1.3) using the m-gon $\mathbf{P}_m$. Take four vertices $i < j < k < l$ of $\mathbf{P}_m$, cf. Figure 1.2. Then the relation (1.3) is reminiscent of the classical Ptolemy Theorem which asserts that for an inscribed quadrilateral, the products of the lengths of two pairs of opposite sides add up to the product of the two diagonals.

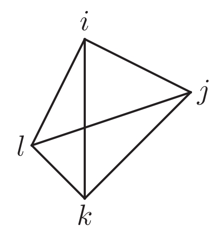


Figure 1.2. Three-term Grassmann-Plücker (or Ptolemy) relation (1.3).

Now Theorem 1.2.3 is an immediate consequence of the following three facts:

(1) Every Plücker coordinate appears as an element of an extended cluster $\tilde{\mathbf{x}}(T)$ for some triangulation T of the polygon $\mathbf{P}_m$.

(2) Any two triangulations of $\mathbf{P}_m$ can be transformed into each other by a sequence of *flips*. Each flip removes a diagonal of a triangulation to create a quadrilateral, then replaces it with the other diagonal of the same quadrilateral. See Figure 1.3.

(3) Each flip, say one involving the diagonals ik and jl, acts on extended clusters by exchanging the Plücker coordinates P_{ik} and P_{jl}. This exchange can be viewed as a subtraction-free transformation determined by the corresponding three-term relation (1.3). □

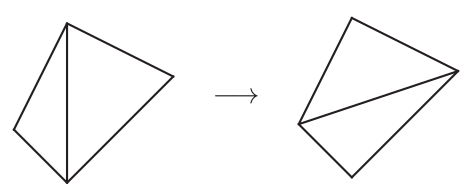


Figure 1.3. A flip of a diagonal in a quadrilateral.

Remark 1.2.4. In fact something stronger than Theorem 1.2.3 holds: every Plücker coordinate can be written as a *Laurent polynomial* with *positive coefficients* in the Plücker coordinates from $\tilde{\mathbf{x}}(T)$. This is an instance of very general phenomena of Laurentness and positivity in cluster algebras, which will be discussed later in the book.

The combinatorics of flips is captured by the graph whose vertices are labeled by the triangulations of the polygon $\mathbf{P}_m$ and whose edges correspond to flips. This is a regular graph: all its vertices have degree $m-3$. Moreover, this graph is the 1-skeleton of an $(m-3)$-dimensional convex polytope (discovered by J. Stasheff [**49**]) called the *associahedron*. See Figure 1.4.

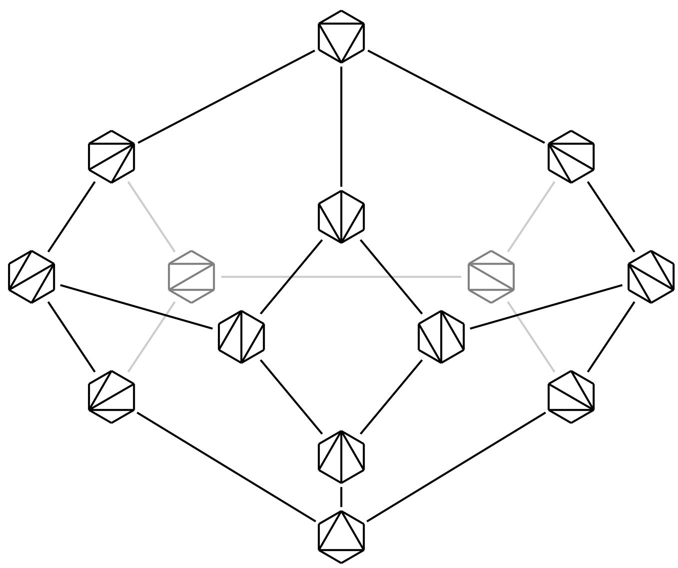


Figure 1.4. The 3-dimensional associahedron.

In the forthcoming terminology of cluster algebras, this graph is an example of an *exchange graph*. Its vertices correspond to extended clusters (all of which have the same cardinality) while its edges correspond to *exchange relations* (1.3): adjacent extended clusters are related to each other by interchanging the cluster variables appearing on the left-hand side of an exchange relation.

Recall that the Plücker ring $R_{2,m}$ is generated by the Plücker coordinates P_{ij} subject to the three-term relations (1.3). The combinatorics of extended clusters provides an elegant way to construct a linear basis for this ring. Define a *cluster monomial* to be a monomial in (i.e., a product of a multiset of) cluster and/or frozen variables all of which belong to one extended cluster; in other words, the corresponding collection of arcs does not contain two diagonals which cross each other. These monomials appeared already in the classical 19th century literature on invariant theory; in particular, it is known [**35, 50**] that they form a linear basis of $R_{2,m}$. Later on we shall discuss a far-reaching generalization of this result in the context of cluster algebras.

As mentioned above, the ideas we have discussed for the Grassmannian $\mathrm{Gr}_{2,m}$ can be generalized to a beautiful theory that works for arbitrary Grassmannians $\mathrm{Gr}_{k,m}$, *cf.* Chapter 9. In Chapter 8, we shall describe a generalization of the $\mathrm{Gr}_{2,m}$ example in a different direction, which involves the combinatorics of flips for triangulations of a Riemann surface with boundary and punctures. That construction has an intrinsic interpretation in hyperbolic geometry where an analogue of the Ptolemy relation holds for the exponentiated hyperbolic distances between horocycles drawn around vertices of a polygon with geodesic sides and cusps at the vertices (the "Penner coordinates" on the corresponding *decorated Teichmüller space* [**41**]).

1.3. The basic affine space

We next turn our attention to total positivity criteria for square matrices. In view of Lemma 1.1.1, it makes sense to study lower- and upper-triangular matrices first, and then proceed to the whole group SL_n. We choose a related but different strategy: first study matrices for which a certain subset of "flag minors" are positive, then treat the general case.

Definition 1.3.1. The *flag minor* P_J labeled by a nonempty proper subset $J \subsetneq \{1,\dots,n\}$ is a function on the special linear group $G = \mathrm{SL}_n$ defined by

$$P_J : z = (z_{ij}) \mapsto \det(z_{ij} \mid i \le |J|, j \in J).$$

Since an n-element set has $2^n - 2$ proper nonempty subsets, a matrix in SL_n has $2^n - 2$ flag minors; each of them occupies first several rows and an arbitrary subset of columns (of the same cardinality).

Let $U \subset G$ be the subgroup of unipotent lower-triangular matrices, i.e. the lower-triangular matrices with 1's on the diagonal. The group U acts on G by multiplication on the left. It consequently acts on $\mathbb{C}[G]$, the ring of polynomials in the matrix entries of a generic matrix of determinant 1. It is easy to see that each flag minor P_J is an invariant of this action: for any $z \in G$ and $y \in U$, we have $P_J(yz) = P_J(z)$. Similarly to the case of Plücker coordinates in a Plücker ring, we have (thanks to the appropriate versions of the First and Second Fundamental Theorems of invariant theory):

(1) the flag minors generate the ring $\mathbb{C}[G]^U$ of U-invariant polynomials in the matrix entries, and

(2) the ideal of relations among the flag minors is generated by certain quadratic *generalized Plücker relations*.

The ring $\mathbb{C}[G]^U$ plays an important role in the representation theory of the semisimple Lie group G: it naturally carries all irreducible representations of G, each with multiplicity 1. (We will not rely on this in what follows.) This ring is the coordinate ring of the *basic affine space*, the (Geometric Invariant Theory) quotient $U\backslash G$. This space is also known as the *base affine space*, the *fundamental affine space*, and the *principal affine space*. In this section, this space plays the role analogous to the role that the Grassmannians played in Section 1.2. As before, we can state everything we need in an elementary fashion, in terms of matrices and their minors.

Definition 1.3.2. An element $z \in G$ is *flag totally positive* (FTP) if all flag minors P_J take positive values at z.

We would like to detect flag total positivity in an efficient way, by testing as few of the $2^n - 2$ flag minors as possible. It turns out that the optimal test of this kind probes only $\frac{(n-1)(n+2)}{2}$ flag minors. We note that $\frac{(n-1)(n+2)}{2} = n^2 - 1 - \binom{n}{2}$ is the dimension of the basic affine space.

Following [**3**], we construct a family of tests for flag total positivity labeled by combinatorial objects called *wiring diagrams* which play the same role as triangulations did in Section 1.2.

This notion is best explained by an example such as the one in Figure 1.5. A wiring diagram consists of a family of n piecewise-straight lines which can be viewed as graphs of n continuous piecewise-linear functions defined on the same interval. The lines are labeled $1, \ldots, n$ as shown in Figure 1.5. The key requirement is that each pair of lines intersects exactly once.

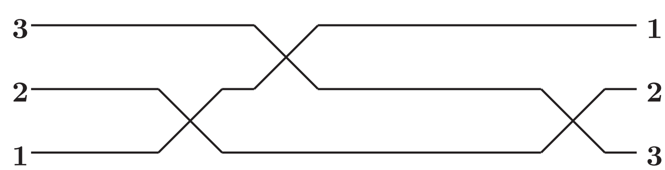


Figure 1.5. A wiring diagram.

We then assign to each *chamber* of a wiring diagram (i.e., a connected component of its complement inside the vertical strip) a subset of $[1, n] = \{1, \ldots, n\}$ indicating which lines pass *below* that chamber; cf. Figure 1.6.

Thus every chamber is naturally associated with a flag minor P_J, called a *chamber minor*, that occupies the columns specified by the set J in the chamber, and the rows $1, 2, \ldots, |J|$. The total number of chamber minors is always $\frac{(n-1)(n+2)}{2}$.

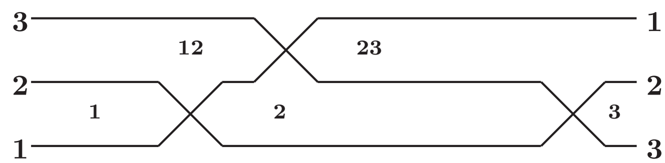


Figure 1.6. Chamber minors P_1, P_2, P_3, P_{12}, and P_{23}.

The chamber minors associated to a wiring diagram make up an *extended cluster*. Such an extended cluster will always contain the $2n-2$ flag minors associated to the unbounded chambers:

$$P_1, P_{1,2}, \ldots, P_{1,2,\ldots,n-1} \text{ and } P_n, P_{n-1,n}, \ldots, P_{2,3,\ldots,n};$$

these are the *frozen variables*. The $\binom{n-1}{2}$ chamber minors associated with bounded chambers are the *cluster variables*; they form the corresponding *cluster*.

Theorem 1.3.3 ([**3**])**.** *Every flag minor can be written as a subtraction-free rational expression in the chamber minors of any given wiring diagram. Thus, if these $\frac{(n-1)(n+2)}{2}$ chamber minors evaluate positively at a matrix $z \in \mathrm{SL}_n$, then z is FTP.*

Proof. Theorem 1.3.3 is implied by the following three facts:

(1) Each flag minor appears as a chamber minor in some wiring diagram.

(2) Any two wiring diagrams can be transformed into each other by a sequence of local *braid moves* of the form shown in Figure 1.7.

(3) Under each braid move, the corresponding collections of chamber minors are obtained from each other by exchanging the minors Y and Z (cf. Figure 1.7), and these minors satisfy the identity

$$YZ = AC + BD\ . \tag{1.4}$$

Statement (1) is easily checked by direct inspection. Statement (2) is a theorem of G. Ringel [**44**]; it is also equivalent to a well known property of reduced words in the symmetric group (Tits' Lemma): any two such words are related by braid moves. Finally, formula (1.4) is one of the aforementioned generalized Plücker relations; its proof is exercise 1.3.4 below. □

Exercise 1.3.4. Prove (1.4). Hint: use Proposition 1.3.6.

Proposition 1.3.5 (Muir's Law of extensible minors)**.** [**39**] *Suppose there is a polynomial identity (I) involving determinants $\Delta_{A,B}$ of a generic matrix, such that every term in (I) consists of the same number of determinants. Let R and C be finite sets of positive integers such that R is disjoint from*

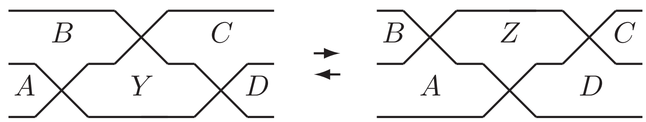


Figure 1.7. A braid move.

every row set A appearing in a determinant in (I), and C is disjoint from every column set B appearing in a determinant in (I). Then one can get a new identity (I') from I by replacing each term $\Delta_{A,B}$ by $\Delta_{A\cup R, B\cup C}$.

Proposition 1.3.6. *Suppose there is a polynomial identity (I) involving flag minors P_B of a generic matrix, such that every term in (I) consists of the same number of flag minors. Let C be a finite set of positive integers such that C is disjoint from every column set B appearing in a flag minor P_B in (I). Then one can get a new identity (I'') from (I) by replacing each term P_B by $P_{B\cup C}$.*

Proof. Let $b = |B|$ and $c = |C|$. Recall that the flag minor P_B is equal to $\Delta_{\{1,2,\dots,b\},B}$. Note that if the identity (I) is true, then the identity (I') is true, where (I') is obtained from (I) by replacing each term $\Delta_{\{1,2,\dots,b\},B}$ by the term $\Delta_{\{c+1,c+2,\dots,c+b\},B}$. But now if we apply Muir's Law with $R = \{1, 2, \dots, c\}$ and C, then we get a new identity (I'') from (I') by replacing each term $\Delta_{\{c+1,c+2,\dots,c+b\},B}$ by $\Delta_{\{1,2,\dots,c+b\},B\cup C} = P_{B\cup C}$. □

Remark 1.3.7. Just as in the case of $\mathrm{Gr}_{2,m}$ (cf. Remark 1.2.4), something much stronger than Theorem 1.3.3 is true: each flag minor can be written as a *Laurent polynomial with positive coefficients* in the chamber minors of a given wiring diagram.

Before moving on to our final example (total positivity for square matrices), let us pause to make a few observations based on our study of total positivity in the Grassmannian $\mathrm{Gr}_{2,m}$, flag total positivity in $G = \mathrm{SL}_n$, and the related rings $R_{2,m}$ and $\mathbb{C}[G]^U$. In both cases, we observed the following common features:

- a family of distinguished generators of the ring (Plücker coordinates and flag minors, respectively);
- a finite subset of *frozen* generators;
- a grouping of the generators into overlapping *extended clusters* all of which have the same size; each extended cluster contains the frozen generators;
- combinatorial data accompanying each extended cluster (triangulations and wiring diagrams, respectively);

- *exchange relations* that can be written using those data; these relations lead to subtraction-free birational maps relating extended clusters to each other;
- a "mutation rule" for producing new combinatorial data from the current one (flips of triangulations and braid moves, respectively).

In the case of the Grassmannian $\mathrm{Gr}_{2,m}$, we defined a graph whose vertices are indexed by the set of triangulations of $\mathbf{P}_m$, and whose edges correspond to flips. This graph is *regular*, i.e., all its vertices have the same degree. Indeed, in any triangulation, we can flip any of the participating diagonals. Put another way, given an extended cluster and a cluster variable within it, there is a unique way to construct a new extended cluster by replacing that cluster variable by another one.

We could construct an analogous graph related to the flag TP elements of SL_n, with vertices corresponding to wiring diagrams, and edges corresponding to braid moves. However, this graph is not regular for $n \geq 4$. The framework of cluster algebras will rectify this issue by providing a recipe for constructing the missing cluster variables and clusters.

Another important property that we observed for the Grassmannian $\mathrm{Gr}_{2,m}$ concerned cluster monomials. Recall that a cluster monomial is a product of cluster and frozen (=coefficient) variables (not necessarily distinct) all of which belong to one extended cluster. Cluster monomials form a linear basis for the Plücker ring $R_{2,m}$. Unfortunately the analogous statement does not hold for the ring $\mathbb{C}[G]^U$. However, the cluster monomials are still linearly independent, and hence can be included in an additive basis for the ring. Explicit constructions of such additive bases that possess "nice" properties (e.g., various versions of positivity) remain at the center of current research on cluster algebras.

1.4. The general linear group

We now turn to total positivity criteria for the general linear group, or equivalently, for square matrices of arbitrary size. It turns out that to test whether a given $n \times n$ matrix is TP it suffices to check the positivity of only n^2 special minors.

For an $n \times n$ matrix z, let $\Delta_{I,J}(z)$ denote the minor of z determined by the row set I and the column set J; here I and J are nonempty subsets of $[1,n] = \{1,\dots,n\}$ of the same cardinality. Thus z is TP if and only if $\Delta_{I,J}(z) > 0$ for all such I and J.

Following [**20, 21**] we construct a family of "optimal" tests for total positivity, labeled by combinatorial objects called *double wiring diagrams*. They generalize the wiring diagrams we saw in the previous section. A double wiring diagram is basically a superposition of two ordinary wiring diagrams, each colored in its own color ('thin' or 'thick'); see Figure 1.8.

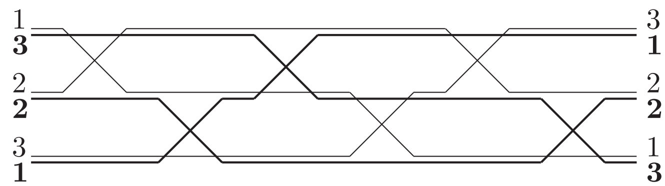


Figure 1.8. A double wiring diagram.

The lines in a double wiring diagram are numbered separately within each color. (Note the difference in the numbering schemes for the two colors.) We then assign to every *chamber* of a diagram a pair of subsets of $[1, n]$: each subset indicates which lines of the corresponding color pass below that chamber; see Figure 1.9. Thus every chamber is naturally associated with a minor $\Delta_{I,J}$ (again called a *chamber minor*) that occupies the rows and columns of an $n \times n$ matrix specified by the sets I and J written into that chamber. The total number of chamber minors is always n^2.

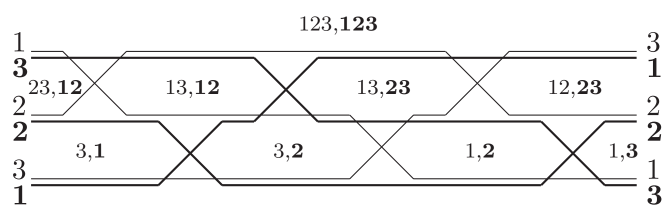


Figure 1.9. This double wiring diagram has $3^2 = 9$ chamber minors: $\Delta_{3,1}$, $\Delta_{3,2}$, $\Delta_{1,2}$, $\Delta_{1,3}$, $\Delta_{23,12}$, $\Delta_{13,12}$, $\Delta_{13,23}$, $\Delta_{12,23}$, and $\Delta_{123,123}$.

Theorem 1.4.1 ([**20**]). *Every minor of a square matrix can be written as a subtraction-free rational expression in the chamber minors of a given double wiring diagram. Thus, if these n^2 chamber minors evaluate positively at a given $n \times n$ matrix z, then z is totally positive.*

By now the reader can guess the strategy for proving this theorem.

Proof. Theorem 1.4.1 is a consequence of the following facts:

(1) Every minor is a chamber minor for some double wiring diagram.

(2) Any two double wiring diagrams are related to each other via a sequence of *local moves* of three different kinds, shown in Figure 1.10.

(3) Under each local move, the corresponding collections of chamber minors transform by exchanging the minors Y and Z, and these minors satisfy the identity

$$YZ = AC + BD \ . \tag{1.5}$$

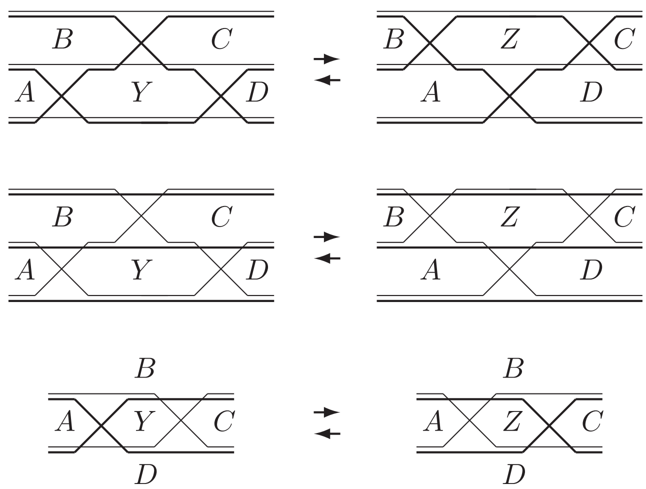


Figure 1.10. Local "moves."

Statements (1) and (2) can be easily derived from their counterparts for ordinary wiring diagrams, which appeared in the proof of Theorem 1.3.3. Each instance of the relation (1.5) is a well known determinantal identity; the reader may enjoy finding its proof on her/his own. The identities corresponding to the top two cases in Figure 1.10 are nothing but the three-term relation (1.4) which we discussed earlier; the third one is sometimes called the "Lewis Carroll identity," due to the role it plays in C. L. Dodgson's condensation method [**14**, pp. 170–180]. All of these identities were proved by P. Desnanot as early as in 1819, see [**39**, pp. 140–142]. □

Exercise 1.4.2. A minor $\Delta_{I,J}$ is called *solid* if both I and J consist of several consecutive indices. It is easy to see that an $n \times n$ matrix z has n^2 solid minors $\Delta_{I,J}$ such that $I \cup J$ contains 1 (see Figure 1.11). Show that z is TP if and only if all these n^2 minors are positive.

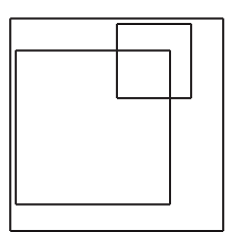


Figure 1.11. Solid minors $\Delta_{I,J}$ with $1 \in I \cup J$.

Remark 1.4.3. Similarly to Remarks 1.2.4 and 1.3.7, Theorem 1.4.1 can be strengthened as follows: every minor of a square matrix can be written as a *Laurent polynomial with positive coefficients* in the chamber minors of a given double wiring diagram.

The algebraic/combinatorial construction described above possesses the same key features that we identified in the two previous settings. The collections of chamber minors associated to double wiring diagrams are again examples of *extended clusters* in the future cluster algebra setup. As noted above, all these extended clusters have the same cardinality n^2. Each of them contains the $2n-1$ minors of the form $\Delta_{[1,p],[n-p+1,n]}$ and/or $\Delta_{[n-p+1,n],[1,p]}$ (for $p \in [1,n]$), which correspond to unbounded chambers. These $2n-1$ minors are the *frozen variables*. Removing them from an extended cluster, we obtain a *cluster* consisting of $(n-1)^2$ *cluster variables*, the chamber minors associated with the bounded chambers. A "mutation" of one of the three kinds depicted in Figure 1.10 replaces a single cluster variable in a cluster by a new one; the product of these two cluster variables (minors) appears on the left-hand side of the corresponding *exchange relation* (1.5).

For $n = 3$, there are 34 clusters corresponding to double wiring diagrams. They are shown in Figure 1.12 as vertices of a graph whose edges correspond to local moves. Looking closely at this graph, we see that it is not regular: of the 34 vertices, 18 have degree 4, and 16 have degree 3. Thus, for each of the 16 clusters corresponding to vertices of degree 3, there is one minor that cannot be exchanged from this cluster to form another cluster. This "irregularity" can be repaired using two additional polynomials in the matrix entries, see Exercise 1.4.4.

Exercise 1.4.4. For a 3×3 matrix $z = (z_{ij})$, let

$$K(z) = z_{33}\Delta_{12,12}(z) - \det(z), \tag{1.6}$$

$$L(z) = z_{11}\Delta_{23,23}(z) - \det(z). \tag{1.7}$$

Use K and L to add 16 more clusters to the graph in Figure 1.12. The resulting graph will be regular of degree 4. As an example, the cluster $\{e, f, g, A\}$ at the bottom of Figure 1.12 will be joined to the new cluster $\{e, f, g, K\}$ by an edge corresponding to a new exchange relation

$$\Delta_{23,23}K = \Delta_{12,23}\Delta_{23,12}z_{33} + \det(z)z_{23}z_{32}\,. \tag{1.8}$$

This construction will yield 16 additional TP tests for 3×3 matrices.

The theory of cluster algebras, to be developed in subsequent chapters, will unify the three examples we have treated here, and will provide a systematic way to produce the "missing" clusters and exchange relations, thereby generating a large class of new total positivity tests.

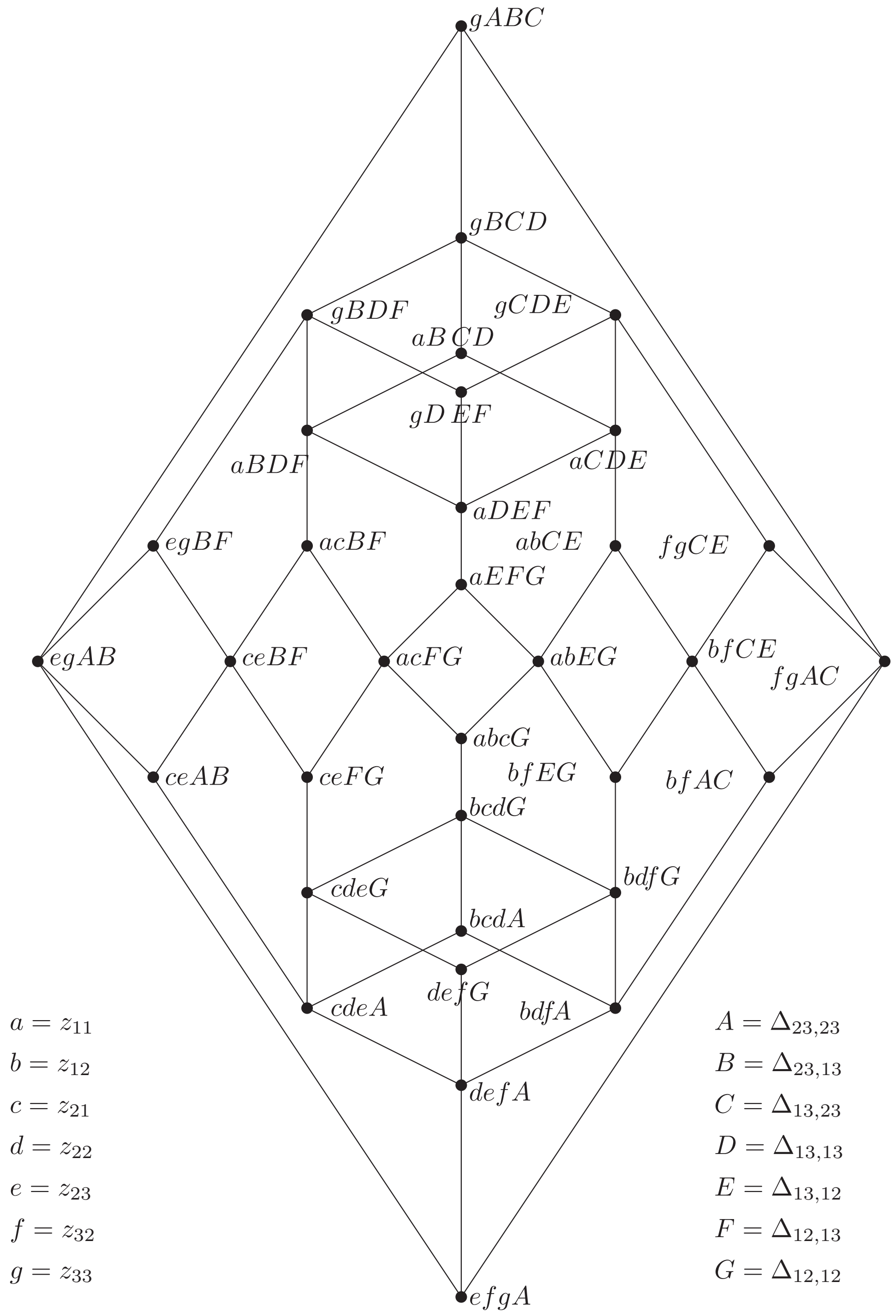


Figure 1.12. Total positivity tests for a 3×3 matrix $z=(z_{ij})$. Each test checks 9 minors. The frozen minors $z_{31}, z_{13}, \Delta_{23,12}(z), \Delta_{12,23}(z), \det(z)$ are common to all tests. The remaining 4 minors form a cluster shown near the corresponding vertex. To illustrate, the test derived from Figure 1.9 involves the cluster $bfCE = \{z_{32}, z_{12}, \Delta_{13,12}, \Delta_{13,23}\}$. The edges of the graph correspond to local moves.

Chapter 2

Mutations of quivers and matrices

In this chapter we discuss mutations of quivers and of skew-symmetrizable matrices. These constructions lie at the heart of the combinatorial framework underlying the general theory of cluster algebras.

Quivers (or more generally, skew-symmetrizable matrices) are the combinatorial data which accompany (extended) clusters and determine exchange relations between them. The notion of mutation generalizes many examples of "local" transformations of combinatorial objects, including those discussed in Chapter 1: flips in triangulations, braid moves in wiring diagrams, etc.

In some guise, quiver mutation appeared in the work of theoretical physicists (cf. [**9, 47**] and the discussion in [**15**, Section 6]) several years before its discovery by mathematicians [**23**]. However, the systematic study of the combinatorics of mutations has only begun with the advent of cluster algebras.

2.1. Quiver mutation

Definition 2.1.1. A *quiver* is a finite oriented graph. We allow multiple edges (called *arrows*) but not loops (i.e., an arrow may not connect a vertex to itself) nor oriented 2-cycles (i.e., no arrows of opposite orientation may connect the same pair of vertices). A quiver does not have to be connected.

In what follows, we will need a slightly richer notion, with some vertices in a quiver designated as *frozen*. The remaining vertices are called *mutable*. We will always assume that there are no arrows between pairs of frozen vertices. (Such arrows would make no difference in the future construction of a cluster algebra associated with a quiver.)

The terminology in Definition 2.1.1 anticipates the role that quiver mutations play in the forthcoming definition of a cluster algebra, wherein the vertices of a quiver are labeled by the elements of an extended cluster, so that the frozen vertices correspond to frozen variables, and the mutable vertices to the cluster variables. In this chapter, all of this remains in the background.

Definition 2.1.2. Let k be a mutable vertex in a quiver Q. The *quiver mutation* μ_k transforms Q into a new quiver $Q' = \mu_k(Q)$ via a sequence of three steps:

(1) For each oriented two-arrow path $i \to k \to j$, add a new arrow $i \to j$ (unless both i and j are frozen, in which case do nothing).

(2) Reverse the direction of all arrows incident to the vertex k.

(3) Repeatedly remove oriented 2-cycles until unable to do so.

An example is given in Figure 2.1.

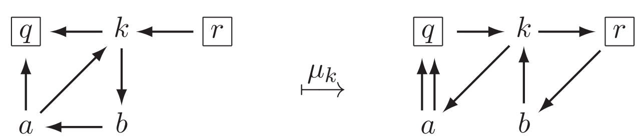


Figure 2.1. A quiver mutation μ_k. Vertices q and r are frozen. Step 1 adds three arrows $a \to b$, $a \to q$, and $r \to b$. Step 2 reverses four arrows connecting k to a, b, q, r. Step 3 removes the arrows $a \to b$ and $b \to a$.

Remark 2.1.3. If a vertex k of a quiver is a sink or a source, then mutation at k reverses the orientations of all arrows incident to k, and does nothing else. This operation was first considered in the context of quiver representation theory (the reflection functors of Bernstein-Gelfand-Ponomarev [**5**]).

We next formulate some simple but important properties of quiver mutation.

Exercise 2.1.4. Verify the following properties of quiver mutation:

(1) Mutation is an involution: $\mu_k(\mu_k(Q)) = Q$.

(2) Mutation commutes with the simultaneous reversal of orientations of all arrows of a quiver.

(3) Let k and ℓ be two mutable vertices which have no arrows between them (in either direction). Then mutations at k and ℓ commute with each other: $\mu_\ell(\mu_k(Q)) = \mu_k(\mu_\ell(Q))$.

In particular, mutations in different connected components of a quiver do not interact with each other.

2.2. Triangulations of polygons

Triangulations of polygons were discussed in Section 1.2 in the context of studying total positivity in the Grassmannian of 2-planes in m-space.

We now associate a quiver to each triangulation of a convex m-gon $\mathbf{P}_m$, and explain how flips of such triangulations correspond to quiver mutations.

Definition 2.2.1. Let T be a triangulation of the polygon $\mathbf{P}_m$ by pairwise noncrossing diagonals. The quiver $Q(T)$ associated to T is defined as follows. The frozen vertices of $Q(T)$ are labeled by the sides of $\mathbf{P}_m$, and the mutable vertices of $Q(T)$ are labeled by the diagonals of T. If two diagonals, or a diagonal and a boundary segment, belong to the same triangle, we connect the corresponding vertices in $Q(T)$ by an arrow whose orientation is determined by the clockwise orientation of the boundary of the triangle. See Figure 2.2.

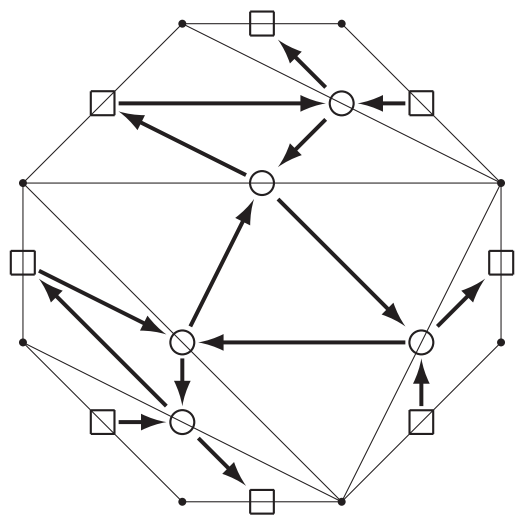


Figure 2.2. The quiver $Q(T)$ associated to a triangulation T of an octagon.

Exercise 2.2.2. Let T be a triangulation of $\mathbf{P}_m$ as above. Let T' be the triangulation obtained from T by flipping a diagonal γ. Verify that the quiver $Q(T')$ is obtained from $Q(T)$ by mutating at the vertex labeled by γ.

The construction of Definition 2.2.1 can be generalized to triangulations of more general oriented surfaces with boundary and punctures; this will be discussed in Chapter 8. Another generalization [**16**] was developed in the study of cluster structures arising in *higher Teichmüller theory*; a very special case is described in the exercise below.

Exercise 2.2.3. To each triangulation T of a convex polygon by noncrossing diagonals, let us associate a quiver $Q_3(T)$ as follows. Place two mutable vertices of $Q_3(T)$ on each diagonal of T, two frozen vertices on each side of the polygon, and one mutable vertex in the interior of each triangle of T. For a triangle in T, let $A_1, A_2, B_1, B_2, C_1, C_2$ be the vertices of $Q_3(T)$ lying on the boundary of the triangle, listed clockwise so that A_1 and A_2 (resp., B_1 and B_2, C_1 and C_2) lie on the same side of the triangle. Let K denote the vertex of $Q_3(T)$ lying inside the triangle. Draw the arrows $A_1 \to K \to B_2 \to C_1 \to K \to A_2 \to B_1 \to K \to C_2 \to A_1$. Doing so for each triangle of T, and removing the arrows between frozen vertices, yields the quiver $Q_3(T)$. Show that if T and T' are connected by a flip, then $Q_3(T)$ and $Q_3(T')$ are connected by a mutation. See Figure 2.3.

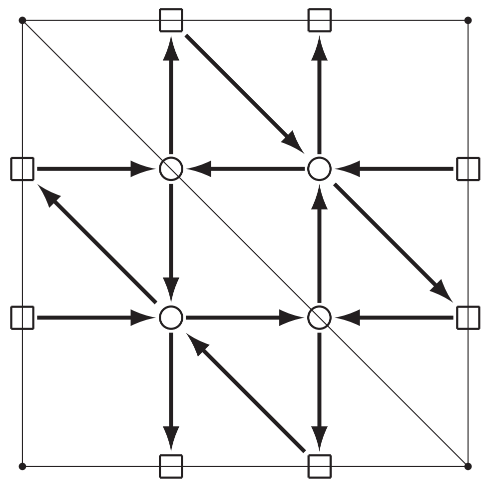


Figure 2.3. The quiver $Q_3(T)$ associated to a triangulation T of a quadrilateral.

2.3. Wiring diagrams

Wiring diagrams were introduced in Section 1.3 in the context of studying total positivity in basic affine spaces. We also explained how to label each chamber of a wiring diagram by a subset of $[1, n]$, cf. Figure 1.6.

We now associate a quiver to each wiring diagram, and demonstrate that braid moves in wiring diagrams translate into quiver mutations.

The *left end* (resp., *right end*) of a chamber is a crossing point of two wires located at the very left (resp., right) of the chamber. Each bounded chamber has two ends; an unbounded chamber has one.

Definition 2.3.1. The quiver $Q(D)$ associated with a wiring diagram D is defined as follows. The vertices of $Q(D)$ are labeled by the chambers of D.

The bounded chambers correspond to mutable vertices; the unbounded chambers correspond to frozen vertices. Let c and c' be two chambers at least one of which is bounded. Then there is an arrow $c \to c'$ in $Q(D)$ if and only if one of the following conditions is met:

(i) the right end of c coincides with the left end of c';

(ii) the left end of c lies directly above c', and the right end of c' lies directly below c;

(iii) the left end of c lies directly below c', and the right end of c' lies directly above c.

An example is shown in Figure 2.4.

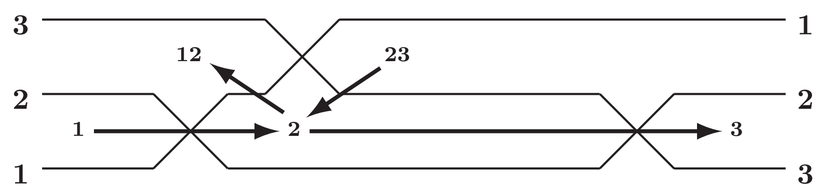


Figure 2.4. A quiver associated with a wiring diagram. All vertices but **2** are frozen (the latter corresponds to the only bounded chamber). Consequently the quiver does not include the arrow **12** $\to$ **23**.

The somewhat technical construction of Definition 2.3.1 is justified by the fact that braid moves on wiring diagrams translate into mutations of associated quivers:

Proposition 2.3.2. *Let D and D' be wiring diagrams related by a braid move at chamber Y (cf. Figure 1.7). Then $Q(D') = \mu_Y(Q(D))$.*

We leave the proof of Proposition 2.3.2 as an exercise for the reader.

Exercise 2.3.3. Draw the wiring diagrams corresponding to the quivers in Figure 2.5. Verify that these wiring diagrams are related by a braid move, and that the quivers are related by a quiver mutation.

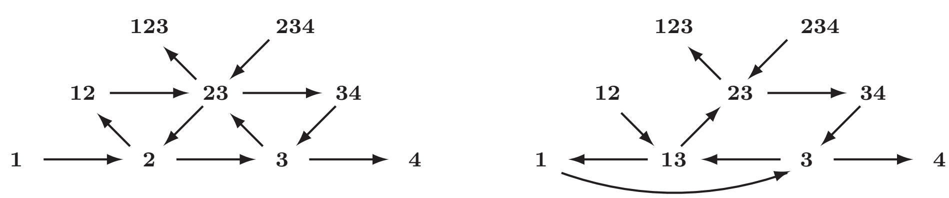


Figure 2.5. Quivers for two wiring diagrams related by a braid move.

2.4. Double wiring diagrams

We next extend the constructions of Section 2.3 to the double wiring diagrams discussed in Section 1.4.

Recall that each chamber of a double wiring diagram D is labeled by a pair of subsets of $[1, n]$, cf. Figure 1.9. Similarly to the case of ordinary wiring diagrams, each chamber of D has either one or two "ends," and each end is either "thick" or "thin" (formed by two thick lines or two thin lines).

Definition 2.4.1. The quiver $Q(D)$ associated with a double wiring diagram D is defined as follows. The vertices of $Q(D)$ are labeled by the chambers of D. The bounded chambers correspond to mutable vertices; the unbounded chambers correspond to frozen vertices. Let c and c' be two chambers, at least one of which is bounded. Then there is an arrow $c \to c'$ in $Q(D)$ if and only if one of the following conditions is met (cf. Figure 2.6):

(i) the right (resp., left) end of c is thick (resp., thin), and coincides with the left (resp., right) end of c';

(ii) the left end of c' is thin, the right end of c' is thick, and the entire chamber c' lies directly above or directly below c;

(iii) the left end of c is thick, the right end of c is thin, and the entire chamber c lies directly above or directly below c';

(iv) the left (resp., right) end of c' is above c and the right (resp., left) end of c is below c' and both ends are thin (resp., thick);

(v) the left (resp., right) end of c is above c' and the right (resp., left) end of c' is below c and both ends are thick (resp., thin).

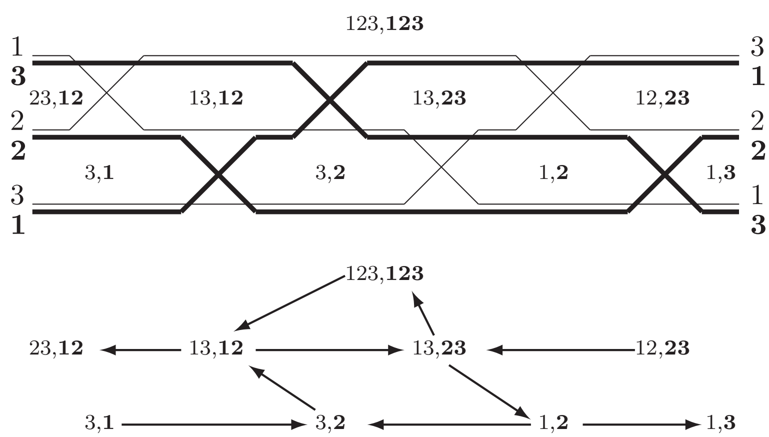


Figure 2.6. A double wiring diagram D and the corresponding quiver $Q(D)$.

Remark 2.4.2. One can check that the quiver $Q(D)$ defined as above depends only on the isotopy type of the double wiring diagram D.

As before, local moves translate into quiver mutations:

Proposition 2.4.3. *Suppose that double wiring diagrams D and D' are related by a local move (cf. Figure 1.10) at chamber Y. Then $Q(D') = \mu_Y(Q(D))$.*

The verification of this statement, which generalizes Proposition 2.3.2, is left as an exercise for the reader.

2.5. Urban renewal

Urban renewal [**33**] is an operation on bipartite graphs which arises in several different contexts including statistical mechanics (spider moves in dimer models [**30**]), gauge theory (Seiberg duality action on brane tilings [**27**]), and total positivity (square moves in dual graphs of Postnikov diagrams [**42**]). In Chapter 9, urban renewal will play an important role in the study of cluster structures on Grassmannians.

Definition 2.5.1. Let G be a connected planar bipartite graph, properly embedded in a disk, and considered up to isotopy. More precisely, we require the following:

- each vertex in G is colored either white or black, and lies either in the interior of the disk or on its boundary;
- each edge in G connects two vertices of different color, and is represented by a simple curve whose interior is disjoint from the other edges and from the boundary;
- each boundary vertex has degree 1, and each internal vertex has degree at least 2.
- the closure of each *face* (i.e, a connected component of the complement of G) is simply connected.

To such a bipartite planar graph G, we associate a quiver $Q(G)$, as follows. The vertices of $Q(G)$ are labeled by the faces of G. A vertex of $Q(G)$ is frozen if the corresponding face is incident to the boundary of the disk, and is mutable otherwise. For each edge e in G, we introduce an arrow connecting the (distinct) faces separated by e; this arrow is oriented so that it "sees the white endpoint of e to the left and the black endpoint to the right" as it crosses over e, see Figure 2.7. We then remove oriented 2-cycles from the resulting quiver, one by one, to get $Q(G)$.

We assume that the quiver $Q(G)$ is connected. One simple observation is that if G has a vertex v of degree 2, and we construct $\tilde{G}$ from G by deleting v and contracting its two incident edges, then the associated quiver does not change: $Q(G) = Q(\tilde{G})$.

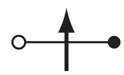


Figure 2.7. Constructing a quiver associated to a bipartite graph on a surface.

Definition 2.5.2. *Urban renewal* is a local transformation of a bipartite graph G as above that takes a quadrilateral face each of whose vertices has degree at least 3, and adds or removes four "legs" as shown in Figure 2.8.

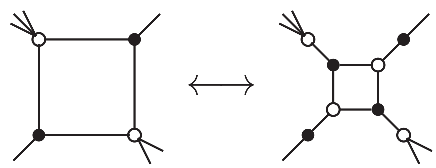


Figure 2.8. Urban renewal.

Exercise 2.5.3. Verify that if bipartite graphs as above are related via urban renewal, then the corresponding quivers are related by a mutation.

Definition 2.5.2 can be generalized to bipartite graphs properly embedded into an oriented surface.

2.6. Mutation equivalence

Definition 2.6.1. Two quivers Q and Q' are called *mutation equivalent* if Q can be transformed into a quiver isomorphic to Q' by a sequence of mutations. (Equivalently, Q' can be transformed into a quiver isomorphic to Q.) The *mutation equivalence class* $[Q]$ of a quiver Q is the set of all quivers (up to isomorphism) which are mutation equivalent to Q.

Example 2.6.2. The *Markov quiver* is a quiver Q of the form shown in Figure 2.9. Mutating Q at any of its 3 vertices produces a quiver isomorphic to Q, so $[Q]$ consists of a single element (up to isomorphism).

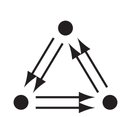


Figure 2.9. The Markov quiver. All three vertices are mutable.

Exercise 2.6.3. Show that all orientations of a tree (with no frozen vertices) are mutation equivalent to each other via mutations at sinks and sources.

Exercise 2.6.4. Which orientations of an n-cycle are mutation equivalent?

An $a \times b$ *grid quiver* is an orientation of an $a \times b$ grid in which each 4-cycle is oriented either clockwise or counterclockwise; see Figure 2.10. All vertices are mutable.

Exercise 2.6.5. Show that a grid quiver is mutation equivalent to the corresponding *triangulated grid quiver* (see Figure 2.10).

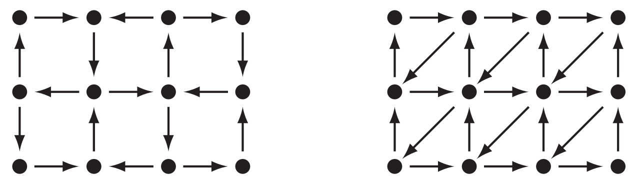


Figure 2.10. The 3×4 grid quiver, and the corresponding triangulated grid quiver.

Exercise 2.6.6. Verify that in each row of Figure 2.11, the quiver on the left is mutation equivalent to any orientation of the Dynkin diagram on the right.

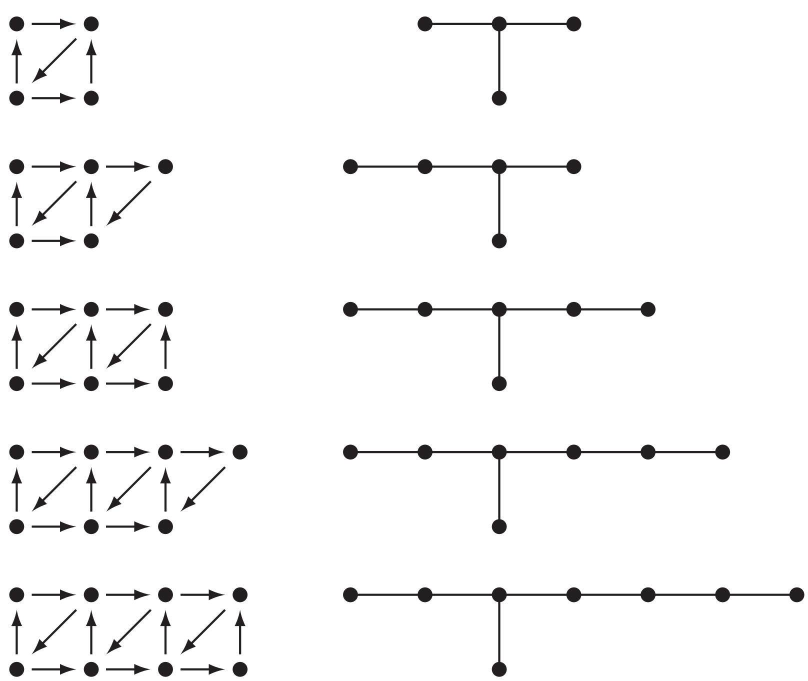


Figure 2.11. Quivers mutation equivalent to orientations of Dynkin diagrams of types D_4, D_5, E_6, E_7, E_8.

A *triangular grid quiver* with k vertices on each side is a quiver with $\binom{k+1}{2}$ vertices and $3\binom{k}{2}$ arrows that has the form shown in Figure 2.12. All vertices are mutable.

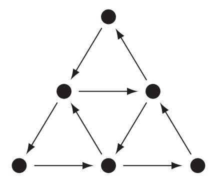


Figure 2.12. A triangular grid quiver.

Exercise 2.6.7. Show that the triangular grid quiver with three vertices on each side (see Figure 2.12) is mutation equivalent to an orientation of a tree.

Exercise 2.6.8. Show that the $k\times(2k+1)$ grid quiver is mutation equivalent to the triangular grid quiver with $2k$ vertices on each side.

Definition 2.6.9. A quiver Q (respectively, an extended skew-symmetrizable matrix $\tilde{B}$) is said to have *finite mutation type* if the mutation equivalence class $[Q]$ of Q (respectively, $\tilde{B}$) is finite.

Quivers of finite mutation type can be completely classified in explicit combinatorial terms. This classification is described in Chapter 8.

We conclude this section by stating, without proof, an innocent-looking but rather nontrivial result about quiver mutation.

A quiver is called *acyclic* if it has no oriented cycles.

Theorem 2.6.10 (see [**7**])**.** *Let Q and Q' be acyclic quivers mutation equivalent to each other. Then Q can be transformed into a quiver isomorphic to Q' via a sequence of mutations at sources and sinks. Consequently (cf. Remark 2.1.3), all acyclic quivers in a given mutation equivalence class have the same underlying undirected graph.*

Corollary 2.6.11. *An acyclic quiver which is mutation equivalent to an orientation of a tree is itself an orientation of the same tree. In particular, orientations of non-isomorphic trees are not mutation equivalent.*

It would be very interesting to find a purely combinatorial proof of Corollary 2.6.11 (or better yet, Theorem 2.6.10).

In general, it can be very hard to determine whether two quivers are mutation equivalent to each other.

Problem 2.6.12. Design a (reasonably efficient) algorithm for deciding whether two quivers are mutation equivalent or not.

2.7. Matrix mutation

In this section, we extend the notion of mutation from quivers to a certain class of matrices. We begin by explaining how matrices can be viewed as generalizations of quivers.

Definition 2.7.1. Let Q be a quiver (as in Definition 2.1.1) with m vertices, n of them mutable. Let us label the vertices of Q by the indices $1, \ldots, m$ so that the mutable vertices are labeled $1, \ldots, n$. The *extended exchange matrix* of Q is the $m \times n$ matrix $\tilde{B}(Q) = (b_{ij})$ defined by

$$b_{ij} = \begin{cases} \ell & \text{if there are } \ell \text{ arrows from vertex } i \text{ to vertex } j \text{ in } Q; \\ -\ell & \text{if there are } \ell \text{ arrows from vertex } j \text{ to vertex } i \text{ in } Q; \\ 0 & \text{otherwise.} \end{cases}$$

The *exchange matrix* $B(Q)$ is the $n \times n$ skew-symmetric submatrix of $\tilde{B}(Q)$ occupying the first n rows:

$$B(Q) = (b_{ij})_{i,j \in [1,n]} \, .$$

To illustrate, consider the Markov quiver Q shown in Figure 2.9. Then

$$\tilde{B}(Q) = B(Q) = \pm \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix},$$

where the sign depends on the labeling of the vertices.

Remark 2.7.2. Ostensibly, the definition of $\tilde{B}(Q)$ depends on the choice of labeling of the vertices of Q by the integers $1, \ldots, m$. To remove this dependence, one may consider extended exchange matrices up to a simultaneous relabeling of rows and columns $1, 2, \ldots, n$, and a relabeling of the rows $n+1, n+2, \ldots, m$.

The proof of the following lemma is straightforward.

Lemma 2.7.3. *Let k be a mutable vertex of a quiver Q. The extended exchange matrix $\tilde{B}(\mu_k(Q)) = (b'_{ij})$ of the mutated quiver $\mu_k(Q)$ is given by*

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + b_{ik} b_{kj} & \text{if } b_{ik} > 0 \text{ and } b_{kj} > 0; \\ b_{ij} - b_{ik} b_{kj} & \text{if } b_{ik} < 0 \text{ and } b_{kj} < 0; \\ b_{ij} & \text{otherwise.} \end{cases} \tag{2.1}$$

We next move from skew-symmetric matrices to a more general class of matrices.

Definition 2.7.4. An $n \times n$ matrix $B = (b_{ij})$ with integer entries is called *skew-symmetrizable* if $d_i b_{ij} = -d_j b_{ji}$ for some positive integers $d_1, \dots, d_n$. In other words, a matrix is skew-symmetrizable if it differs from a skew-symmetric matrix by a rescaling of its rows by positive scalars.

An $m \times n$ integer matrix, with $m \geq n$, whose top $n \times n$ submatrix is skew-symmetrizable is called an *extended skew-symmetrizable* matrix.

Exercise 2.7.5. Show that the class of matrices B described in Definition 2.7.4 would not change if instead of rescaling the rows of B, we allow to rescale its columns; alternatively, we could allow conjugation of B by a diagonal matrix with positive real diagonal entries.

We are now ready to define the notion of matrix mutation.

Definition 2.7.6. Let $\tilde{B} = (b_{ij})$ be an $m \times n$ extended skew-symmetrizable integer matrix. For $k \in [1, n]$, the *matrix mutation* μ_k *in direction* k transforms $\tilde{B}$ into the $m \times n$ matrix $\mu_k(\tilde{B}) = (b'_{ij})$ whose entries are given by (2.1).

By Lemma 2.7.3, matrix mutation generalizes quiver mutation.

Exercise 2.7.7. Under the conventions of Definitions 2.7.4 and 2.7.6, verify that

(1) the mutated matrix $\mu_k(\tilde{B})$ is again extended skew-symmetrizable, with the same choice of $d_1, \dots, d_n$;
(2) $\mu_k(\mu_k(\tilde{B})) = \tilde{B}$;
(3) $\mu_k(-\tilde{B}) = -\mu_k(\tilde{B})$;
(4) $\mu_k(B^T) = (\mu_k(B))^T$, where B^T denotes the transpose of B;
(5) if $b_{ij} = b_{ji} = 0$, then $\mu_i(\mu_j(\tilde{B})) = \mu_j(\mu_i(\tilde{B}))$.

For $b \in \mathbb{R}$, let $\operatorname{sgn}(b)$ be 1, 0, or -1, depending on whether b is positive, zero, or negative.

Definition 2.7.8. Let B be a skew-symmetrizable matrix. The skew-symmetric matrix $S(B) = (s_{ij})$ defined by

$$s_{ij} = \operatorname{sgn}(b_{ij}) \sqrt{|b_{ij} b_{ji}|} \tag{2.2}$$

is called the *skew-symmetrization* of B. Note that $S(B)$ has real (not necessarily integer) entries. The following result shows that skew-symmetrization commutes with mutation (extended verbatim to matrices with real entries).

Exercise 2.7.9. For any skew-symmetrizable matrix B and any k, we have

$$S(\mu_k(B)) = \mu_k(S(B)). \tag{2.3}$$

Definition 2.7.10. The *diagram* of a skew-symmetrizable $n \times n$ matrix $B = (b_{ij})$ is the weighted directed graph $\Gamma(B)$ with the vertices $1, \dots, n$ such that there is a directed edge from i to j if and only if $b_{ij} > 0$, and this edge is assigned the weight $|b_{ij}b_{ji}|$. In particular, if $b_{ij} \in \{-1, 0, 1\}$ for all i and j, then $\Gamma(B)$ is a quiver whose exchange matrix is B.

To illustrate Definition 2.7.10, consider $B = \left[\begin{smallmatrix} 0 & 2 & -2 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{smallmatrix}\right]$. Then $\Gamma(B)$ is an oriented cycle with edge weights 2, 4, and 2.

More generally, we use the term *diagram* in the rest of this chapter to mean a finite directed graph Γ (no loops, multiple edges, or 2-cycles allowed) whose edges are assigned positive real weights.

We note that the diagram $\Gamma(B)$ does *not* determine B: for instance, the matrix $(-B^T)$ has the same diagram as B. Here is another example:

$$\Gamma\left(\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\right) = \Gamma\left(\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\right).$$

Note that the diagram $\Gamma(B)$ and the skew-symmetric matrix $S(B)$ encode the same information about B: having an edge $i \to j$ in $\Gamma(B)$ supplied with weight c is the same as saying that $s_{ij} = \sqrt{c}$ and $s_{ji} = -\sqrt{c}$.

Proposition 2.7.11. *For a skew-symmetrizable matrix B, the diagram $\Gamma' = \Gamma(\mu_k(B))$ is uniquely determined by the diagram $\Gamma = \Gamma(B)$ and an index k.*

Proof. By Exercise 2.7.9, $S(\mu_k(B)) = \mu_k(S(B))$. It remains to translate this property into the language of diagrams. □

In the situation of Proposition 2.7.11, we write $\Gamma' = \mu_k(\Gamma)$, and call the transformation μ_k a *diagram mutation* in direction k. A detailed description of diagram mutation can be found in [**24**, Proposition 8.1]. Two diagrams Γ and Γ' related by a sequence of mutations are called *mutation equivalent,* and we write $\Gamma \sim \Gamma'$.

Remark 2.7.12. While the entries of B are integers, the entries of $S(B)$ may be irrational, as the weights of $\Gamma(B)$ may not be perfect squares. On the other hand, one can deduce from the skew-symmetrizability of B that the product of weights over the edges of any cycle in the underlying graph of $\Gamma(B)$ is a perfect square.

Lemma 2.7.13. *If the diagram $\Gamma(B)$ of an $n \times n$ skew-symmetrizable matrix B is connected, then the skew-symmetrizing vector $(d_1, \dots, d_n)$ is unique up to rescaling.*

Proof. Let $(d_1, \dots, d_n)$ and $(d'_1, \dots, d'_n)$ be two skew-symmetrizing vectors. We have $d_i b_{ij} = -d_j b_{ji}$ and $d'_i b_{ij} = -d'_j b_{ji}$ for all i and j. So if b_{ij} is nonzero, then $\frac{b_{ij}}{b_{ji}} = \frac{-d_j}{d_i} = \frac{-d'_j}{d'_i}$ and hence $\frac{d_j}{d'_j} = \frac{d_i}{d'_i}$. Since $\Gamma(B)$ is connected, there exists an ordering $\ell_1, \ell_2, \dots, \ell_n$ of its vertices such that every vertex ℓ_j with $2 \le j \le n$ is connected by an edge in $\Gamma(B)$ to a vertex ℓ_i with $i < j$; in other words, $b_{\ell_i \ell_j} \ne 0$. It follows that $\frac{d_{\ell_1}}{d'_{\ell_1}} = \frac{d_{\ell_2}}{d'_{\ell_2}} = \cdots = \frac{d_{\ell_n}}{d'_{\ell_n}}$, as desired. □

2.8. Invariants of matrix mutations

The following notion is a straightforward extension of Definition 2.6.1.

Definition 2.8.1. Two skew-symmetrizable matrices B and B' are *mutation equivalent* if one can get from B to B' by a sequence of mutations, possibly followed by simultaneous renumbering of rows and columns. The *mutation equivalence class* $[B]$ of B is the set of all matrices mutation equivalent to B. These notions generalize to extended skew-symmetrizable matrices in an obvious way.

It is natural to extend Problem 2.6.12 to the setting of matrix mutations:

Problem 2.8.2. Find an effective way to determine whether two given $n \times n$ skew-symmetrizable matrices are mutation equivalent.

Problem 2.8.2 remains wide open, even in the case of skew-symmetric matrices (or equivalently quivers). For $n = 2$, the question is trivial, since mutation simply negates the entries of the matrix. For $n = 3$, there is an explicit algorithm for determining whether two skew-symmetric matrices are mutation equivalent, see [**2**].

Problem 2.8.2 is closely related to the problem of identifying explicit nontrivial invariants of matrix (or quiver) mutation. Unfortunately, very few invariants of this kind are known at present.

Theorem 2.8.3 ([**4**, Lemma 3.2])**.** *Mutations preserve the rank of a matrix.*

Proof. Let $\tilde{B}$ be an $m \times n$ extended skew-symmetrizable integer matrix. Fix an index $k \in [1, n]$ and a sign $\varepsilon \in \{1, -1\}$. The rule (2.1) describing the matrix mutation in direction k can be rewritten as follows:

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \max(0, -\varepsilon b_{ik})\, b_{kj} + b_{ik} \max(0, \varepsilon b_{kj}) & \text{otherwise.} \end{cases} \tag{2.4}$$

(To verify this, examine the four possible sign patterns for b_{ik} and b_{kj}.) Next observe that (2.4) can be restated as

$$\begin{aligned} \mu_k(\tilde{B}) &= J_{m,k}\, \tilde{B} J_{n,k} + J_{m,k}\, \tilde{B} F_k + E_k\, \tilde{B} J_{n,k} \\ &= (J_{m,k} + E_k)\, \tilde{B}\, (J_{n,k} + F_k) \end{aligned} \tag{2.5}$$

where

- $J_{m,k}$ (respectively, $J_{n,k}$) denotes the diagonal matrix of size $m \times m$ (respectively, $n \times n$) whose diagonal entries are all 1, except for the (k, k) entry, which is -1;
- $E_k = (e_{ij})$ is the $m \times m$ matrix with $e_{ik} = \max(0, -\varepsilon b_{ik})$, and all other entries equal to 0;
- $F_k = (f_{ij})$ is the $n \times n$ matrix with $f_{kj} = \max(0, \varepsilon b_{kj})$, and all other entries equal to 0.

(Here we used that $E_k \tilde{B} F_k = 0$ because $b_{ii} = 0$ for all i.) Since

$$\det(J_{m,k} + E_k) = \det(J_{n,k} + F_k) = -1, \tag{2.6}$$

it follows that $\operatorname{rank}(\mu_k(\tilde{B})) = \operatorname{rank}(\tilde{B})$. □

Theorem 2.8.4. *The determinant of a skew-symmetrizable matrix is invariant under mutation.*

Proof. This follows from (2.5) and (2.6) (taking $m = n$ and $B = \tilde{B}$). □

Another invariant of matrix mutations is the greatest common divisor of the matrix elements of B.

Remark 2.8.5. For skew-symmetric matrices (equivalently, quivers with no frozen vertices), formulas (2.5) and (2.6) allow us to interpret mutation as a transformation of a skew-symmetric bilinear form over the integers under a particular unimodular change of basis. One can then use the general theory of invariants of such transformations (the *skew Smith normal form*, see [**40**, Section IV.3])) to identify some invariants of quiver mutation. Unfortunately this approach does not yield much beyond the facts established above.

Chapter 3

Clusters and seeds

This chapter introduces cluster algebras of *geometric type*. A more general construction of cluster algebras over an arbitrary semifield will be discussed in Chapter 11.

3.1. Basic definitions

Let us recall the three motivating examples discussed in Chapter 1: Grassmannians of 2-planes, affine base spaces, and general linear groups. In each of these examples, we manipulated two kinds of data:

- combinatorial data (triangulations, wiring diagrams) and
- algebraic data (Plücker coordinates, chamber minors).

Accordingly, transformations applied to these data occurred on two levels:

- on the "primary" level, we saw the combinatorial data evolve via local moves (flips in triangulations, braid moves in wiring diagrams); as shown in Chapter 2, a unifying description of this dynamics can be given using the language of quiver mutations;
- on the "secondary" level, we saw the algebraic data evolve in a way that was "driven" by the combinatorial dynamics, with subtraction-free birational transformations encoded by the current combinatorial data.

An attempt to write the exchange relations in terms of the quiver at hand naturally leads to the axiomatic setup of cluster algebras of geometric type, which we now describe.

Let m and n be two positive integers such that $m \geq n$. As an *ambient field* for a cluster algebra, we take a field $\mathcal{F}$ isomorphic to the field of rational functions over $\mathbb{C}$ (alternatively, over $\mathbb{Q}$) in m independent variables.

Definition 3.1.1. A *labeled seed* of geometric type in $\mathcal{F}$ is a pair $(\tilde{\mathbf{x}},\tilde{B})$ where

- $\tilde{\mathbf{x}} = (x_1, \ldots, x_m)$ is an m-tuple of elements of $\mathcal{F}$ forming a *free generating set*; that is, $x_1, \ldots, x_m$ are algebraically independent, and $\mathcal{F} = \mathbb{C}(x_1, \ldots, x_m)$;
- $\tilde{B} = (b_{ij})$ is an $m \times n$ extended skew-symmetrizable integer matrix, see Definition 2.7.4.

We shall use the following terminology:

- $\tilde{\mathbf{x}}$ is the (labeled) *extended cluster* of the labeled seed $(\tilde{\mathbf{x}}, \tilde{B})$;
- the n-tuple $\mathbf{x} = (x_1, \ldots, x_n)$ is the (labeled) *cluster* of this seed;
- the elements $x_1, \ldots, x_n$ are its *cluster variables*;
- the remaining elements $x_{n+1}, \ldots, x_m$ of $\tilde{\mathbf{x}}$ are the *frozen variables* (or *coefficient variables*);
- the matrix $\tilde{B}$ is the *extended exchange matrix* of the seed;
- its top $n \times n$ submatrix B is the *exchange matrix*.

Definition 3.1.2. Let $(\tilde{\mathbf{x}}, \tilde{B})$ be a labeled seed as above. Take an index $k \in \{1, \ldots, n\}$. The *seed mutation* μ_k in direction k transforms $(\tilde{\mathbf{x}}, \tilde{B})$ into the new labeled seed $\mu_k(\tilde{\mathbf{x}}, \tilde{B}) = (\tilde{\mathbf{x}}', \tilde{B}')$ defined as follows:

- $\tilde{B}' = \mu_k(\tilde{B})$ (cf. Definition 2.7.6).
- the extended cluster $\tilde{\mathbf{x}}' = (x'_1, \ldots, x'_m)$ is given by $x'_j = x_j$ for $j \neq k$, whereas $x'_k \in \mathcal{F}$ is determined by the *exchange relation*

$$x_k\, x'_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}. \tag{3.1}$$

Exercise 3.1.3. Consider each of the three settings that we discussed in Sections 1.2, 1.3, and 1.4. Construct a seed $(\tilde{\mathbf{x}}, \tilde{B}(Q))$ where Q is a quiver associated with a particular triangulation, wiring diagram, or double wiring diagram (see Definitions 2.2.1, 2.3.1, and 2.4.1, respectively), and $\tilde{\mathbf{x}}$ is the extended cluster consisting of the corresponding Plücker coordinates or chamber minors. Verify that applying the recipe (3.1) to these data recovers the appropriate exchange relations (1.3), (1.4), and (1.5), respectively.

To keep track of the various labeled seeds one can obtain by mutation from a given one, we introduce the following combinatorial setup.

Definition 3.1.4. Let $\mathbb{T}_n$ denote the *n-regular tree* whose edges are labeled by the numbers $1, \dots, n$, so that the n edges incident to each vertex receive different labels. See Figure 3.1.

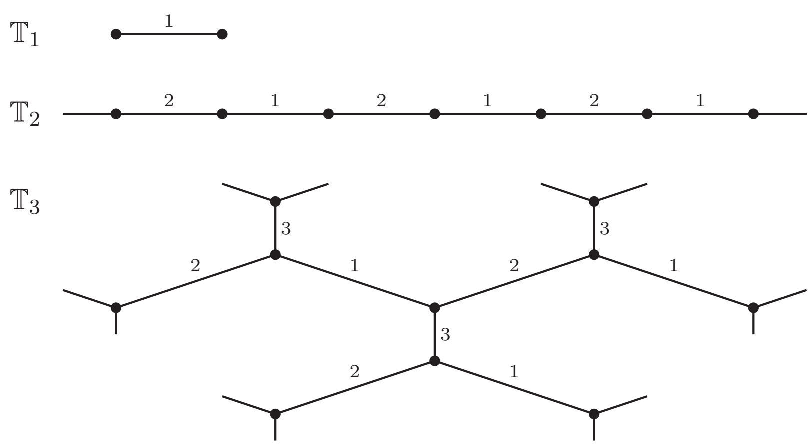


Figure 3.1. The n-regular trees $\mathbb{T}_n$ for $n = 1, 2, 3$.

We shall write $t \overset{k}{\text{———}} t'$ to indicate that vertices $t, t' \in \mathbb{T}_n$ are joined by an edge labeled by k.

Definition 3.1.5. A *seed pattern* is defined by assigning a labeled seed $(\tilde{\mathbf{x}}(t), \tilde{B}(t))$ to every vertex $t \in \mathbb{T}_n$, so that the seeds assigned to the endpoints of any edge $t \overset{k}{\text{———}} t'$ are obtained from each other by the seed mutation in direction k. A seed pattern is uniquely determined by any one of its seeds. See Figure 3.2.

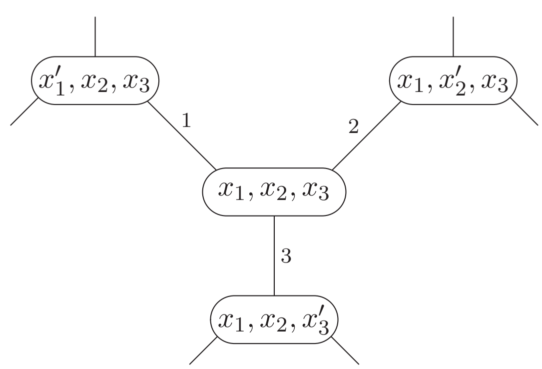


Figure 3.2. Clusters in a seed pattern.

Now everything is in place for defining cluster algebras.

Definition 3.1.6. Let $(\tilde{\mathbf{x}}(t), \tilde{B}(t))_{t\in\mathbb{T}_n}$ be a seed pattern as above, and let

$$\mathcal{X} = \bigcup_{t\in\mathbb{T}_n} \mathbf{x}(t)$$

be the set of all cluster variables appearing in its seeds. We let the *ground ring* be $R = \mathbb{C}[x_{n+1}, \dots, x_m]$, the polynomial ring generated by the frozen variables. (A common alternative is to take $R = \mathbb{C}[x_{n+1}^{\pm 1}, \dots, x_m^{\pm 1}]$, the ring of Laurent polynomials in the frozen variables, see Section 7.6 for an example. Sometimes the scalars are restricted to $\mathbb{Q}$, or even to $\mathbb{Z}$.)

The *cluster algebra* $\mathcal{A}$ (of geometric type, over R) associated with the given seed pattern is the R-subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables: $\mathcal{A} = R[\mathcal{X}]$. To be more precise, a cluster algebra is the R-subalgebra $\mathcal{A}$ as above together with a fixed seed pattern in it.

A common way to describe a cluster algebra is to pick an *initial (labeled) seed* $(\tilde{\mathbf{x}}_\circ, \tilde{B}_\circ)$ in $\mathcal{F}$ and build a seed pattern from it. The corresponding cluster algebra, denoted $\mathcal{A}(\tilde{\mathbf{x}}_\circ, \tilde{B}_\circ)$, is generated over the ground ring R by all cluster variables appearing in the seeds mutation equivalent to $(\tilde{\mathbf{x}}_\circ, \tilde{B}_\circ)$.

Remark 3.1.7 (cf. Exercise 3.1.3)**.** It can be shown that applying this construction in each of the three settings discussed in Chapter 1, one obtains cluster algebras naturally identified with the Plücker ring $R_{2,m}$ (cf. Section 7.7), the ring of invariants $\mathbb{C}[\mathrm{SL}_k]^U$ (cf. Section 7.5), and the polynomial ring $\mathbb{C}[z_{11}, \dots, z_{kk}]$ (cf. Section 7.6), respectively.

Remark 3.1.8. It is often more natural to work with *(unlabeled) seeds*, which differ from the labeled ones in that we identify two seeds $(\tilde{\mathbf{x}}, \tilde{B})$ and $(\tilde{\mathbf{x}}', \tilde{B}')$ in which $\mathbf{x}'$ is a permutation of $\mathbf{x}$, and $\tilde{B}'$ is obtained from $\tilde{B}$ by the corresponding permutation of rows and columns. Note that ignoring the labeling does not affect the resulting cluster algebra in a meaningful way.

Remark 3.1.9. Many questions arising in cluster algebra theory and its applications do not really concern cluster algebras as such. These are questions which are not about commutative rings carrying a cluster structure; rather, they are about seed patterns and the birational transformations that relate extended clusters to each other. For those questions, the choice of the ground ring is immaterial: the formulas remain the same regardless.

Remark 3.1.10. Since any free generating collection of m elements in $\mathcal{F}$ can be mapped to any other such collection by an automorphism of $\mathcal{F}$, the choice of the initial extended cluster $\tilde{\mathbf{x}}_\circ$ is largely inconsequential: the cluster algebra $\mathcal{A}(\tilde{\mathbf{x}}_\circ, \tilde{B}_\circ)$ is determined, up to an isomorphism preserving all the matrices $\tilde{B}(t)$, by the initial extended exchange matrix $\tilde{B}_\circ$, and indeed by its mutation equivalence class. Also, replacing $\tilde{B}_\circ$ by $-\tilde{B}_\circ$ yields essentially the same cluster algebra (all matrices $\tilde{B}(t)$ change their sign).

Remark 3.1.11. The same commutative ring (or two isomorphic rings) can carry very different cluster structures. One can construct two seed patterns whose sets of exchange matrices are disjoint from each other, yet the two rings generated by their respective sets of cluster variables are isomorphic. We will give concrete examples later on.

Remark 3.1.12. We will soon encounter many examples in which different vertices of the tree $\mathbb{T}_n$ correspond to identical labeled or unlabeled seeds. In spite of that, the set $\mathcal{X}$ of cluster variables will typically be infinite. Note that this does not preclude a cluster algebra $\mathcal{A}$ from being finitely generated (which is often the case). We shall also see that even when $\mathcal{X}$ is finite, the exchange relations (3.1) do not always generate the defining ideal of $\mathcal{A}$.

3.2. Examples of rank 1 and 2

The *rank* of a cluster algebra (or its underlying seed pattern) is the cardinality of each of its clusters (denoted above by n). In this section, we look at some examples of cluster algebras of small rank.

Rank 1. This case is very simple. The tree $\mathbb{T}_1$ has two vertices, so we only have two seeds, and two clusters (x_1) and (x_1'). The extended exchange matrix $\tilde{B}_\circ$ can be any $m \times 1$ matrix whose top entry is 0. The single exchange relation has the form $x_1\, x_1' = M_1 + M_2$ where M_1 and M_2 are monomials in the frozen variables $x_2, \dots, x_m$ which do not share a common factor x_i. The cluster algebra is generated by $x_1, x_1', x_2, \dots, x_m$, subject to this relation, and lies inside the ambient field $\mathcal{F} = \mathbb{C}(x_1, x_2, \dots, x_m)$.

Simple as they might be, cluster algebras of rank 1 do arise “in nature,” cf. Examples 1.1.2 and 1.1.3. We will discuss these and other examples in Chapter 7.

Rank 2. Any 2×2 skew-symmetrizable matrix looks like this:

$$\pm \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}, \tag{3.2}$$

for some integers b and c which are either both positive, or both equal to 0. Applying a mutation μ_1 or μ_2 to a matrix of the form (3.2) simply changes its sign.

Example 3.2.1. In the case $b = c = 0$, the two mutations commute, and the story reduces to the rank 1 case. We get four cluster variables x_1, x_2, x_1', x_2', two exchange relations of the form $x_1 x_1' = M_1 + M_2$ and $x_2 x_2' = M_3 + M_4$ where M_1, M_2, M_3, M_4 are monomials in the frozen variables, and four clusters (x_1, x_2), (x_1', x_2), (x_1, x_2'), and (x_1', x_2').

For the rest of this section, we assume that $b > 0$ and $c > 0$. We denote the cluster variables in our cluster algebra $\mathcal{A}$ of rank 2 by

$$\ldots, z_{-2}, z_{-1}, z_0, z_1, z_2, \ldots,$$

so that the seed pattern looks like this:

$$\cdots \overset{1}{\text{———}} \begin{array}{c}(z_1 \quad z_0)\\ \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix}\end{array} \overset{2}{\text{———}} \begin{array}{c}(z_1 \quad z_2)\\ \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}\end{array} \overset{1}{\text{———}} \begin{array}{c}(z_3 \quad z_2)\\ \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix}\end{array} \overset{2}{\text{———}} \begin{array}{c}(z_3 \quad z_4)\\ \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}\end{array} \overset{1}{\text{———}} \cdots$$

where we placed each cluster on top of the corresponding exchange matrix. (The extended exchange matrix may have additional rows.)

We denote by $\mathcal{A} = \mathcal{A}(b, c)$ a cluster algebra of rank 2 which has exchange matrices $\pm \left[\begin{smallmatrix} 0 & b \\ -c & 0 \end{smallmatrix}\right]$ and no frozen variables. (Cluster algebras without frozen variables are generally said to have *trivial coefficients*.) The exchange relations in $\mathcal{A}(b, c)$ are, in the notation introduced above:

$$z_{k-1}\, z_{k+1} = \begin{cases} z_k^c + 1 & \text{if } k \text{ is even;} \\ z_k^b + 1 & \text{if } k \text{ is odd.} \end{cases} \tag{3.3}$$

Example 3.2.2. The cluster variables in the cluster algebra $\mathcal{A}(1, 1)$ with trivial coefficients satisfy the recurrence

$$z_{k-1}\, z_{k+1} = z_k + 1. \tag{3.4}$$

Expressing everything in terms of the initial cluster (z_1, z_2), we get:

$$z_3 = \frac{z_2 + 1}{z_1}, \quad z_4 = \frac{z_1 + z_2 + 1}{z_1 z_2}, \quad z_5 = \frac{z_1 + 1}{z_2}, \quad z_6 = z_1, \quad z_7 = z_2, \ \ldots,$$

so the sequence is 5-periodic! Thus in this case, we have only 5 distinct cluster variables. In the seed pattern, we will have:

$$\cdots \overset{2}{\text{———}} \begin{array}{c}(z_1 \quad z_2)\\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\end{array} \overset{1}{\text{———}} \begin{array}{c}(z_3 \quad z_2)\\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\end{array} \overset{2}{\text{———}} \cdots \overset{1}{\text{———}} \begin{array}{c}(z_7 \quad z_6)\\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\end{array} \overset{2}{\text{———}} \cdots$$

Note that even though the labeled seeds containing the clusters (z_1, z_2) and (z_7, z_6) are different, the corresponding unlabeled seeds coincide. Just switch z_6 and z_7, and interchange the rows and the columns in the associated exchange matrix. Thus, this exchange pattern has 5 distinct (unlabeled) seeds.

Remark 3.2.3. The recurrence (3.4) arises in different mathematical contexts such as dilogarithm identities (cf., e.g., bibliographical pointers in [**19**, Section 1.1]), the Napier-Gauss *Pentagramma Mirificum* (cf. [**8**] and [**11**, Section 12.7]) and Coxeter's frieze patterns [**10**].

Example 3.2.4. We now keep the same exchange matrices but introduce a single frozen variable y. Consider a seed pattern which looks like this:

$$\cdots \overset{\begin{matrix} z_1 & z_2 \end{matrix}}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ p & q \end{bmatrix}} \overset{1}{\text{---}} \overset{\begin{matrix} z_3 & z_2 \end{matrix}}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -p & p{+}q \end{bmatrix}} \overset{2}{\text{---}} \overset{\begin{matrix} z_3 & z_4 \end{matrix}}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ q & -p{-}q \end{bmatrix}} \overset{1}{\text{---}} \overset{\begin{matrix} z_5 & z_4 \end{matrix}}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -q & -p \end{bmatrix}} \overset{2}{\text{---}} \overset{\begin{matrix} z_5 & z_6 \end{matrix}}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -q & p \end{bmatrix}} \overset{1}{\text{---}} \overset{\begin{matrix} z_7 & z_6 \end{matrix}}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ q & p \end{bmatrix}} \cdots,$$

where p and q are nonnegative integers. Relabeling the rows and columns to keep the 2×2 exchange matrices invariant, we get

$$\cdots \overset{\begin{matrix} z_1 & z_2 \end{matrix}}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ p & q \end{bmatrix}} \text{---} \overset{\begin{matrix} z_2 & z_3 \end{matrix}}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ p{+}q & -p \end{bmatrix}} \text{---} \overset{\begin{matrix} z_3 & z_4 \end{matrix}}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ q & -p{-}q \end{bmatrix}} \text{---} \overset{\begin{matrix} z_4 & z_5 \end{matrix}}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -p & -q \end{bmatrix}} \text{---} \overset{\begin{matrix} z_5 & z_6 \end{matrix}}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -q & p \end{bmatrix}} \text{---} \overset{\begin{matrix} z_6 & z_7 \end{matrix}}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ p & q \end{bmatrix}} \cdots,$$

so the sequence of extended exchange matrices remains 5-periodic. We then compute the cluster variables:

$$z_3 = \frac{z_2 + y^p}{z_1}, \; z_4 = \frac{y^{p+q} z_1 + z_2 + y^p}{z_1 z_2}, \; z_5 = \frac{y^q z_1 + 1}{z_2}, \; z_6 = z_1, \; z_7 = z_2\,;$$

the 5-periodicity persists! Just as in the case of trivial coefficients, there are five distinct cluster variables overall, and five distinct unlabeled seeds.

The above computations were based on the assumption that both entries in the third row of the initial extended exchange matrix are nonnegative. In fact, this condition is not required for 5-periodicity. Note that we could start with an initial seed containing the cluster (z_i, z_{i+1}), for any $i \in \{1,2,3,4,5\}$, and we would get the same 5-periodic behaviour. Since any row vector has the form (p,q), $(p{+}q,-p)$, $(q,-p{-}q)$, $(-p,-q)$, or $(-q,p)$, for some $p,q \geq 0$ (see Figure 3.3), we conclude that any seed pattern with extended exchange matrices of the form $\pm \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \\ * & * \end{smallmatrix}\right]$ has exactly five seeds.

As we shall later see, the general case of a seed pattern with exchange matrices $\pm \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and an arbitrary number of frozen variables exhibits the same qualitative behaviour: there will still be five cluster variables and five seeds.

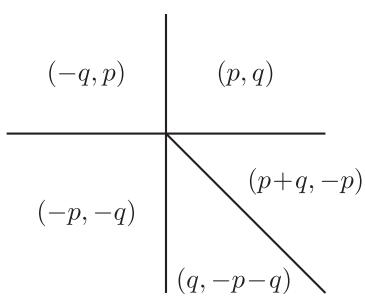


Figure 3.3. Five types of "frozen rows" in extended exchange matrices with top rows $(0,1)$ and $(-1,0)$. Within each of the five cones, the points are parameterized by $p,q \geq 0$.

Example 3.2.5. The cluster variables in the cluster algebra $\mathcal{A}(1,2)$ satisfy the recurrence

$$z_{k-1}\, z_{k+1} = \begin{cases} z_k^2+1 & \text{if } k \text{ is even;} \\ z_k+1 & \text{if } k \text{ is odd.} \end{cases}$$

Expressing everything in terms of the initial cluster (z_1, z_2), we get:

$$z_3 = \frac{z_2^2+1}{z_1}, \quad z_4 = \frac{z_2^2+z_1+1}{z_1 z_2}, \quad z_5 = \frac{z_1^2+z_2^2+2z_1+1}{z_1 z_2^2}, \quad z_6 = \frac{z_1+1}{z_2},$$

and then $z_7 = z_1$ and $z_8 = z_2$, so the sequence is 6-periodic! Thus in this case, we have only 6 distinct cluster variables, and 6 distinct seeds.

Exercise 3.2.6. Compute the cluster variables for the cluster algebra with the initial extended exchange matrix $\left[\begin{smallmatrix} 0 & 1 \\ -2 & 0 \\ p & q \end{smallmatrix}\right]$.

Exercise 3.2.7. Compute the cluster variables of the cluster algebra $\mathcal{A}(1,3)$. (Start by evaluating them in the specialization $z_1 = z_2 = 1$; notice that all the numbers will be integers.)

Example 3.2.8. Consider the cluster algebra $\mathcal{A}(1,4)$. Setting $z_1 = z_2 = 1$ and applying the recurrence (3.3), we see that the cluster variables $z_3, z_4, \dots$ specialize to the following values:

$$2, 3, 41, 14, 937, 67, 21506, 321, 493697, 1538, 11333521, 7369, 260177282, \dots$$

This does not look like a periodic sequence, so the (unspecialized) sequence of cluster variables is not periodic either!

The good news is that all these numbers are integers. Why does this happen? To understand this, let us recursively compute the cluster variables $z_3, z_4, \dots$ in terms of z_1 and z_2:

$$\begin{aligned}
z_3 &= \frac{z_2^4+1}{z_1}, \\
z_4 &= \frac{z_3+1}{z_2} = \frac{z_2^4+z_1+1}{z_1 z_2}, \\
z_5 &= \frac{z_4^4+1}{z_3} \\
&= \frac{z_2^{12}+4z_1z_2^8+3z_2^8+6z_1^2z_2^4+8z_1z_2^4+z_1^4+3z_2^4+4z_1^3+6z_1^2+4z_1+1}{z_1^3 z_2^4}, \\
z_6 &= \frac{z_5+1}{z_4} = \frac{z_2^8+3z_1z_2^4+2z_2^4+z_1^3+3z_1^2+3z_1+1}{z_1^2 z_2^3}, \text{ etc.}
\end{aligned}$$

Now we see what is going on: the evaluations of these expressions at $z_1 = z_2 = 1$ are integers because they are *Laurent polynomials* in z_1 and z_2, i.e., their denominators are monomials. (This is by no means to be expected: for example, the computation of z_6 involves dividing by $z_4 = \frac{z_2^4+z_1+1}{z_1 z_2}$.)

3.3. Laurent phenomenon

The examples of Laurentness that we have seen before are special cases of the following general phenomenon.

Theorem 3.3.1. *In a cluster algebra of geometric type, each cluster variable can be expressed as a Laurent polynomial with integer coefficients in the elements of any extended cluster.*

The rest of this section is devoted to the proof of Theorem 3.3.1. First, we state a simple auxiliary lemma which can be obtained by direct inspection of the exchange relations (3.1).

Lemma 3.3.2. *Let $\tilde{B}_\circ$ be an $m \times n$ extended exchange matrix. Let $\tilde{B}'_\circ$ be the matrix obtained from $\tilde{B}_\circ$ by deleting the rows labeled by a subset $I \subset \{n+1, \dots, m\}$. Then the formulas expressing the cluster variables in a cluster algebra $\mathcal{A}(\tilde{\mathbf{x}}'_\circ, \tilde{B}'_\circ)$ in terms of the initial extended cluster $\tilde{\mathbf{x}}'_\circ$ can be obtained from their counterparts for $\mathcal{A}(\tilde{\mathbf{x}}_\circ, \tilde{B}_\circ)$ by specializing the frozen variables x_i $(i \in I)$ to 1, and relabeling the remaining variables accordingly.*

Remark 3.3.3. A specialization of the kind described in Lemma 3.3.2 sends Laurent polynomials to Laurent polynomials. This means that if we add extra frozen variables to the initial seed and establish Laurentness of some distant cluster variable in this modified setting, it would then imply its Laurentness for the original seed pattern.

Let us set up the notation needed for the proof of Theorem 3.3.1:

- $t_\circ \in \mathbb{T}_n$ is an (arbitrarily chosen) initial vertex;
- $(\tilde{\mathbf{x}}_\circ, \tilde{B}_\circ)$ is the initial seed;
- $\tilde{\mathbf{x}}_\circ = (x_1, \dots, x_m)$ is the initial extended cluster;
- $\tilde{B}_\circ = (b_{ij})$ is the initial $m \times n$ extended exchange matrix;
- $t \in \mathbb{T}_n$ is an arbitrary vertex;
- $x \in \mathbf{x}(t)$ is a cluster variable at t;
- $t_\circ \overset{j}{\text{---}} t_1 \overset{k}{\text{---}} t_2 - \cdots - t$ is the unique path in $\mathbb{T}_n$ connecting $t_\circ$ to t;
- d is the length of this path, i.e., the distance in $\mathbb{T}_n$ between $t_\circ$ and t;
- $\tilde{\mathbf{x}}(t_1) = (\tilde{\mathbf{x}}(t_\circ) - \{x_j\}) \cup \{x'_j\}$;
- $\tilde{\mathbf{x}}(t_2) = (\tilde{\mathbf{x}}(t_1) - \{x_k\}) \cup \{x'_k\}$.

We will prove the Laurentness of x, viewed as a function of $\mathbf{x}_\circ$, by induction on d. (More precisely, the statement we prove by induction concerns arbitrary seeds at distance d from each other in arbitrary cluster algebras of geometric type.) The base cases $d = 1$ and $d = 2$ are trivial.

There are two possibilities to consider.

Case 1: $b_{jk} = b_{kj} = 0$. Let t_3 be the vertex in $\mathbb{T}_n$ connected to $t_\circ$ by an edge labeled k. Since μ_j and μ_k commute at $t_\circ$ (cf. Exercise 2.7.7(5)), each of the two seeds attached to t_1 and t_3, respectively, lies at distance $d-1$ from a seed containing x, and $\tilde{\mathbf{x}}(t_3) = (\tilde{\mathbf{x}}(t_\circ) - \{x_k\}) \cup \{x'_k\}$.

By the induction assumption, the cluster variable x is expressed as a Laurent polynomial in terms of the extended cluster $\tilde{\mathbf{x}}(t_1) = (x_1, \dots, x'_j, \dots, x_m)$. Also, $x'_j = \frac{M_1+M_2}{x_j}$, where M_1 and M_2 are monomials in $x_1, \dots, x_m$. Substituting this into the aforementioned Laurent polynomial, we obtain a formula expressing x in terms of $\tilde{\mathbf{x}}_\circ$. Another such formula is obtained by taking the Laurent polynomial expression for x in terms of $\tilde{\mathbf{x}}(t_3)$, and substituting $x'_k = \frac{M_3+M_4}{x_k}$, with M_3 and M_4 some monomials in $x_1, \dots, x_m$. Removing common factors, we obtain (necessarily identical) expressions for x as a ratio of coprime polynomials in $x_1, \dots, x_m$, with monic denominator.

Note that in the first computation, all non-monomial factors that can potentially remain in the denominator must come from $M_1 + M_2$; in the second one, they can only come from M_3+M_4. If M_1+M_2 and M_3+M_4 were coprime to each other, the Laurentness of x would follow. This coprimality however does not hold in general. (For example, if columns j and k of $\tilde{B}_\circ$ are equal to each other, then $M_1 + M_2 = M_3 + M_4$.) We can however use a trick based on Lemma 3.3.2, cf. Remark 3.3.3. Let us introduce a new frozen variable x_{m+1} and extend the matrix $\tilde{B}_\circ$ by an extra row in which the $(m+1, j)$-entry is 1, and all other entries are 0. Now $M_1 + M_2$ has become a binomial which has degree 1 in the variable x_{m+1}. Hence $M_1 + M_2$ is irreducible; moreover it cannot divide M_3+M_4 as the latter does not depend on x_{m+1}. So the argument goes through, and we are done with Case 1.

Case 2: $b_{jk}b_{kj} < 0$. This case is much harder. The general shape of the proof remains the same: we use induction on d together with a coprimality argument assisted by the introduction of additional frozen variables. One new aspect of the proof is that we need to separately consider the case $d = 3$ since the induction step relies on it.

Without loss of generality we assume that $b_{jk} < 0$ and $b_{kj} > 0$. Otherwise, change the signs of all extended exchange matrices; this will not affect the formulas relating extended clusters to each other, see Exercise 2.7.7(3).

We denote by $t_3 \in \mathbb{T}_n$ the vertex connected to t_2 by an edge labeled j, and introduce notation

$$\tilde{\mathbf{x}}(t_3) = (\tilde{\mathbf{x}}(t_2) - \{x'_j\}) \cup \{x''_j\} = (x_1, \dots, x''_j, \dots, x'_k, \dots, x_m).$$

(Whether $j < k$ or $k < j$ is immaterial.) See Figure 3.4.

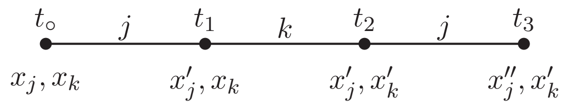


Figure 3.4. Cluster variables obtained via successive mutations μ_j, μ_k, μ_j.

Note that among cluster variables obtained by at most three mutations from the initial seed, those of the form x_j'' are the only ones whose Laurentness is not obvious.

Lemma 3.3.4. *The cluster variable x_j'' is a Laurent polynomial in $\tilde{\mathbf{x}}_\circ$.*

Proof. Let $\mu_j(\tilde{B}_\circ) = \tilde{B}(t_1) = (b_{ij}^{(1)})$ and $\mu_k(\tilde{B}(t_1)) = \tilde{B}(t_2) = (b_{ij}^{(2)})$ be the extended exchange matrices at t_1 and t_2, respectively. Our assumption $b_{jk} < 0$ implies that $b_{jk}^{(1)} > 0$ and $b_{kj}^{(2)} > 0$.

We will view each of the cluster variables x_j', x_k', x_j'' as a rational function in the elements $x_1, \dots, x_m$ of the initial extended cluster $\tilde{\mathbf{x}}_\circ$. The notation $P \sim Q$ will mean that P and Q differ by a monomial factor, i.e., $P = QM$ where M is a Laurent monomial in $x_1, \dots, x_m$.

The relevant instances of the exchange relation (3.1) imply that

$$x_j' \sim x_j^{-1}\left(\prod_i x_i^{b_{ij}^{(1)}} + 1\right), \tag{3.5}$$

$$x_k' = x_k^{-1}\left((x_j')^{b_{jk}^{(1)}} \prod_{\substack{b_{ik}^{(1)}>0 \\ i\neq j}} x_i^{b_{ik}^{(1)}} + \prod_{b_{ik}^{(1)}<0} x_i^{-b_{ik}^{(1)}}\right), \tag{3.6}$$

$$x_j'' \sim (x_j')^{-1}\left((x_k')^{b_{kj}^{(2)}} \prod_{i\neq k} x_i^{b_{ij}^{(2)}} + 1\right). \tag{3.7}$$

To establish that x_j'' is a Laurent polynomial in $x_1, \dots, x_m$, we need to show that the second factor in (3.7) is divisible by

$$P_j = P_j(x_1, \dots, x_m) = \prod_i x_i^{b_{ij}^{(1)}} + 1,$$

the numerator in (3.5). Using the notation $Q \equiv R$ to express the property that $Q - R = P_j S$ for some Laurent polynomial S, we then get, modulo P_j:

$$
\begin{aligned}
x_k' &\equiv x_k^{-1} \prod_{b_{ik}^{(1)}<0} x_i^{-b_{ik}^{(1)}},\\
(x_k')^{b_{kj}^{(2)}} \prod_{i\neq k} x_i^{b_{ij}^{(2)}} + 1 &\equiv \Bigg(x_k^{-1} \prod_{b_{ik}^{(1)}<0} x_i^{-b_{ik}^{(1)}}\Bigg)^{b_{kj}^{(2)}} \prod_{i\neq k} x_i^{b_{ij}^{(2)}} + 1\\
&= x_k^{b_{kj}^{(1)}} \prod_{b_{ik}^{(1)}<0} x_i^{b_{ik}^{(1)} b_{kj}^{(1)}} \prod_{i\neq k} x_i^{b_{ij}^{(2)}} + 1 = \prod_i x_i^{b_{ij}^{(1)}} + 1 \equiv 0,
\end{aligned}
$$

as desired. In the last line, we used the fact that $b_{kj}^{(1)} < 0$ and consequently

$$
b_{ij}^{(2)} = \begin{cases} b_{ij}^{(1)} - b_{ik}^{(1)} b_{kj}^{(1)} & \text{if } b_{ik}^{(1)} < 0;\\ b_{ij}^{(1)} & \text{if } b_{ik}^{(1)} \geq 0 \text{ and } i \neq k. \end{cases} \quad \square
$$

Lemma 3.3.5. *Suppose that distinct indices $q, r \in \{n+1, \dots, m\}$ are such that $b_{qj} = 1$ and $b_{rk} = 1$, and moreover all other entries in rows q and r of $\tilde{B}_\circ$ are equal to* 0. *Then x_j' is coprime to both x_k' and x_j''.*

Here "coprime" means that those cluster variables, viewed as Laurent polynomials in $\mathbf{x}_\circ$, have no common non-monomial factor.

Proof. Let us denote $b_{jk} = -b$ and $b_{kj} = c$. Recall that $b_{kj} > 0$, so $b, c > 0$. The local structure of the extended exchange matrices at $t_\circ$, t_1, and t_2 at the intersections of rows j, k, r, s and columns j, k is as follows:

$$
\begin{array}{c|ccccc}
 & & j & k & & \\
\hline
 & & \vdots & \vdots & & \\
j & \cdots & 0 & -b & \cdots \\
k & \cdots & c & 0 & \cdots \\
 & & \vdots & \vdots & & \\
q & \cdots & 1 & 0 & \cdots \\
r & \cdots & 0 & 1 & \cdots
\end{array}
\qquad
\begin{array}{c|ccccc}
 & & j & k & & \\
\hline
 & & \vdots & \vdots & & \\
j & \cdots & 0 & b & \cdots \\
k & \cdots & -c & 0 & \cdots \\
 & & \vdots & \vdots & & \\
q & \cdots & -1 & 0 & \cdots \\
r & \cdots & 0 & 1 & \cdots
\end{array}
\qquad
\begin{array}{c|ccccc}
 & & j & k & & \\
\hline
 & & \vdots & \vdots & & \\
j & \cdots & 0 & -b & \cdots \\
k & \cdots & c & 0 & \cdots \\
 & & \vdots & \vdots & & \\
q & \cdots & -1 & 0 & \cdots \\
r & \cdots & 0 & -1 & \cdots
\end{array}
$$

$$
\tilde{B}_\circ \qquad\qquad \mu_j(\tilde{B}) \qquad\qquad \mu_k(\mu_j(\tilde{B}))
$$

We then have

$$
\begin{aligned}
x_j' &= x_j^{-1}(x_k^c\, x_q M_1 + M_2)\\
x_k' &= x_k^{-1}((x_j')^b\, x_r M_3 + M_4)\\
x_j'' &= (x_j')^{-1}(x_q M_5 + (x_k')^c M_6),
\end{aligned}
$$

where $M_1, \dots, M_6$ are monomials in the x_i's, with $i \notin \{j,k,q,r\}$. We see that x'_j is linear in x_q and hence irreducible (as a Laurent polynomial in $\tilde{\mathbf{x}}$), i.e., it cannot be written as a product of two non-monomial factors. Since x'_j does not depend on x_r, we conclude that x'_k is linear in x_r and hence irreducible, and moreover coprime with x'_j.

It remains to show that x'_j and x''_j are coprime. Note that x'_k and x''_j can be regarded as polynomials in x_r; we denote by $x'_k(0)$ and $x''_j(0)$ their specializations at $x_r = 0$. If we show that $x''_j(0)$ is coprime to $x'_j = x'_j(0)$, then we'll be done. To this end, note that $x'_k(0) = x_k^{-1} M_4$. Therefore

$$x''_j(0) = x_j \, \frac{x_q M_5 + (x_k^{-1} M_4)^c M_6}{x_k^c \, x_q M_1 + M_2} \, .$$

Here both the numerator and denominator are linear in x_q, and therefore the denominator (essentially, x'_j) cannot divide the numerator more than once. Also, x'_j is irreducible. Hence $x''_j(0)$ and x'_j are coprime, as desired. □

We are now ready to complete the proof of Case 2 of Theorem 3.3.1. We begin by augmenting the initial extended exchange matrix $\tilde{B}_\circ$ by two additional rows corresponding to two new frozen variables x_q and x_r. We set

$$b_{qi} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j; \end{cases} \qquad b_{ri} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases}$$

so as to satisfy the conditions of Lemma 3.3.5.

By the induction assumption, the cluster variable x is expressed as a Laurent polynomial in terms of each of the extended clusters $\tilde{\mathbf{x}}(t_1)$ and $\tilde{\mathbf{x}}(t_3)$. The only elements of these clusters which do not appear in $\tilde{\mathbf{x}}_\circ$ are x'_j, x'_k, and x''_j, so

$$x = \frac{\text{Laurent polynomial in } \tilde{\mathbf{x}}_\circ}{(x'_j)^a} = \frac{\text{Laurent polynomial in } \tilde{\mathbf{x}}_\circ}{(x'_k)^b (x''_j)^c} \, ,$$

for some $a, b, c \in \mathbb{Z}$. By Lemma 3.3.4, x'_j, x'_k, and x''_j are Laurent polynomials in $\tilde{\mathbf{x}}_\circ$. By Lemma 3.3.5, x'_j is coprime to both x'_k and x''_j. The theorem now follows by the same argument (based on Lemma 3.3.2) as the one used in Case 1. □

Theorem 3.3.1 can be sharpened as follows.

Theorem 3.3.6. *In a cluster algebra of geometric type, frozen variables do not appear in the denominators of the Laurent polynomials expressing cluster variables in terms of an initial extended cluster.*

In our standard notation, each cluster variable is a Laurent polynomial in the initial cluster variables $x_1, \dots, x_n$, with coefficients in $\mathbb{Z}[x_{n+1}, \dots, x_m]$.

Proof. We borrow the notation from the proof of Theorem 3.3.1 above. Let x be a cluster variable from a distant seed, and x_r a frozen variable ($n < r \le m$). We will think of x as a Laurent polynomial $x(x_r)$ whose coefficients are integral Laurent polynomials in the variables x_i, with $i \neq r$. We want to show that x is in fact a polynomial in x_r; Theorem 3.3.6 will then follow by varying r.

We will make use of the following trivial lemma.

Lemma 3.3.7. *Let P and Q be two polynomials (in any number of variables) with coefficients in a domain S, and with nonzero constant terms a and b, respectively. If the ratio P/Q is a Laurent polynomial over S, then it is in fact a polynomial over S with the constant term a/b.*

Our proof of Theorem 3.3.6 proceeds by induction on d, the smallest distance in $\mathbb{T}_n$ between the initial seed and a seed containing x. We will inductively prove the following strengthening of the desired statement:

> *$x(x_r)$ is a polynomial in x_r whose constant term $x(0)$ can be written as a subtraction-free rational expression in the elements of $\tilde{\mathbf{x}}_\circ - \{x_r\}$; in particular, $x(0) \neq 0$.*

If $d = 0$, then $x \in \tilde{\mathbf{x}}_\circ$, and there is nothing to prove. If $d > 0$, then x appears on the left-hand side of an exchange relation (3.1) in which all other cluster variables involved come from a seed located at distance $d-1$ from $\tilde{\mathbf{x}}_\circ$. Applying the inductive assumption to all these cluster variables and using Lemma 3.3.7 together with the fact that x_r contributes to at most one of the monomials on the right-hand side of this exchange relation, we obtain our claim for x. □

3.4. Connections to number theory

Example 3.4.1 (*Markov triples*)**.** Consider the cluster algebra defined by the Markov quiver given in Figure 2.9. Since the quiver is invariant under mutations, exchange relations for any cluster (x_1, x_2, x_3) will look the same:

$$\begin{aligned} x_1'\, x_1 &= x_2^2 + x_3^2, \\ x_2'\, x_2 &= x_1^2 + x_3^2, \\ x_3'\, x_3 &= x_1^2 + x_2^2. \end{aligned}$$

If we start with a triple $(1, 1, 1)$ and mutate in all possible directions, we will get an infinite set of triples in $\mathbb{Z}^3$, including those shown in Figure 3.5.

We next observe that all these triples satisfy the diophantine *Markov equation*

$$x_1^2 + x_2^2 + x_3^2 = 3x_1x_2x_3.$$

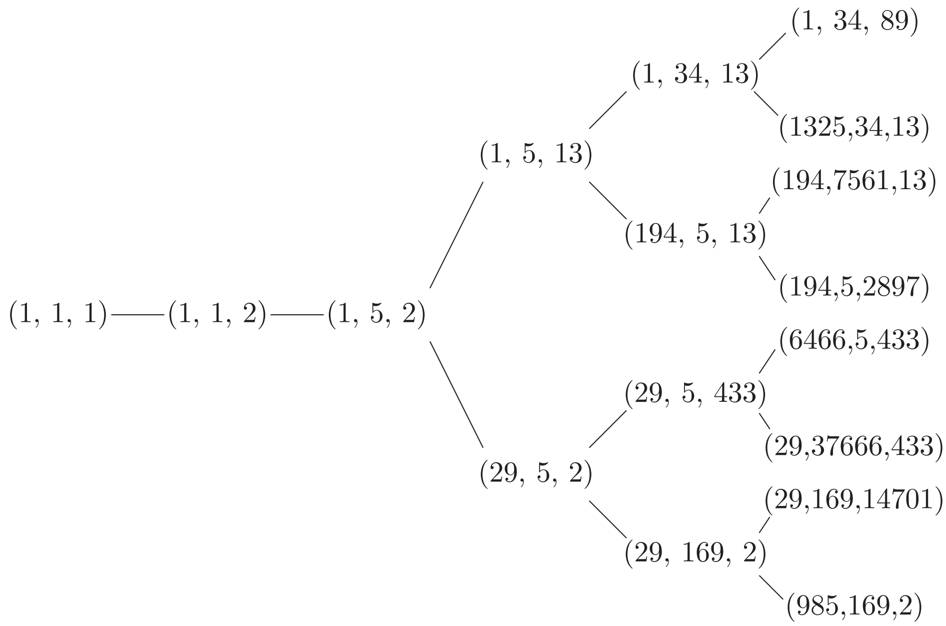


Figure 3.5. Markov triples.

To see this, verify that each mutation in our cluster algebra transforms a solution of this equation (a *Markov triple*) into another solution. (This is an instance of *Vieta jumping*, which replaces one root of a quadratic equation by another root.) Starting with the solution $(1, 1, 1)$, we get the tree of Markov triples above. In fact, every Markov triple appears in this tree. The celebrated (still open) Uniqueness Conjecture asserts that the maximal elements of Markov triples are all distinct. See [**1**] for a detailed account.

Returning to the cluster-algebraic interpretation of this example, we note that more generally, the quantity

$$\frac{x_1^2 + x_2^2 + x_3^2}{x_1 x_2 x_3}$$

is invariant under mutations in this seed pattern. This is closely related to *integrability* of the Markov recurrence; cf. Chapter 10.

Example 3.4.2 (*Fermat numbers*)**.** Sometime in the 1640's, Pierre Fermat conjectured that for every positive integer n, the number $F_n = 2^{2^n} + 1$ is prime. This was disproved in 1732 by Leonhard Euler, who discovered that

$$F_5 = 2^{32} + 1 = 641 \cdot 6700417.$$

Curiously, this factorization can be obtained by observing cluster mutations.

Consider the rank 2 cluster algebra with the initial seed $(\tilde{\mathbf{x}}, \tilde{B})$ where

$$\tilde{\mathbf{x}} = (x_1, x_2, x_3), \quad \tilde{B} = \begin{bmatrix} 0 & 4 \\ -1 & 0 \\ 1 & -3 \end{bmatrix}.$$

The mutation μ_1 produces a new extended cluster $\mathbf{x}' = (x_1', x_2, x_3)$ where

$$x_1' \, x_1 = x_2 + x_3 \,.$$

Taking the specialization

$$(x_1, x_2, x_3) = (3, -1, 16),$$

we see that the mutated extended cluster specializes to

$$(x_1', x_2, x_3) = (5, -1, 16).$$

Applying the sharp version of the Laurent phenomenon (Theorems 3.3.1 and 3.3.6) to the initial extended cluster $\mathbf{x}$, we see that every cluster variable specializes to an integer (possibly) divided by a power of 3; applying the same result to $\mathbf{x}'$, we conclude that every cluster variable is an integer (possibly) divided by a power of 5. Thus, every cluster variable specializes to an integer! Now let us see which integers we get. Alternately applying the mutations μ_1 and μ_2, we obtain the following sequence, with specialized cluster variables written on top of the extended exchange matrices:

$$\begin{array}{rr} 3 & -1 \\ \hline 0 & 4 \\ -1 & 0 \\ 1 & -3 \end{array} \xrightarrow{\mu_1} \begin{array}{rr} 5 & -1 \\ \hline 0 & -4 \\ 1 & 0 \\ -1 & 1 \end{array} \xrightarrow{\mu_2} \begin{array}{rr} 5 & -641 \\ \hline 0 & 4 \\ -1 & 0 \\ 0 & -1 \end{array}$$

$$\xrightarrow{\mu_1} \begin{array}{rr} -128 & -641 \\ \hline 0 & -4 \\ 1 & 0 \\ 0 & -1 \end{array} \xrightarrow{\mu_2} \begin{array}{rr} -128 & \frac{-F_5}{641} \\ \hline 0 & 4 \\ -1 & 0 \\ 0 & 1 \end{array}.$$

This shows that $F_5/641$ is an integer, reproducing Euler's discovery.

Example 3.4.3. The *Somos-4 sequence* $z_0, z_1, z_2, \ldots$ is defined by the initial conditions $z_0 = z_1 = z_2 = z_3 = 1$ and the recurrence

$$z_{m+2} z_{m-2} = z_{m+1} z_{m-1} + z_m^2 \,.$$

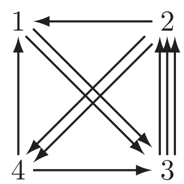
This sequence is named after M. Somos who discovered it (and its various generalizations) sometime in the 1980s; see, e.g., [**6, 31**] and references therein.

The first several terms of the Somos-4 sequence are

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, 620297, 7869898, \ldots$$

—all integers! An explanation of the integrality of this sequence can be given using cluster algebras.

Consider the quiver



(all four vertices are mutable). It is easy to check that mutating at the vertex labeled 1 produces a quiver that differs from the original one by clockwise rotation by $\pi/2$. It follows that subsequent quiver mutations at $2, 3, 4, 1, 2, 3, 4, 1, \ldots$ will generate a sequence of cluster variables satisfying the Somos-4 recurrence above. In view of the Laurent phenomenon, the initial conditions $z_0 = z_1 = z_2 = z_3 = 1$ will result in a sequence of integers.

Remark 3.4.4. An alternative approach to establishing integrality of the Somos-4 and other related sequences is based on explicit combinatorial interpretations of their terms. In particular, the numbers z_m defined above can be shown to count *perfect matchings* in certain planar bipartite graphs, see [**48**] and Figure 3.6.

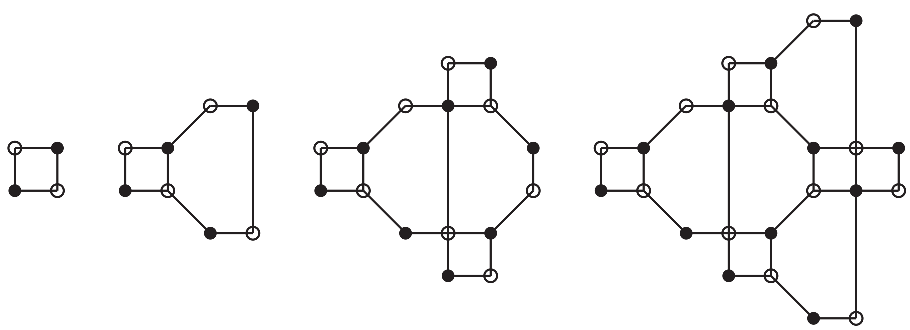


Figure 3.6. The number of perfect matchings in each of these bipartite graphs is 2, 3, 7, 23, respectively (cf. the Somos-4 sequence).

Remark 3.4.5. Somos sequences and their various generalizations are intimately related to the arithmetic of elliptic curves; see, e.g., [**32**] and references therein. Here is a typical result, stated here without proof (this version is due to D. Speyer). Consider the elliptic curve

$$y^2 = 1 - 8x + 12x^2 - 4x^3,$$

and let x_m be the x-coordinate of the point $P + mQ$ where $P = (0, 1)$, $Q = (1, -1)$, and we are using the standard group law of the elliptic curve.

Then

$$x_m = \frac{z_{m-1}z_{m+1}}{z_m^2}$$

where (z_m) is the Somos-4 sequence above.

Exercise 3.4.6. Use the Laurent phenomenon to show that the sequence $z_0, z_1, z_2, \ldots$ defined by the initial conditions $z_0 = z_1 = z_2 = 1$ and the recurrence

$$z_{m+3}z_m = z_{m+2}z_{m+1} + 1$$

consists entirely of integers.

Exercise 3.4.7. Show that all elements of the sequence $z_0, z_1, z_2, \ldots$ defined by the generalized Somos-4 recurrence

$$z_{m+2}z_{m-2} = az_{m+1}z_{m-1} + bz_m^2$$

are Laurent polynomials in z_0, z_1, z_2, z_3, with coefficients in $\mathbb{Z}[a, b]$.

Example 3.4.8. The *Somos-5* sequence

$$1, 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, 1217, 6161, 22833, 165713, \ldots$$

is defined by the recurrence relation

$$z_m z_{m+5} = z_{m+1}z_{m+4} + z_{m+2}z_{m+3} \quad (m = 1, 2, \ldots) \tag{3.8}$$

with the initial conditions $z_1 = \cdots = z_5 = 1$. *A priori*, one expects the numbers z_m to be rational—but in fact, all of them are integers. Once again, this is a consequence of a stronger statement: viewed as a function of $z_1, \ldots, z_5$, every z_m is a Laurent polynomial with integer coefficients. To prove this, we need to find a cluster algebra with an initial cluster $(z_1, \ldots, z_5)$ (no frozen variables) which has all relations (3.8) among its exchange relations, so that all the z_m are among its cluster variables.

Exercise 3.4.9. Establish the integrality of all terms of the Somos-5 sequence by examining the sequence of mutations

$$\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \ldots,$$

in the cluster algebra with the initial exchange matrix

$$B = \tilde{B} = \begin{bmatrix} 0 & -1 & 1 & 1 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ -1 & 2 & 0 & -2 & 1 \\ -1 & 0 & 2 & 0 & -1 \\ 1 & -1 & -1 & 1 & 0 \end{bmatrix}. \tag{3.9}$$

Example 3.4.10 (see [**26**])**.** Fix a positive integer n, and let $(a_1, \dots, a_{n-1})$ be a *palindromic* integer vector, that is, $a_i = a_{n-i}$ for $i = 1, \dots, n-1$. Consider the sequence $z_1, z_2, \dots$ given by the recurrence

$$z_m z_{m+n} = \prod_{i=1}^{n-1} z_{m+i}^{[a_i]_+} + \prod_{i=1}^{n-1} z_{m+i}^{[-a_i]_+} \quad (m = 1, 2, \dots) \tag{3.10}$$

with indeterminates $z_1, \dots, z_n$ as the initial terms; here we use the notation

$$[a]_+ = \max(a, 0). \tag{3.11}$$

(The recurrence (3.8) is a special case with $n = 5$ and $(a_1, \dots, a_{n-1}) = (1, -1, -1, 1)$.) Then all the terms z_m are integer Laurent polynomials in $z_1, \dots, z_n$. To show this, we find an $n \times n$ skew-symmetric integer matrix $B = (b_{ij})$ such that $\mu_1(B)$ is obtained from B by the cyclic permutation of its rows and columns, and such that its first column is given by $b_{i1} = a_{i-1}$ for $i = 2, \dots, n$. It is not hard to see that setting

$$-b_{ji} = b_{ij} = a_{i-j} + \sum_{k=1}^{j-1} ([-a_{i-k}]_+ [a_{j-k}]_+ \ - \ [a_{i-k}]_+ [-a_{j-k}]_+)$$

for $1 \le j < i \le n$ works as desired.

3.5. Y-patterns

We keep the notational conventions used in Section 3.3.

One of our goals is to show that many structural properties of a seed pattern are determined by (the mutation class formed by) its $n \times n$ exchange matrices $B(t)$, and do not depend on the bottom parts of the matrices $\tilde{B}(t)$.

Theorem 3.5.1 below concerns certain Laurent monomials in the elements of a given seed; each of these Laurent monomials is simply a ratio of the two terms appearing on the right-hand side of an exchange relation (3.1). Surprisingly, the evolution of these ratios is completely controlled by the matrices $B(t)$. That is, the laws governing this evolution do not depend on the bottom parts of the matrices $\tilde{B}(t)$.

Theorem 3.5.1. *Let $(\tilde{\mathbf{x}}, \tilde{B})$ and $(\tilde{\mathbf{x}}', \tilde{B}')$ be two labeled seeds related by mutation at k, with extended clusters*

$$\tilde{\mathbf{x}} = (x_1, \dots, x_m), \quad \tilde{\mathbf{x}}' = (x'_1, \dots, x'_m)$$

and $m \times n$ extended exchange matrices

$$\tilde{B} = (b_{ij}), \quad \tilde{B}' = (b'_{ij}).$$

Define the n-tuples $\hat{\mathbf{y}} = (\hat{y}_1, \ldots, \hat{y}_n)$ and $\hat{\mathbf{y}}' = (\hat{y}'_1, \ldots, \hat{y}'_n)$ by

$$\hat{y}_j = \prod_{i=1}^{m} x_i^{b_{ij}}, \quad \hat{y}'_j = \prod_{i=1}^{m} (x'_i)^{b'_{ij}}. \tag{3.12}$$

Then

$$\hat{y}'_j = \begin{cases} \hat{y}_k^{-1} & \text{if } j = k; \\ \hat{y}_j\,(\hat{y}_k + 1)^{-b_{kj}} & \text{if } j \neq k \text{ and } b_{kj} \leq 0; \\ \hat{y}_j\,(\hat{y}_k^{-1} + 1)^{-b_{kj}} & \text{if } j \neq k \text{ and } b_{kj} \geq 0. \end{cases} \tag{3.13}$$

Proof. We check (3.13) case by case. The case $j = k$ is easy:

$$\hat{y}'_k = \prod_i (x'_i)^{b'_{ik}} = \prod_{i \neq k} x_i^{b'_{ik}} = \prod_{i \neq k} x_i^{-b_{ik}} = \hat{y}_k^{-1}.$$

If $j \neq k$ and $b_{kj} \leq 0$, then

$$\begin{aligned} \hat{y}'_j =& (x'_k)^{b'_{kj}} \prod_{i \neq k} x_i^{b'_{ij}} = (x'_k)^{-b_{kj}} \prod_{i \neq k} x_i^{b_{ij}} \prod_{b_{ik}<0} x_i^{-b_{ik}b_{kj}} \\ =& x_k^{b_{kj}} \Bigg(\prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}} \Bigg)^{-b_{kj}} \prod_{i \neq k} x_i^{b_{ij}} \prod_{b_{ik}<0} x_i^{-b_{ik}b_{kj}} \\ =& \hat{y}_j (\hat{y}_k + 1)^{-b_{kj}}\,. \end{aligned}$$

If $j \neq k$ and $b_{kj} \geq 0$, then

$$\begin{aligned} \hat{y}'_j =& (x'_k)^{b'_{kj}} \prod_{i \neq k} x_i^{b'_{ij}} = (x'_k)^{-b_{kj}} \prod_{i \neq k} x_i^{b_{ij}} \prod_{b_{ik}>0} x_i^{b_{ik}b_{kj}} \\ =& x_k^{b_{kj}} \Bigg(\prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}} \Bigg)^{-b_{kj}} \prod_{i \neq k} x_i^{b_{ij}} \prod_{b_{ik}>0} x_i^{b_{ik}b_{kj}} \\ =& \hat{y}_j (1 + \hat{y}_k^{-1})^{-b_{kj}}\,. \end{aligned}$$

□

Theorem 3.5.1 suggests the following definitions.

Definition 3.5.2. A Y*-seed* of rank n in a field $\mathcal{F}$ is a pair (Y, B) where

- Y is an n-tuple of elements of $\mathcal{F}$;
- B is a skew-symmetrizable $n \times n$ integer matrix.

We say that two Y-seeds (Y, B) and (Y', B') of rank n are related by a Y*-seed mutation* μ_k in direction k (here $1 \leq k \leq n$) if

- the matrices $B = (b_{ij})$ and $B' = (b'_{ij})$ are related via mutation at k;

- the n-tuple $Y'=(Y'_1,\dots,Y'_n)$ is obtained from $Y=(Y_1,\dots,Y_n)$ by

$$Y'_j = \begin{cases} Y_k^{-1} & \text{if } j=k;\\ Y_j\,(Y_k+1)^{-b_{kj}} & \text{if } j\neq k \text{ and } b_{kj}\le 0;\\ Y_j\,(Y_k^{-1}+1)^{-b_{kj}} & \text{if } j\neq k \text{ and } b_{kj}\ge 0. \end{cases} \tag{3.14}$$

It is easy to check that mutating (Y',B') at k recovers (Y,B).

A *Y-pattern* of rank n is a collection of Y-seeds $(Y(t),B(t))_{t\in\mathbb{T}_n}$ labeled by the vertices of the n-regular tree $\mathbb{T}_n$, such that for any edge $t \overset{k}{\text{———}} t'$ in $\mathbb{T}_n$, the Y-seeds $(Y(t),B(t))$ and $(Y(t'),B(t'))$ are related to each other by the Y-seed mutation in direction k.

Remark 3.5.3. In Definition 3.5.2, we do not require the elements Y_i to be algebraically independent, one reason being that this condition does not always hold for the monomials $\hat{y}_j$ in Theorem 3.5.1. Consequently, one can not *a priori* guarantee that the mutation process can propagate to all vertices in $\mathbb{T}_n$ (what if $Y_k=0$ in (3.14)?). To ensure the existence of a Y-pattern with a given initial seed (Y,B), one can for example require all elements of Y to be given by subtraction-free expressions in some set of variables, or alternatively take positive values under a particular specialization of these variables. As each of these conditions reproduces under mutations of Y-seeds, the mutation process can then proceed without hindrance.

Using the terminology introduced in Definition 3.5.2, we can state the following direct corollary of Theorem 3.5.1.

Corollary 3.5.4. *Let $(\tilde{\mathbf{x}}(t),\tilde{B}(t))_{t\in\mathbb{T}_n}$ be a seed pattern in $\mathcal{F}$, with*

$$\tilde{\mathbf{x}}(t)=(x_{1;t},\dots,x_{m;t}),\quad \tilde{B}(t)=(b^t_{ij}).$$

Let $B(t)=(b^t_{ij})_{i,j\le n}$ denote the exchange matrix at a vertex $t\in\mathbb{T}_n$, and let $\hat{\mathbf{y}}(t)=(\hat{y}_{1;t},\dots,\hat{y}_{n;t})$ be the n-tuple of elements in $\mathcal{F}$ given by

$$\hat{y}_{k;t}=\prod_{i=1}^{m} x_{i;t}^{b^t_{ik}}\,. \tag{3.15}$$

Then $(\hat{\mathbf{y}}(t),B(t))_{t\in\mathbb{T}_n}$ is a Y-pattern in $\mathcal{F}$.

Remark 3.5.5. The rules governing the evolution of Y-seeds may seem simpler than the corresponding rules of seed mutation:

- Y-seed mutations are driven by the $n\times n$ matrices B whereas ordinary seed mutations require the extended $m\times n$ matrices $\tilde{B}$;
- in the Y-seed setting, there are no frozen variables;

- each recurrence (3.14) only involves two variables Y_j and Y_k whereas the exchange relation (3.1) potentially involves all cluster variables of the current seed.

On the other hand,

- a seed mutation only changes one cluster variable whereas a Y-seed mutation may potentially change all the variables $Y_1, \dots, Y_n$;
- consequently, we end up getting "more" Y-variables than cluster variables (if the number of seeds is finite, then this is a precise statement);
- the Y-pattern recurrences do not, generally speaking, exhibit the Laurent phenomenon.

Remark 3.5.6. In many applications, the cluster algebra under investigation has a distinguished (multi-)grading, and its exchange relations are all (multi-)homogeneous. It follows that the rational expressions $\hat{y}_{k;t}$ defined by (3.15) have (multi-)degree 0. It is not surprising, then, that Y-patterns naturally arise in the study of configurations (of points, flags, etc.) in projective spaces. See Examples 3.5.7 and 3.5.9 below.

Example 3.5.7 (*Configurations of points on the projective line*)**.** The *cross-ratio* is a quantity associated with an ordered quadruple of collinear points, particularly points on the projective line $\mathbb{P}^1$ (say over $\mathbb{C}$). For our purposes, it will be convenient to use the following version of the cross-ratio. Let $P_1, P_2, P_3, P_4 \in \mathbb{P}^1$ be four distinct points on the projective line, with projective coordinates $(a_1 : b_1)$, $(a_2 : b_2)$, $(a_3 : b_3)$, and $(a_4 : b_4)$, respectively. We then define

$$Y(P_1, P_2, P_3, P_4) = \frac{P_{14}\, P_{23}}{P_{12}\, P_{34}}, \tag{3.16}$$

where we use the notation

$$P_{ij} = \det \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix} = a_i b_j - a_j b_i \,. \tag{3.17}$$

This quantity is related to the conventional cross ratio via

$$Y(P_1, P_2, P_3, P_4) = -(P_1, P_3; P_4, P_2).$$

The cross-ratio is essentially the only projective invariant of a quadruple of points. The action of the symmetric group $\mathcal{S}_4$ permuting the points in a generic quadruple has the Klein four-group as its kernel; it produces six different cross-ratios all of which are uniquely determined by any one of them.

It is well known (see, e.g., [**43**, Section 7.4]) that any rational function of an ordered m-tuple of points on the projective line which is invariant under projective transformations can be expressed in terms of cross-ratios associated to various quadruples of points. In fact, one only needs cross-ratios associated with $m-3$ quadruples to get all $\binom{m}{4}$ of them. One way to make this explicit is by using the machinery of Y-patterns. A configuration of m distinct ordered points $P_1, \dots, P_m \in \mathbb{P}^1$ with projective coordinates $(a_1 : b_1), \dots, (a_m : b_m)$ can be encoded by a $2 \times m$ matrix

$$z = \begin{bmatrix} a_1 & a_2 & \dots & a_m \\ b_1 & b_2 & \dots & b_m \end{bmatrix}.$$

Recall that the Plücker coordinates $P_{ij} = P_{ij}(z)$ are defined by the formula (3.17), for $1 \le i < j \le m$.

We now associate a Y-seed to an arbitrary triangulation T of a convex m-gon $\mathbf{P}_m$ (cf. Sections 1.2 and 2.2) by $m-3$ pairwise noncrossing diagonals. Recall that $\mathbf{P}_m$ has m vertices labeled $1, \dots, m$, in clockwise order. We label the diagonals of T by the numbers $1, \dots, m-3$, and define the exchange matrix B_T to be the $(m-3) \times (m-3)$ matrix associated to the mutable part of the quiver $Q(T)$, see Definition 2.2.1. (Ignore the frozen vertices associated with the sides of the polygon.) Consider a diagonal of T labeled d. This diagonal triangulates a quadrilateral with vertices labeled i, j, k, ℓ in clockwise order, connecting vertices i and k, cf. Figure 1.2. Define $Y_d = Y(P_i, P_j, P_k, P_\ell)$, cf. (3.16). Note that since $Y(P_i, P_j, P_k, P_\ell) = Y(P_k, P_\ell, P_i, P_j)$, there is no ambiguity in this definition. Finally, define the Y-seed associated with T to be the pair (Y_T, B_T), where $Y_T = (Y_1, \dots, Y_{m-3})$.

It is now an exercise to verify that these Y-seeds transform under flips by the Y-seed mutation rule (3.14). Note that this example is nothing but the application of the construction in Theorem 3.5.1 to the seed pattern from the $\mathrm{Gr}(2, m)$ example!

Exercise 3.5.8. Given six points $P_1, \dots, P_6$ on the projective line, express the cross-ratios for the quadruples $\{P_i, P_4, P_5, P_6\}$ in terms of the cross-ratios for the quadruples $\{P_i, P_1, P_2, P_3\}$.

Example 3.5.9 (*The pentagram map*)**.** The *pentagram map*, introduced in [**46**], is a transformation of generic projective polygons (i.e., cyclically ordered tuples of points on the projective plane $\mathbb{P}^2$) defined by the following construction: given a polygon A as input, draw all of its "shortest" diagonals, and output the "smaller" polygon A' which they cut out. See Figure 3.7.

As shown in [**29**], the pentagram map is related to Y-seed mutation. To explain the connection, one needs to describe the pentagram map in properly chosen coordinates. We shall view a polygon with n vertices as an n-periodic sequence $A = (A_i)_{i \in \mathbb{Z}}$ of points in $\mathbb{P}^2$. Given two polygons related by the pentagram map, it is convenient to index the points of one of them by the integers $\mathbb{Z}$ and the points of the other by the half-integers $\mathbb{Z} + \frac{1}{2}$, as shown at the right in Figure 3.7.

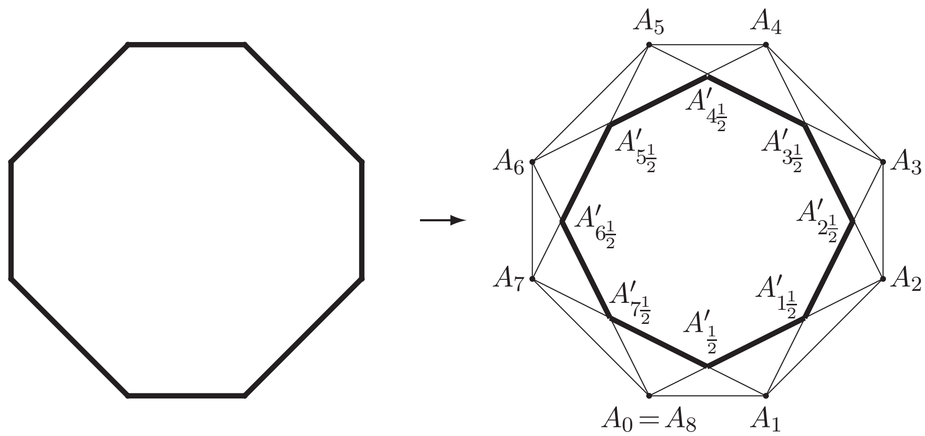


Figure 3.7. The pentagram map.

Recall the definition (3.16) of the projective invariant $Y(P_1, P_2, P_3, P_4)$ (a negative cross-ratio) associated with a quadruple of distinct collinear points P_1, P_2, P_3, P_4. One can associate a similar invariant to a quadruple of distinct concurrent lines L_1, L_2, L_3, L_4 in $\mathbb{P}^2$ passing through a point Q: any line L not passing through Q intersects these lines in four distinct points P_1, P_2, P_3, P_4, and the number $Y(L_1, L_2, L_3, L_4) = Y(P_1, P_2, P_3, P_4)$ does not depend on the choice of the line L.

Definition 3.5.10. Let A be a polygon indexed either by $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$. The *y-parameters of* A are the numbers $y_j(A)$ defined by

$$
\begin{aligned}
y_{2k}(A) &= Y(\overleftrightarrow{A_kA_{k-1}}, \overleftrightarrow{A_kA_{k+2}}, \overleftrightarrow{A_kA_{k+1}}, \overleftrightarrow{A_kA_{k-2}})^{-1} \\
y_{2k+1}(A) &= Y(A_k, \overleftrightarrow{A_{k+2}A_{k+3}} \cap L, A_{k+1}, \overleftrightarrow{A_{k-2}A_{k-1}} \cap L),
\end{aligned}
\tag{3.18}
$$

where $L = \overleftrightarrow{A_kA_{k+1}}$. (Here $\overleftrightarrow{A_iA_j}$ denotes the line passing through A_i and A_j.) See Figure 3.8.

We next define the $2n \times 2n$ exchange matrix $B = (b_{ij})$ by

$$
b_{ij} = \begin{cases} (-1)^j & \text{if } i - j \equiv \pm 1 \mod 2n; \\ (-1)^{j+1} & \text{if } i - j \equiv \pm 3 \mod 2n; \\ 0 & \text{otherwise.} \end{cases}
\tag{3.19}
$$

We set $Y(A) = (y_1(A), \dots, y_n(A))$. Thus $(Y(A), B)$ is a Y-seed of rank $2n$.

The following result, obtained in [**29**], is included without proof.

Proposition 3.5.11. *Let A be an n-gon indexed by $\mathbb{Z}$, and let A' be the n-gon (indexed by $\mathbb{Z} + \frac{1}{2}$) obtained from A via the pentagram map. Then applying the composition of Y-seed mutations*

$$\mu_{even} = \mu_2 \circ \mu_4 \circ \cdots \circ \mu_{2n}$$

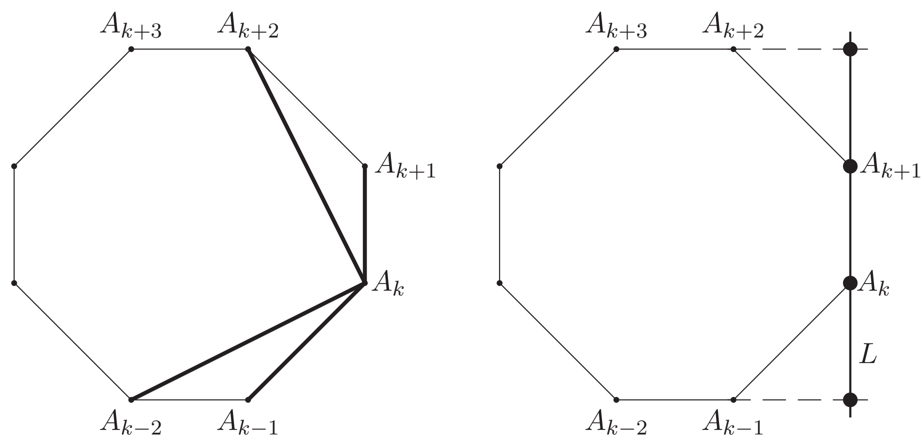


Figure 3.8. The y-parameters of a polygon.

to the Y-seed $(Y(A), B)$ (cf. (3.18)–(3.19)*) produces the Y-seed $(Y(A'), -B)$.*

Similarly, let A' be an n-gon indexed by $\mathbb{Z} + \frac{1}{2}$. Then applying

$$\mu_{odd} = \mu_1 \circ \mu_3 \circ \cdots \circ \mu_{2n-1}$$

to the Y-seed $(Y(A'), -B)$ produces the Y-seed $(Y(A''), B)$ associated with the n-gon A'' obtained from A' via the pentagram map.

(Note that the individual mutations in each of μ_{even} and μ_{odd} commute.)

Y-patterns have arisen in many other mathematical contexts. An incomplete list includes:

- Thurston's *shear coordinates* in Teichmüller spaces and their generalizations (see, e.g., [**17**] and references therein);
- recursively defined sequences of points on *elliptic curves*, and associated Somos-like sequences, cf. Example 3.4.3 and Remark 3.4.5;
- *wall-crossing formulas* for motivic Donaldson-Thomas invariants of M. Kontsevich and Y. Soibelman (see, e.g., [**34**]), and related wall-crossing phenomena for BPS states in theoretical physics;
- Fock-Goncharov varieties [**16**], including moduli spaces of point configurations in basic affine spaces;
- Zamolodchikov's Y-systems [**25, 52**] in the theory of the Thermodynamic Bethe Ansatz.

We will return to some of the aforementioned applications in the subsequent chapters.

Example 3.5.12 (*Y-pattern of type A_2*)**.** Consider the Y-pattern of rank 2

$$\cdots \overset{2}{\text{———}} (Y(0),B(0)) \overset{1}{\text{———}} (Y(1),B(1)) \overset{2}{\text{———}} (Y(2),B(2)) \overset{1}{\text{———}} \cdots$$

with the exchange matrices

$$B(t) = (-1)^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3.20}$$

The rule (3.14) of Y-seed mutation gives the following recurrence for the Y-seeds $Y(t) = (Y_{1;t}, Y_{2;t})$. For t even, we have

$$Y(t+1) = \mu_1(Y(t)), \quad Y_{1;t+1} = Y_{1;t}^{-1}, \quad Y_{2;t+1} = Y_{2;t}\,(Y_{1;t}^{-1}+1)^{-1},$$

whereas for t odd, we have

$$Y(t+1) = \mu_2(Y(t)), \quad Y_{1;t+1} = Y_{1;t}\,(Y_{2;t}^{-1}+1)^{-1}, \quad Y_{2;t+1} = Y_{2;t}^{-1}.$$

We then recursively obtain the pairs $Y(t) = (Y_{1;t}, Y_{2;t})$ listed in Figure 3.9.

t	$Y_{1;t}$	$Y_{2;t}$
0	y_1	y_2
1	y_1^{-1}	$y_1\,y_2(y_1+1)^{-1}$
2	$y_2(y_1y_2+y_1+1)^{-1}$	$(y_1+1)\,y_1^{-1}y_2^{-1}$
3	$(y_1y_2+y_1+1)\,y_2^{-1}$	$y_1^{-1}(y_2+1)^{-1}$
4	y_2^{-1}	$y_1(y_2+1)$
5	y_2	y_1
6	$y_1y_2(y_1+1)^{-1}$	y_1^{-1}
7	$(y_1+1)\,y_1^{-1}y_2^{-1}$	$y_2(y_1y_2+y_1+1)^{-1}$
8	$y_1^{-1}(y_2+1)^{-1}$	$(y_1y_2+y_1+1)\,y_2^{-1}$
9	$y_1(y_2+1)$	y_2^{-1}
10	y_1	y_2

Figure 3.9. The Y-seeds $(Y(t),B(t)) = ((Y_{1;t},Y_{2;t}),B(t))$ in type A_2. The exchange matrices $B(t)$ are given by (3.20). The initial Y-seed is $(Y(0),B(0))$, with $Y(0) = (Y_{1;0},Y_{2;0}) = (y_1,y_2)$. This sequence of Y-seeds is 10-periodic: $Y(t+10) = Y(t)$.

3.6. Tropical semifields

In this section, we re-examine the combinatorics of matrix mutations, relating it to the concept of Y-seeds and their mutations discussed in Section 3.5. We begin by introducing the notion of semifield, and in particular, the tropical semifield, which will give us an important alternative way to encode the bottom part of an extended exchange matrix $\tilde{B}$.

Definition 3.6.1. A *semifield* is an abelian group P, written multiplicatively, endowed with an operation of "auxiliary addition" $\oplus$ which is required to be commutative and associative, and satisfy the distributive law with respect to the multiplication in P.

We emphasize that $(P, \oplus)$ does not have to be a group, just a semigroup. Also, every element of P has a multiplicative inverse, so unless P is trivial, it does not contain an additive identity (or "zero") element.

Definition 3.6.2. Let $\mathrm{Trop}(q_1, \dots, q_\ell)$ denote the multiplicative group of Laurent monomials in the variables $q_1, \dots, q_\ell$. We equip $\mathrm{Trop}(q_1, \dots, q_\ell)$ with the binary operation of *tropical addition* $\oplus$ defined by

$$\prod_{i=1}^{\ell} q_i^{a_i} \oplus \prod_{i=1}^{\ell} q_i^{b_i} = \prod_{i=1}^{\ell} q_i^{\min(a_i, b_i)} \ . \tag{3.21}$$

Lemma 3.6.3. *Tropical addition is commutative and associative, and satisfies the distributive law with respect to the ordinary multiplication:* $(p \oplus q)r = pr \oplus qr$.

Thus $\mathrm{Trop}(q_1, \dots, q_\ell)$ is a semifield, which we call the *tropical semifield* generated by $q_1, \dots, q_\ell$.

Remark 3.6.4. The above terminology differs from the one used in *tropical geometry* by what is essentially a notational convention: replacing Laurent monomials by the corresponding vectors of exponents, one gets a semifield in which multiplication is the ordinary addition, and auxiliary addition amounts to taking the minimum.

The formalism of the tropical semifield and its auxiliary addition allows us to restate the rules of matrix mutation in the following way.

Let $\tilde{B}$ be an $m \times n$ extended exchange matrix. As before, let $x_{n+1}, \dots, x_m$ be formal variables. We encode the bottom $(m-n) \times n$ submatrix of $\tilde{B}$ by the *coefficient tuple* $\mathbf{y} = (y_1, \dots, y_n) \in \mathrm{Trop}(x_{n+1}, \dots, x_m)^n$ defined by

$$y_j = \prod_{i=n+1}^{m} x_i^{b_{ij}} \quad (j \in \{1, \dots, n\}) \ . \tag{3.22}$$

Thus the matrix $\tilde{B}$ contains the same information as its top $n \times n$ submatrix B together with the coefficient tuple $\mathbf{y}$.

Proposition 3.6.5. *Let $\tilde{B} = (b_{ij})$ and $\tilde{B}'$ be two extended skew-symmetrizable matrices related by a mutation μ_k, and let $\mathbf{y} = (y_1, \dots, y_n)$ and $\mathbf{y}' = (y_1', \dots, y_n')$ be the corresponding coefficient tuples (cf.* (3.22)*). Then*

$$y_j' = \begin{cases} y_k^{-1} & \text{if } j = k; \\ y_j (y_k \oplus 1)^{-b_{kj}} & \text{if } j \neq k \text{ and } b_{kj} \leq 0; \\ y_j (y_k^{-1} \oplus 1)^{-b_{kj}} & \text{if } j \neq k \text{ and } b_{kj} \geq 0. \end{cases} \tag{3.23}$$

Comparing (3.23) with (3.14), we can informally say that the coefficient tuple $\mathbf{y}$ undergoes a "tropical Y-seed mutation" at k.

Proposition 3.6.5 can be proved by translating the rules of matrix mutation into the language of the tropical semifield. We outline a different proof which explains the connection between the formulas (3.13)–(3.14) and (3.23), and introduces some notions that will be useful in the sequel.

Definition 3.6.6. Let $\mathbb{Q}_{\mathrm{sf}}(x_1, \dots, x_m)$ denote the set of nonzero rational functions in $x_1, \dots, x_m$ which can be written as subtraction-free rational expressions in these variables (with positive rational coefficients). Thus, each element of $\mathbb{Q}_{\mathrm{sf}}(x_1, \dots, x_m)$ can be written in the form $\frac{P(x_1,\dots,x_m)}{Q(x_1,\dots,x_m)}$, where P and Q are polynomials with positive coefficients. The set $\mathbb{Q}_{\mathrm{sf}}(x_1, \dots, x_m)$ is a semifield with respect to the ordinary operations of addition and multiplication. We call it the *universal semifield* generated by $x_1, \dots, x_m$.

This terminology is justified by the following easy lemma, whose proof we omit (see [**3**, Lemma 2.1.6]). Informally speaking, this lemma says that the generators $x_1, \dots, x_m$ of the semifield $\mathbb{Q}_{\mathrm{sf}}(x_1, \dots, x_m)$ do not satisfy any relations, save for those which are implied by the axioms of a semifield.

Lemma 3.6.7. *For any semifield $\mathcal{S}$, any map $f : \{x_1, \dots, x_m\} \to \mathcal{S}$ extends uniquely to a semifield homomorphism $\mathbb{Q}_{\mathrm{sf}}(x_1, \dots, x_m) \to \mathcal{S}$.*

Proof of Proposition 3.6.5. Let $\tilde{\mathbf{x}} = (x_1, \dots, x_m)$ be a collection of indeterminates. Define the semifield homomorphism

$$f : \mathbb{Q}_{\mathrm{sf}}(x_1, \dots, x_m) \to \mathrm{Trop}(x_{n+1}, \dots, x_m)$$

by setting (cf. Lemma 3.6.7)

$$f(x_i) = \begin{cases} 1 & \text{if } i \leq n; \\ x_i & \text{if } i > n. \end{cases}$$

Applying mutation at k to the seed $(\tilde{\mathbf{x}}, \tilde{B})$, we get a new seed $(\tilde{\mathbf{x}}', \tilde{B}')$, in which the only new cluster variable x_k' satisfies an exchange relation of the form $x'_k\, x_k = M_1 + M_2$. The two monomials M_1 and M_2 are coprime, and in particular do not share a frozen variable x_i. Applying the semifield homomorphism f, we obtain $1 \cdot f(x'_k) = f(M_1) \oplus f(M_2) = 1$, so $f(x'_k) = 1$.

Now let $\hat{\mathbf{y}}$ and $\hat{\mathbf{y}}'$ (resp., $\mathbf{y}$ and $\mathbf{y}'$) be defined by (3.12) (resp., (3.22)). Since all cluster variables in $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{x}}'$ are sent to 1 by f, we conclude that $f(\hat{\mathbf{y}}) = \mathbf{y}$ and $f(\hat{\mathbf{y}}') = \mathbf{y}'$. Therefore applying f to (3.13) yields (3.23). □

Let $(\tilde{\mathbf{x}}, \tilde{B})$ be a labeled seed as before. Since the extended exchange matrix $\tilde{B}$ contains the same information as the exchange matrix B together with the coefficient tuple $\mathbf{y}$ defined by (3.22), we can identify the seed $(\tilde{\mathbf{x}}, \tilde{B})$ with the triple $(\mathbf{x}, \mathbf{y}, B)$. Abusing notation, we will also refer to such triples as (labeled) seeds:

Definition 3.6.8. Let $\mathcal{F}$ be a field of rational functions (say over $\mathbb{C}$) in some m variables which include the *frozen variables* $x_{n+1}, \dots, x_m$. A labeled *seed* (of geometric type) of rank n is a triple $\Sigma = (\mathbf{x}, \mathbf{y}, B)$ consisting of

- a *cluster* $\mathbf{x}$, an n-tuple of elements of $\mathcal{F}$ such that the *extended cluster* $\mathbf{x} \cup \{x_{n+1}, \dots, x_m\}$ freely generates $\mathcal{F}$;
- an *exchange matrix* B, a skew-symmetrizable integer matrix;
- a *coefficient tuple* $\mathbf{y}$, an n-tuple of Laurent monomials in the tropical semifield $\mathrm{Trop}(x_{n+1}, \dots, x_m)$.

We can now restate the rules of seed mutation in this language.

Proposition 3.6.9. *Let* $(\mathbf{x}, \mathbf{y}, B)$, *with* $B = (b_{ij})$ *and* $\mathbf{y} = (y_1, \dots, y_n)$, *and* $(\mathbf{x}', \mathbf{y}', B')$, *with* $\mathbf{y}' = (y'_1, \dots, y'_n)$, *be two labeled seeds related by a mutation* μ_k. *Then* $(\mathbf{x}', \mathbf{y}', B')$ *is obtained from* $(\mathbf{x}, \mathbf{y}, B)$ *as follows:*

- $B' = \mu_k(B)$;
- $\mathbf{y}'$ *is given by the "tropical* Y*-seed mutation rule"* (3.23);
- $\mathbf{x}' = (\mathbf{x} - \{x_k\}) \cup \{x'_k\}$, *where* x'_k *is defined by the exchange relation*

$$x_k\, x'_k = \frac{y_k}{y_k \oplus 1} \prod_{b_{ik}>0} x_i^{b_{ik}} + \frac{1}{y_k \oplus 1} \prod_{b_{ik}<0} x_i^{-b_{ik}}\,; \tag{3.24}$$

Proof. The only statement requiring proof is (3.24), which can be easily seen to be a rewriting of (3.1). □

We can now re-define the notion of a labeled seed pattern.

Definition 3.6.10. A *labeled seed pattern* of rank n is obtained by assigning a triple $\Sigma(t) = (\mathbf{x}(t), \mathbf{y}(t), B(t))$ as above to every vertex t in the n-regular tree $\mathbb{T}_n$, and requiring that the triples assigned to adjacent vertices of the tree are related by the corresponding mutation, as described in Proposition 3.6.9.

Example 3.6.11 (*Type* A_2)**.** Consider the seed pattern of rank 2

$$\cdots \overset{2}{\text{———}} \Sigma(0) \overset{1}{\text{———}} \Sigma(1) \overset{2}{\text{———}} \Sigma(2) \overset{1}{\text{———}} \Sigma(3) \overset{2}{\text{———}} \Sigma(4) \overset{1}{\text{———}} \Sigma(5) \overset{2}{\text{———}} \cdots$$

formed by the seeds $\Sigma(t) = (\mathbf{x}(t), \mathbf{y}(t), B(t))$, for $t \in \mathbb{T}_2 \cong \mathbb{Z}$, with the exchange matrices

$$B(t) = (-1)^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3.25}$$

(cf. Example 3.5.12). Note that we do not specify the bottom part of the initial exchange matrix, nor even the number of frozen variables. Still, we can express all the seeds in terms of the initial one using the language of the tropical semifield, and following the recipe formulated in Proposition 3.6.9. The results of the computation are shown in Figure 3.10.

t	$\mathbf{y}(t)$		$\mathbf{x}(t)$	
0	y_1	y_2	x_1	x_2
1	$\dfrac{1}{y_1}$	$\dfrac{y_1y_2}{y_1 \oplus 1}$	$\dfrac{y_1 + x_2}{x_1(y_1 \oplus 1)}$	x_2
2	$\dfrac{y_2}{y_1y_2 \oplus y_1 \oplus 1}$	$\dfrac{y_1 \oplus 1}{y_1y_2}$	$\dfrac{y_1 + x_2}{x_1(y_1 \oplus 1)}$	$\dfrac{x_1y_1y_2 + y_1 + x_2}{(y_1y_2 \oplus y_1 \oplus 1)x_1x_2}$
3	$\dfrac{y_1y_2 \oplus y_1 \oplus 1}{y_2}$	$\dfrac{1}{y_1(y_2 \oplus 1)}$	$\dfrac{x_1y_2 + 1}{x_2(y_2 \oplus 1)}$	$\dfrac{x_1y_1y_2 + y_1 + x_2}{(y_1y_2 \oplus y_1 \oplus 1)x_1x_2}$
4	$\dfrac{1}{y_2}$	$y_1(y_2 \oplus 1)$	$\dfrac{x_1y_2 + 1}{x_2(y_2 \oplus 1)}$	x_1
5	y_2	y_1	x_2	x_1

Figure 3.10. Seeds in type A_2. The exchange matrices are given by (3.25). We denote the initial cluster by $\mathbf{x}_\circ = (x_1, x_2)$, and the initial coefficient tuple by $\mathbf{y}_\circ = (y_1, y_2)$. The formulas for the coefficient tuples $\mathbf{y}(t)$ are the tropical versions of the formulas in Figure 3.9. Note that the labeled seed $\Sigma(5)$ is obtained from $\Sigma(0)$ by interchanging the indices 1 and 2; the sequence then continues by obvious periodicity (so that $\Sigma(10)$ is identical to $\Sigma(0)$, etc.).

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