COMBINATORICS OF THE TWO-SPECIES ASEP AND KOORNWINDER MOMENTS
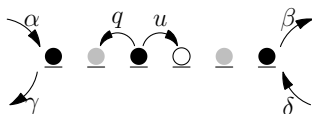
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ABSTRACT. In previous work [CW10, CW11, CW13], the first and third authors introduced staircase tableaux, which they used to give combinatorial formulas for the stationary distribution of the asymmetric simple exclusion process (ASEP) and for the moments of the Askey-Wilson weight function. The fact that the ASEP and Askey-Wilson moments are related at all is unexpected, and is due to [USW04] (see also [CSSW12]). The ASEP is a model of particles hopping on a one-dimensional lattice of N sites with open boundaries; particles can enter and exit at both left and right borders. It was introduced around 1970 [MGP68, Spi70] and is cited as a model for both traffic flow and translation in protein synthesis. Meanwhile, the Askey-Wilson polynomials are a family of orthogonal polynomials in one variable which sit at the top of the hierarchy of classical orthogonal polynomials. So from this previous work, we have the relationship

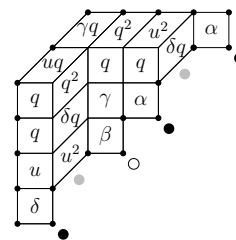ASEP $--$ staircase tableaux $--$ Askey-Wilson moments.

The Askey-Wilson polynomials can be viewed as the one-variable case of the multivariate Koornwinder polynomials, which are also known as the Macdonald polynomials for the type BC root system. It is natural then to ask whether one can generalize the relationships among the ASEP, Askey-Wilson moments, and staircase tableaux, in such a way that Koornwinder moments replace Askey-Wilson moments. In [CW15], we made a precise link between Koornwinder moments and the *two-species ASEP*, a generalization of the ASEP which has two species of particles with different "weights." In this article we introduce *rhombic staircase tableaux*, and show that we have the relationship

2-species ASEP $--$ rhombic staircase tableaux $--$ Koornwinder moments.

In particular, we give formulas for the stationary distribution of the two-species ASEP and for Koornwinder moments, in terms of rhombic staircase tableaux.

Tableaux formulae
Describe hopping particles
Please excuse our proofs

Contents

1. Introduction

Introduced in the late 1960's [MGP68, Spi70], the asymmetric exclusion process (ASEP) is a model of interacting particles hopping left and right on a one-dimensional lattice of N sites. In the most general form of the ASEP with open boundaries, particles may enter and exit at the left with probabilities α and γ, and they may exit and enter at the right with probabilities β and δ. In the bulk, the probability of hopping left is q times the probability of hopping right. Since its introduction, there has been a huge amount of activity on the ASEP and its variants for a number of reasons: for example, it exhibits boundary-induced phase transitions [Lig75], [Lig99, page 221], it is connected to the XXZ model [SS94, ER96], and it is regarded as a model for translation in protein synthesis [MGP68], the nuclear pore complex [ZTC+07, ZTJT+10], traffic flow [SW97], and the formation of shocks [DLS97]. Moreover, dating back to at least 1982 [SZ82], it was realized that there were connections between the ASEP and combinatorics. A main goal was to find a combinatorial description of the stationary distribution – that is, to express each component of the stationary distribution as a generating function for a set of combinatorial objects. Our work [CW10, CW11, CW13] introduced and used *staircase tableaux* to give a complete answer to this question, building on previous partial results of [SZ82, BE04, DS05, Ang06, BCE+06, Cor07, CW07b, CW07a, Vie].

A second reason that combinatorialists became intrigued by the ASEP was its link to orthogonal polynomials [Sas99], and in particular to Askey-Wilson polynomials [USW04], a family of orthogonal polynomials which are at the top of the hierarchy of classical orthogonal polynomials in one variable. Starting in the early 1980's, mathematicians including Flajolet [Fla80], Viennot [Vie85], and Foata [Foa78], initiated a combinatorial approach to orthogonal polynomials, which led to the discovery of various combinatorial formulas for the moments of (the weight functions of) many of the polynomials in the Askey scheme, including q-Hermite, Chebyshev, q-Laguerre, Charlier, Meixner, and Al-Salam-Chihara polynomials, see e.g. [IS98, ISV87, KSZ11, dMSW95, SS96]. By combining our tableaux formulas for the ASEP with the known link between the ASEP and Askey-Wilson moments, we gave the first combinatorial formula for Askey-Wilson moments in [CW10, CW11, CW13]. In summary, we have the relationships of Figure 1.

It is well-known that Askey-Wilson polynomials can be viewed as a specialization of the multivariate *Macdonald-Koornwinder polynomials*, which are also known as *Koornwinder polynomials* [Koo92], or *Macdonald polynomials for the non-reduced*

ASEP ——————[USW04]—————— Askey-Wilson moments

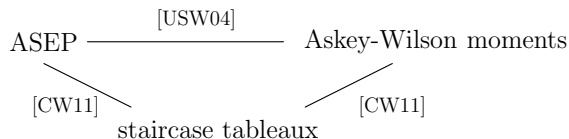[CW11]   [CW11]

staircase tableaux

FIGURE 1.

type BC root system [Mac95]. (For brevity, we will henceforth call them Koornwinder polynomials.) These polynomials are particularly important because the Macdonald polynomials associated to any classical root system can be expressed as limits or special cases of Koornwinder polynomials [vD95]. Since Koornwinder polynomials generalize Askey-Wilson polynomials, it is natural to ask if one can find an analogue of Figure 1 with Koornwinder moments replacing Askey-Wilson moments.

In [CW15], we showed that there is a relationship between the *two-species ASEP* and *homogeneous Koornwinder moments* which is parallel to that between the ASEP and Askey-Wilson moments. The *two-species ASEP* is a generalization of the ASEP which involves two different kinds of particles, *heavy* and *light*. Both types of particles can hop left and right in the lattice, with heavy and light particles interacting exactly as do particles and holes in the usual ASEP, but only the heavy particles can enter and exit the lattice at the boundaries. So in particular, the number r of light particles is conserved. When there are no light particles, the two-species ASEP reduces to the usual ASEP.[1] Recent work on the two-species ASEP includes [Uch08, ALS09, AAMP12, ALS12, MV15]. Meanwhile, *homogeneous Koornwinder moments* are defined to be the integrals of the homogeneous symmetric polynomials when one uses the Koornwinder measure. (This is a natural multivariate generalization of the notion of Askey-Wilson moment μ_n, which is the integral of x^n when one uses the Askey-Wilson measure.) The main result of [CW15] was that the homogeneous Koornwinder moments are proportional to the fugacity partition function of the two-species ASEP.

In this paper, we introduce *rhombic staircase tableaux*, and show that we have a trio of relationships (see Figure 2) which generalize those from Figure 1. In particular,
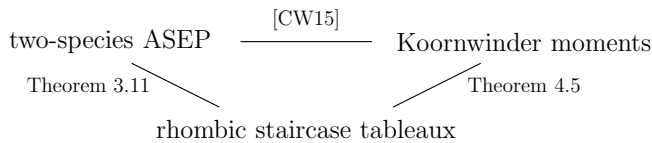
two-species ASEP ——————[CW15]—————— Koornwinder moments

Theorem 3.11   Theorem 4.5

rhombic staircase tableaux

FIGURE 2.

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[1]Since the usual ASEP has been cited as a model for e.g. traffic flow, it is natural to ask about "real life" interpretations of the two-species ASEP. To give one, let us consider a $1D$ lattice of sites, with a chasm at either side. Each site is either empty or occupied by a kangaroo or rabbit. The kangaroos and rabbits can hop left and right; moreover, the kangaroos may hop over the rabbits, and over the chasms. However, the rabbits cannot hop over the chasm, so their number is conserved.

To give another potential application, recall that the usual ASEP models the nuclear pore complex, a multiprotein machine that manages the transport of material into and out of the nucleus [ZTC+07, ZTJT+10], through a single-file pore represented by a $1D$ lattice. One may imagine a generalization in which the nucleus is semipermeable; some materials may travel in and out, while others may not.

our first main result (see Theorem 3.11) is a combinatorial formula for the stationary distribution of the two-species ASEP in terms of rhombic staircase tableaux. Combining this result with [CW15], we obtain our second main result (see Theorem 4.5), which is a formula for each of the homogeneous Koornwinder moments in terms of rhombic staircase tableaux. This formula shows that (up to a simple scalar factor) each homogeneous Koornwinder moment can be written as a polynomial with positive coefficients.
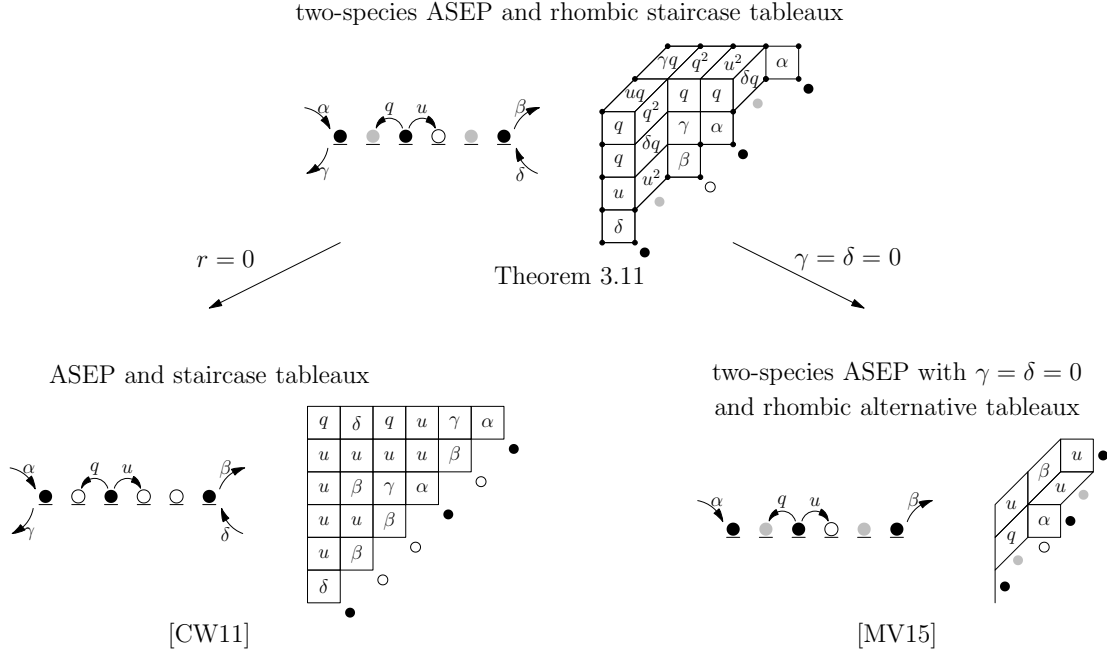

FIGURE 3.

Theorem 3.11 simultaneously generalizes the main results of [CW11] and [MV15], see Figure 3. Our previous work [CW11] treated the ordinary ASEP, which can be thought of as the two-species ASEP when the number r of light particles is equal to zero; the combinatorial objects used there were staircase tableaux, which are certain fillings of a Young diagram of staircase shape. The proofs were technically difficult, with the final step reduced to a computer check, see [CW13]. Meanwhile, [MV15] treated the two-species ASEP but when $\gamma = \delta = 0$, so that particles can enter the lattice only from the left, and exit only to the right. The combinatorial objects used there were rhombic alternative tableaux, which are filings of certain rhombic tilings; the fact that the tiles have three possible slopes is related to the fact that each site in the lattice can be filled by either a hole, a heavy particle, or a light particle. (See [Man15] for a generalization of this idea to the k-species ASEP.) Because of the simple boundary conditions, the proofs in [MV15] were reasonably straightforward and completely combinatorial. In this paper we treat the two-species ASEP with general boundary conditions, using our new rhombic staircase tableaux, which generalize both staircase tableaux and rhombic alternative tableaux. Due to the general boundary conditions, our proofs are necessarily intricate, but we manage to

bypass the computer check in [CW13], giving a completely "human" proof of our main results (and the first human proof of the results of [CW11]).

In recent years there has been a great deal of activity surrounding the combinatorics of (type A) Macdonald polynomials, including the celebrated combinatorial formula of Haglund-Haiman-Loehr [HHL05a, HHL05b, HHL08]. One of the most exciting aspects of Theorem 4.5 is that it gives some hope that there may be a combinatorial formula for Koornwinder polynomials, akin to that for type A Macdonald polynomials. Indeed, combining Theorem 4.5 with [Can15, (76) and (117)] shows that when $\lambda = (1^{N-r}, 0^r)$, and $\mathbf{z} = (z_1, \ldots, z_N)$ with $z_i = 1$ for all i, the Koornwinder polynomial $P_\lambda(\mathbf{z}; a, b, c, d | q, q)$ is (up to a scalar factor) equal to a polynomial in $\alpha, \beta, \gamma, \delta$ and q, with positive integer coefficients, using the change of variables from (3). Moreover that polynomial is a sum over rhombic staircase tableaux!

It would be interesting to relate our work to the *Macdonald processes* of Borodin and Corwin [BC14], which are probability measures on sequences of partitions defined in terms of non-negative specializations of the (type A) Macdonald polynomials. Macdonald processes are related to various interacting particle systems, though not explicitly to the ASEP with open boundaries. Perhaps there exists some more general *Koornwinder processes* which would be connected to the ASEP? It would also be interesting to relate our work to that of Cantini [Can15], who used the affine Hecke algebra of type $\hat{C}_N$ to give a link between the two-species ASEP and Koornwinder polynomials.

The structure of this paper is as follows. In Section 2, we define the two-species ASEP. Then in Section 3, we introduce rhombic staircase tableaux and give our first main result, a combinatorial formula for steady state probabilities of the two-species ASEP in terms of rhombic staircase tableaux. In Section 4, we give our second main result, a combinatorial formula for homogeneous Koornwinder moments. In Section 5 we present a *Matrix Ansatz* for the two-species ASEP (generalizing one given by Uchiyama), which is an algebraic tool for computing steady state probabilities. In Section 6, we define a family of matrices, and explain how they can be thought of as "transfer matrices" for rhombic staircase tableaux. Section 7 is devoted to the proof of our first main result. The proof of our second main result follows reasonably quickly from the first one, and is given in Section 8.

2. The two-species ASEP

We start by defining the asymmetric exclusion process (ASEP) with open boundaries. We will then define the two-species ASEP, which generalizes the usual ASEP. For an introduction to interacting particle systems, see [Lig05].

2.1. The asymmetric exclusion process (ASEP).

Definition 2.1. Let α, β, γ, δ, q, and u be constants such that $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 \leq \gamma \leq 1$, $0 \leq \delta \leq 1$, $0 \leq q \leq 1$, and $0 \leq u \leq 1$. Let B_N be the set of all 2^N words in the language $\{\circ, \bullet\}^*$. The *ASEP* is the Markov chain on B_N with transition probabilities:

- If $X = A \bullet \circ B$ and $Y = A \circ \bullet B$ then $P_{X,Y} = \frac{u}{N+1}$ (particle hops right) and $P_{Y,X} = \frac{q}{N+1}$ (particle hops left).
- If $X = \circ B$ and $Y = \bullet B$ then $P_{X,Y} = \frac{\alpha}{N+1}$ (particle enters from the left).
- If $X = B \bullet$ and $Y = B \circ$ then $P_{X,Y} = \frac{\beta}{N+1}$ (particle exits to the right).
- If $X = \bullet B$ and $Y = \circ B$ then $P_{X,Y} = \frac{\gamma}{N+1}$ (particle exits to the left).
- If $X = B \circ$ and $Y = B \bullet$ then $P_{X,Y} = \frac{\delta}{N+1}$ (particle enters from the right).
- Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.

See Figure 4 for an illustration of the four states, with transition probabilities, for the case $N = 2$. The probabilities on the loops are determined by the fact that the sum of the probabilities on all outgoing arrows from a given state must be 1.
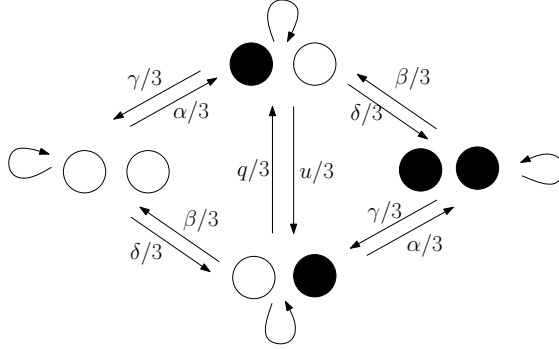

FIGURE 4. The state diagram of the ASEP for $N = 2$

In the long time limit, the system reaches a steady state where all the probabilities $P_N(\tau)$ of finding the system in configurations $\tau = (\tau_1, \tau_2, \ldots, \tau_N)$ are stationary. More specifically, the *stationary distribution* is the unique (up to scaling) eigenvector of the transition matrix of the Markov chain with eigenvalue 1.

2.2. **The two-species ASEP.** The two-species ASEP is a generalization of the ASEP which involves two kinds of particles, *heavy* and *light*. We will denote a heavy particle by a 2 and a light particle by a 1. We will also denote a hole (or the absence of a particle) by a 0. In the two-species ASEP, heavy particles behave exactly as do particles in the usual ASEP: they can hop in and out of the boundary, and they can hop left and right in the lattice (swapping places with a hole or with a heavy particle). Light particles cannot hop in and out of the boundary, but they may hop left and right in the lattice (swapping places with a hole or with a light particle). Therefore the number r of light particles is conserved.

Definition 2.2. Let α, β, γ, δ, q, and u be constants such that $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 \leq \gamma \leq 1$, $0 \leq \delta \leq 1$, $0 \leq q \leq 1$, and $0 \leq u \leq 1$. Let $B_{N,r}$ be the set of all words in $\{0, 1, 2\}^N$ which contain precisely r 1's; note that $|B_{N,r}| = \binom{N}{r} 2^{N-r}$. The *two-species ASEP* is the Markov chain on $B_{N,r}$ with transition probabilities:

- If $X = A21B$ and $Y = A12B$, or if $X = A20B$ and $Y = A02B$, or if $X = A10B$ and $Y = A01B$, then $P_{X,Y} = \frac{u}{N+1}$ and $P_{Y,X} = \frac{q}{N+1}$.
- If $X = 0B$ and $Y = 2B$ then $P_{X,Y} = \frac{\alpha}{N+1}$.
- If $X = B2$ and $Y = B0$ then $P_{X,Y} = \frac{\beta}{N+1}$.
- If $X = 2B$ and $Y = 0B$ then $P_{X,Y} = \frac{\gamma}{N+1}$.
- If $X = B0$ and $Y = B2$ then $P_{X,Y} = \frac{\delta}{N+1}$.
- Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.

See Figure 5 for an illustration of the state diagram of the two-species ASEP on a lattice of $N = 2$ sites, with $r = 1$ light particle.
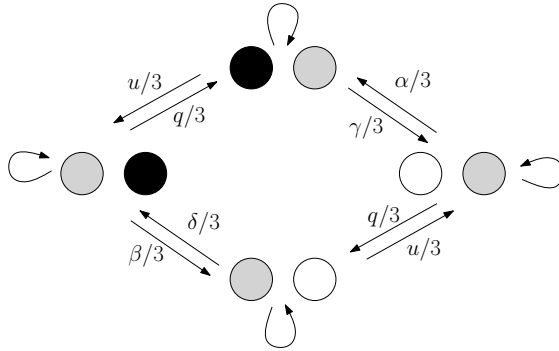

FIGURE 5. The state diagram of the two-species ASEP for $N = 2$ and $r = 1$. Light particles are denoted in grey, and heavy particles are denoted in black.

Note that if $r = 0$, i.e. there are no light particles, then the two-species ASEP is simply the usual ASEP.

3. Rhombic staircase tableaux and the two-species ASEP

In this section we will define *rhombic staircase tableaux*, which are the main combinatorial objects used in this paper. We will then state our main result, which is a formula for the steady state distribution of the two-species ASEP in terms of rhombic staircase tableaux.

Definition 3.1. Let $\tau = (\tau_1, \ldots, \tau_N) \in \{0, 1, 2\}^N$. For $0 \leq i \leq 2$, we let $|\tau|_i$ denote the number of i's in τ. We also let $|\tau|_{02}$ denote the total number of 0's and 2's in τ. Set $r = |\tau|_1$. The *rhombic diagram* $\Gamma(\tau)$ associated to τ is a convex shape whose southeast border is obtained by reading the components of $(\tau_1, \ldots, \tau_N)$ from left to right, and replacing each 0 or 2 by a *corner* (a south step followed by a west step), and replacing each 1 by a diagonal step southeast. The northwest border consists of $N - r$ west steps, followed by r southeast steps, followed by $N - r$ south steps. We say that this rhombic diagram has *size* (N, r).

Example 3.2. Suppose $\tau = (1, 2, 0, 1, 0)$. Then $N = 5$, $r = 2$, and $\Gamma(\tau)$ is as in Figure 6.

Definition 3.3. A *tile* is a *square*, a *short rhombus*, or a *tall rhombus*, as shown at the left of Figure 7.
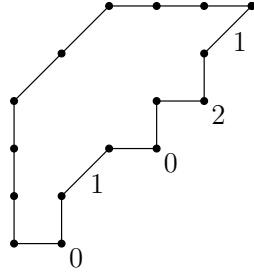
FIGURE 6. The rhombic diagram $\Gamma(1, 2, 0, 1, 0)$.
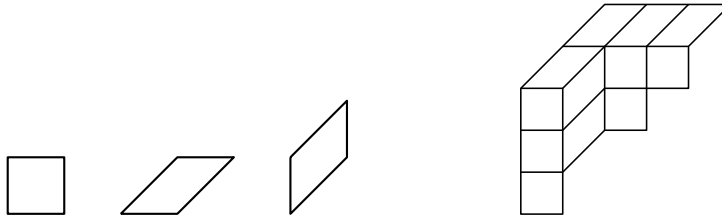
FIGURE 7. At the left: a square, a short rhombus, and a tall rhombus. At the right: the maximal tiling of $\Gamma(1, 2, 0, 1, 0)$.

Definition 3.4. Let $\tau = (\tau_1, \dots, \tau_N) \in \{0, 1, 2\}^N$ and let $r = |\tau|_1$. The *maximal tiling* of $\Gamma(\tau)$ is a tiling of $\Gamma(\tau)$ by squares, short rhombi, and tall rhombi, which is obtained as follows. For each $0 \leq j \leq N$, we construct a column of tiles: if $\tau_j = 1$, this column consists of $|(\tau_1, \dots, \tau_{j-1})|_{02}$ tall rhombi; and if $\tau_j = 0$ or 2, then this column consists of $1 + |(\tau_1, \dots, \tau_{j-1})|_{02}$ squares at the bottom, and $|(\tau_1, \dots, \tau_{j-1})|_1$ short rhombi at the top. We refer to these columns as *1-columns* and *02-columns*, respectively. We then concatenate these columns from right to left so that they comprise a tiling of $\Gamma(\tau)$. See the rightmost picture in Figure 7.

Definition 3.5. A *vertical strip* is a maximal connected set of tiles where two adjacent tiles share a horizontal edge, see the first (leftmost) picture in Figure 8. Note that the 02-columns from Definition 3.4 are all vertical strips. A *horizontal strip* is a maximal connected set of tiles where two adjacent tiles share a vertical edge, see the second picture in Figure 8. The shaded regions in the third and fourth pictures in Figure 8 show a vertical and horizontal strip within the maximal tiling of $\Gamma(1, 2, 0, 1, 0)$.
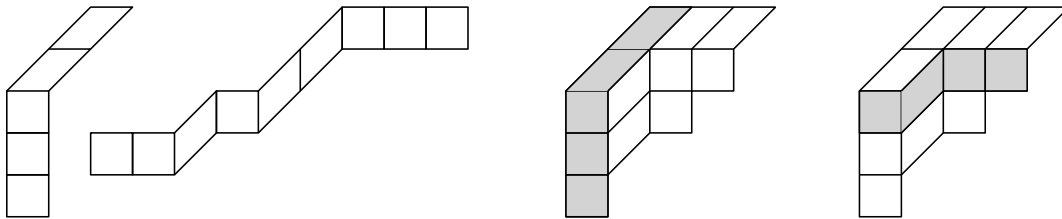

FIGURE 8. Vertical and horizontal strips.

Remark 3.6. The horizontal strips of the maximal tiling of $\Gamma(\tau)$ where $\tau = (\tau_1, \ldots, \tau_N)$ are in bijection with the corners of $\Gamma(\tau)$, or in other words with $\{\tau_i \in \tau \mid \tau_i = 0 \text{ or } 2\}$.

Definition 3.7. Let $\tau = (\tau_1, \ldots, \tau_N) \in \{0, 1, 2\}^N$. A *rhombic staircase tableau* T of *shape* $\Gamma(\tau)$ is a filling of the tiles of the maximal tiling of $\Gamma(\tau)$ such that:
- Each square is either empty, or contains an α, β, γ, or δ.
- Each tall rhombus is either empty, or contains a βu or δq.
- Each short rhombus is either empty, or contains an αu or γq.
- The lowest square (if there is one) in each vertical strip is not empty.
- Every tile which is above, and in the same vertical strip as, an α or γ must be empty.
- Every tile which is left of, and in the same horizontal strip as, a β or δ must be empty.

We say that a rhombic staircase tableau has *size* (N, r) if its shape has size (N, r). Let us number the components (i.e. the corners and diagonal steps) of the southeast border from 1 to N. If for $1 \leq i \leq N$ we have that $\tau_i = 2$ (respectively, $\tau_i = 0$) if and only if the corresponding component of the southeast border is a corner box containing an α or δ (resp. β or γ), then we say that the tableau T has *type* τ, and we write $\text{type}(T) = \tau$.

See the left of Figure 9 for an example of a rhombic staircase tableau of type $(1, 0, 0, 1, 2)$.
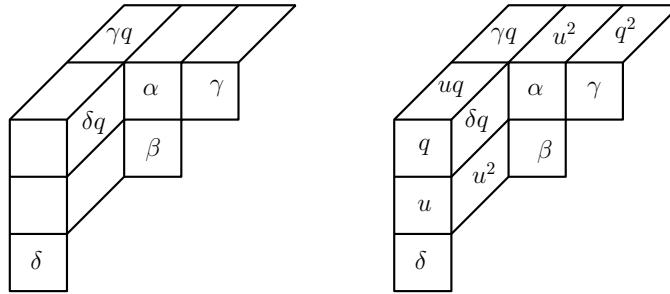
FIGURE 9. At the left: a rhombic staircase tableau T of type $(1, 0, 0, 1, 2)$. At the right: the same rhombic staircase tableau, with empty tiles filled according to Definition 3.8.

Definition 3.8. The *weight* $w(T)$ of a rhombic staircase tableau T is a monomial in $\alpha, \beta, \gamma, \delta, q$, and u, which we obtain as follows. Every empty tile of T is assigned a monomial in q and u, based on the label of the closest labeled tile to its right in the same horizontal strip (if the tile lies in a horizontal strip), and the label of the closest labeled tile below it in the same vertical strip (if the tile lies in a vertical strip), such that:
- each empty square which sees a β (resp. δ) to its right gets assigned a u (resp. q);
- each empty square which sees an α or γ to its right, and an α or δ below, gets a u;
- each empty square which sees an α or γ to its right, and a β or γ below, gets a q;
- each empty tall rhombus which sees a β (resp. δ) to the right gets a u^2 (resp. q^2);
- each empty tall rhombus which sees an α or γ to the right gets a uq;
- each empty short rhombus which sees an α (resp. γ) below it gets a u^2 (resp. q^2);

• each empty short rhombus which sees a β or δ below it gets a uq.

See Figures 10 and 11 for a pictorial summary of these rules. After assigning a monomial in q and u to each empty tile in this way, the *weight* of T is then defined as the product of all labels in all tiles.

FIGURE 10. How to assign a u or q to the empty squares of a rhombic staircase tableau.
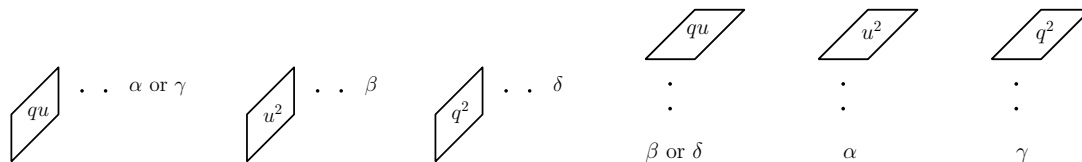
FIGURE 11. How to assign a monomial in u and q to the empty rhombi of a rhombic staircase tableau.
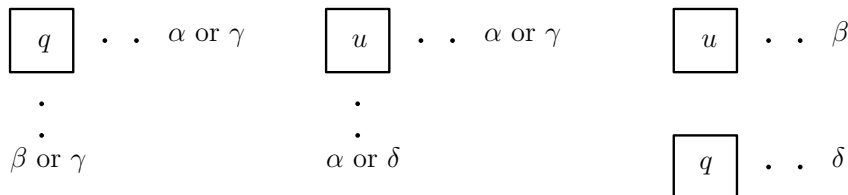
Example 3.9. Let T be the rhombic staircase tableau at the left of Figure 9. Then if we fill the empty boxes according to Definition 3.8, we obtain the filling shown at the right of Figure 9. We find that the weight $w(T)$ of T is $\alpha\beta\gamma^2\delta^2 u^6 q^6$.

Remark 3.10. The weight of a rhombic staircase tableau of size (N, r) always has degree $\frac{(N-r)(N+3r+1)}{2} = \binom{N-r+1}{2} + 2r(N-r)$. One can see this by using Definition 3.4, which implies that the maximal tiling of a shape of size (N, r) contains $\binom{N-r+1}{2}$ squares and $r(N-r)$ rhombi. For convenience, we will sometimes set $u = 1$, since this results in no loss of information.

The first main result of this paper is the following. Note that it will be a direct consequence of Theorem 5.4, Theorem 6.3, and Theorem 7.1.

Theorem 3.11. *Consider the two-species ASEP on a lattice of N sites with precisely r light particles. Let τ be any state; in other words, $\tau = (\tau_1, \ldots, \tau_N) \in \{0, 1, 2\}^N$, and $|\tau|_1 = r$. Set $\mathbf{Z}_{N,r} = \sum_T w(T)$, where the sum is over all rhombic staircase tableaux T of size (N, r). Then the steady state probability that the two-species ASEP is at state τ is precisely*

$$\frac{\sum_T w(T)}{\mathbf{Z}_{N,r}},$$

where the sum is over all rhombic staircase tableaux T of type τ.

Remark 3.12. Note that (up to the scalar factor $\mathbf{Z}_{N,r}$) Theorem 3.11 expresses steady state probabilities as polynomials in $\alpha, \beta, \gamma, \delta, q, u$ with positive coefficients. This is a very strong form of positivity. Since probabilities must be positive real numbers, of course it is necessary that for any physically relevant specialization of

the parameters, our formulas must evaluate to be non-negative real numbers. But a priori there is no reason that the formulas must be polynomials in the parameters with positive coefficients.

Example 3.13. Consider the two-species ASEP on a lattice of 2 sites with precisely 1 light particle. Let τ be the state $(2, 1)$. There are four tableaux of type τ; these are shown in Figure 12. Therefore the steady state probability that the two-species ASEP is at state τ is equal to $\frac{\alpha\beta u + \alpha\delta q + \alpha u q + \delta q^2}{\mathbf{Z}_{2,1}}$.
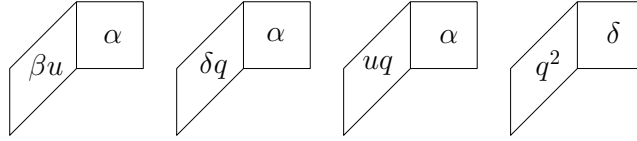

FIGURE 12. The four tableaux of type $(2, 1)$.

Definition 3.14. We also define $\mathbf{Z}_{N,r}(\xi) = \sum_T w(T)\xi^{|T|_2}$, where $|T|_2$ is the number of 2's in type(T), and the sum is over all rhombic staircase tableaux T of size (N, r). In other words, $\mathbf{Z}_{N,r}(\xi)$ is the weight generating function for tableaux T of size (N, r), with an extra variable ξ keeping track of the number of heavy particles in type(T).

In the case that $r = 0$, we write $\mathbf{Z}_N$ and $\mathbf{Z}_N(\xi)$ for $\mathbf{Z}_{N,r}$ and $\mathbf{Z}_{N,r}(\xi)$, respectively.

Remark 3.15. When $r = 0$, rhombic staircase tableaux recover the staircase tableaux of [CW11], and Theorem 3.11 specializes to [CW11, Theorem 3.5]. And when $\gamma = \delta = 0$, rhombic staircase tableaux recover the rhombic alternative tableaux of [MV15], and Theorem 3.11 specializes to [MV15, Theorem 1.2].

We have the following closed formula for $\mathbf{Z}_{N,r}$ when $q = 1$ (and also $u = 1$), which follows easily from [MV15, Theorem 2.19].

Proposition 3.16.

$$\mathbf{Z}_{N,r}(q = 1) = \binom{N}{r} \prod_{i=r}^{N-1} (\alpha + \beta + \gamma + \delta + i(\alpha + \gamma)(\beta + \delta)).$$

Proof. When $\gamma = \delta = 0$, our rhombic staircase tableaux are in bijection with the rhombic alternative tableaux of Mandelshtam and Viennot [MV15]. To obtain a rhombic alternative tableau from a rhombic staircase tableau which contains no γ's or δ's, simply delete every diagonal box containing an α (respectively, β) along with its vertical strip (respectively, horizontal strip). So in the case $\gamma = \delta$, by [MV15, Theorem 2.19], we have that

$$(1) \qquad \mathbf{Z}_{N,r}(q = 1, \gamma = \delta = 0) = \binom{N}{r} \prod_{i=r}^{N-1} (\alpha + \beta + i\alpha\beta).$$

But now note that in the $q = 1$ case, the letters α and γ play an identical role in rhombic staircase tableaux, as do the letters β and δ. Therefore we can compute $\mathbf{Z}_{N,r}(q = 1)$ by replacing each α in (1) by $\alpha + \gamma$ and by replacing each β by $\beta + \delta$. The result follows. $\square$

If we set $\alpha = \beta = \gamma = \delta = 1$ into Proposition 3.16, we obtain the following.

Corollary 3.17. *The number of rhombic staircase tableaux equals* $4^{N-r} \frac{N!}{r!} \binom{N}{r}$.

4. A TABLEAUX FORMULA FOR KOORNWINDER MOMENTS

In this section we will define Koornwinder moments, as in [CW15], and use rhombic staircase tableaux to give an explicit formula for the homogeneous Koornwinder moments. The proof of our formula will be given in Section 8.

Definition 4.1. Let $\mathbf{z} = (z_1, \ldots, z_m)$, $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition, and a, b, c, d, q, t be generic complex parameters. The *Koornwinder polynomials* $P_\lambda(\mathbf{z}; a, b, c, d|q, t)$ are multivariate orthogonal polynomials which are the type BC-case of Macdonald polynomials. More specifically, $P_\lambda(\mathbf{z}; a, b, c, d|q, t)$ is the unique Laurent polynomial which is invariant under permutation and inversion of variables, with leading monomial $\mathbf{z}^\lambda$, and orthogonal with respect to the *Koornwinder density*

$$\prod_{1 \leq i < j \leq m} \frac{(z_i z_j, z_i/z_j, z_j/z_i, 1/z_i z_j; q)_\infty}{(t z_i z_j, t z_i/z_j, t z_j/z_i, t/z_i z_j; q)_\infty} \prod_{1 \leq i \leq m} \frac{(z_i^2, 1/z_i^2; q)_\infty}{(a z_i, a/z_i, b z_i, b/z_i, c z_i, c/z_i, d z_i, d/z_i; q)_\infty}$$

on the unit torus $|z_1| = \cdots = |z_m| = 1$, where the parameters satisfy $|a|, |b|, |c|, |d|, |q|, |t| < 1$.

It is well-known that when $q = t$, we have

$$P_\lambda(\mathbf{z}; a, b, c, d|q, q) = \text{const} \cdot \frac{\det(p_{m-j+\lambda_j}(z_i; a, b, c, d|q))_{i,j=1}^m}{\det(p_{m-j}(z_i; a, b, c, d|q))_{i,j=1}^m},$$

where the p_i's are the Askey-Wilson polynomials.

For orthogonal polynomials in one variable, the kth moment μ_k is defined to be the integral of x^k with respect to the associated density function. For the multivariate Koornwinder polynomials, there are several ways that we could define moments. One way would be to integrate a monomial in $x_1, \ldots, x_m$ (here we set $x_i = \frac{z_i + z_i^{-1}}{2}$) with respect to the Koornwinder density. Following a suggestion of Eric Rains [Rai], we instead define Koornwinder moments by integrating Schur polynomials $s_\lambda(x_1, \ldots, x_m)$ with respect to the Koornwinder density.

Definition 4.2. [CW15, Definition 2.6] Let $I_k(f(x_1, \ldots, x_m); a, b, c, d; q, q)$ denote the result of integrating the function $f(x_1, \ldots, x_m)$ with respect to the Koornwinder density. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition. We define the *Koornwinder moment*

$$M_\lambda = I_k(s_\lambda(x_1, \ldots, x_m); a, b, c, d; q, q).$$

We also define another version of Koornwinder moments as follows.

Definition 4.3. [CW15, Definition 4.2] Set $\mathcal{Z}_N(\xi) = \prod_{i=0}^{N-1} (\alpha\beta - q^i \gamma\delta)^{-1} \mathbf{Z}_N(\xi)$.[2] Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, we define the *Koornwinder moment* at $q = t$ to be

$$(2) \qquad K_\lambda(\xi) = \frac{\det(\mathcal{Z}_{\lambda_i + m - i + m - j}(\xi))_{i,j=1}^m}{\det(\mathcal{Z}_{2m-i-j}(\xi))_{i,j=1}^m}.$$

---

[2]We need to pass from $\mathbf{Z}_N$ to $\mathcal{Z}_N$ because in [CW15] we worked with the version $\mathcal{Z}_N$ of the partition function.

The fact that we refer to both quantities M_λ and $K_\lambda(\xi)$ as Koornwinder moments is justified by the following result.

Proposition 4.4. [CW15, Proposition 4.3] *Let* $\lambda = (\lambda_1, \ldots, \lambda_m)$ *be a partition. We have that*

$$M_\lambda(a, b, c, d|q) = \left(\frac{1-q}{2i}\right)^{|\lambda|} K_\lambda(-1; \alpha, \beta, \gamma, \delta; q),$$

where $|\lambda| = \sum_j \lambda_j$, *and* $\alpha, \beta, \gamma, \delta$ *are related to* a, b, c, d *as follows:*
(3)
$$\alpha = \frac{1-q}{1 - ac + ai + ci}, \ \beta = \frac{1-q}{1 - bd - bi - di}, \ \gamma = \frac{(1-q)ac}{1 - ac + ai + ci}, \ \delta = \frac{(1-q)bd}{1 - bd - bi - di},$$

where i *is the square root of* -1.

We refer to moments of the form K_λ, where $\lambda = (k, 0, \ldots, 0)$ as *homogeneous Koornwinder moments*. Note that in this case, M_λ is the result of integrating the homogeneous symmetric function $h_k(x_1, \ldots, x_m)$ with respect to the Koornwinder density. (Here m is the number of parts of λ, i.e. $m - 1$ is the number of 0's in the partition.) This is quite a natural analogue of the notion of moment for orthogonal polynomials in one variable.

We note also that the homogeneous Koornwinder moments play a role similar to that of the homogeneous symmetric functions: [CW15, Theorem 5.1] gave a Jacobi-Trudi formula that expresses a general Koornwinder moment as a determinant in the homogeneous ones.

The following is the second main result of this paper: it gives a tableaux formula for the homogeneous Koornwinder moments.

Theorem 4.5. *The homogeneous Koornwinder moment* $K_{(N-r,0,0,\ldots,0)}(\xi)$ *(where there are precisely* r *0's in the partition) can be computed in terms of rhombic staircase tableaux via*

$$K_{(N-r,0,0,\ldots,0)}(\xi) = \frac{1}{(1-q)^r \prod_{i=0}^{N-r-1}(\alpha\beta - q^{i+2r}\gamma\delta)} \mathbf{Z}_{N,r}(\xi),$$

where $\mathbf{Z}_{N,r}(\xi)$ *is the weight generating function for rhombic staircase tableaux given in Definition 3.14.*

5. A Matrix Ansatz for the two-species ASEP

In this section we will prove Theorem 5.4, which shows that if one has matrices and vectors satisfying certain relations, then one can use appropriate matrix products to compute steady state probabilities of the two-species ASEP. Our Matrix Ansatz is a generalization of the one given by Uchiyama [Uch08], which in turn generalized the one for the usual ASEP in [DEHP93]. We will be working with a distinguished matrix A together with two families of matrices $\{D^{(i)}\}$ and $\{E^{(i)}\}$ where i ranges over the non-negative integers $\mathbb{N}$. Note that in this section, we have set $u = 1$; setting one parameter equal to 1 does not lose any information.

Before stating the theorem, we need to introduce some terminology.

Definition 5.1. A word $X = X_1 \ldots X_m$ in $\{A, D^{(i)}, E^{(j)} \mid i, j \in \mathbb{N}\}$ is *A-compatible* if $X_k = D^{(i)}$ or $X_k = E^{(i)}$ if and only if there are precisely i A's in $X_1 X_2 \ldots, X_{k-1}$.

More generally, we say that a word $X = X_1 \ldots X_m$ in $\{A, D^{(i)}, E^{(j)} \mid i, j \in \mathbb{N}\}$ is *(A, t)-compatible* if $X_k = D^{(t+i)}$ or $X_k = E^{(t+i)}$ if and only if there are precisely i A's in $X_1 X_2 \ldots, X_{k-1}$.

For example, the word $X = D^{(0)} A E^{(1)} A A D^{(3)}$ is A-compatible, and $X = D^{(3)} A E^{(4)} A A D^{(6)}$ is $(A, 3)$-compatible.

Definition 5.2. Let $|X|$ denote the number of letters in X, and let $|X|_A$ denote the number of A's in the word X. Set $\|X\| = |X| + |X|_A$.

Definition 5.3. Given a word $\tau = (\tau_1, \ldots, \tau_N)$ in the letters $\{0, 1, 2\}$, there is a unique A-compatible word $X(\tau)$ with $|X(\tau)| = N$ which is obtained by replacing each $\tau_i = 0$ by an $E^{(j)}$, each $\tau_i = 1$ by an A, and each $\tau_i = 2$ by a $D^{(j)}$. Conversely, given an A-compatible word X, there is a unique sequence $\tau = \tau(X) = (\tau_1, \ldots, \tau_N)$ in the letters $\{0, 1, 2\}$ such that $X = X(\tau)$.

For example, if $\tau = (0, 2, 1, 1, 2, 1, 0)$, then $X(\tau) = E^{(0)} D^{(0)} A A D^{(2)} A E^{(3)}$. Conversely if $X = E^{(0)} D^{(0)} A A D^{(2)} A E^{(3)}$ then $\tau(X) = (0, 2, 1, 1, 2, 1, 0)$.

Theorem 5.4. *Let $\langle W|$ and $|V\rangle$ be vectors and let A, $\{D^{(i)} \mid i \in \mathbb{N}\}$, and $\{E^{(i)} \mid i \in \mathbb{N}\}$ be matrices. Let $\{\lambda_n\}$ be a family of constants indexed by the non-negative integers. Suppose that for any A-compatible word X with $|X|_A = t$, any (A, t)-compatible word Y, and any $(A, t + 1)$-compatible word Y', the following relations hold:*

(I) $\langle W|X(D^{(t)} E^{(t)} - q E^{(t)} D^{(t)}) Y |V\rangle = \lambda_{\|XY\|+2} \langle W|X(D^{(t)} + E^{(t)}) Y |V\rangle$

(II) $\beta \langle W|X D^{(t)}|V\rangle - \delta \langle W|X E^{(t)}|V\rangle = \lambda_{\|X\|+1} \langle W|X|V\rangle$

(III) $\alpha \langle W|E^{(0)} X|V\rangle - \gamma \langle W|D^{(0)} X|V\rangle = \lambda_{\|X\|+1} \langle W|X|V\rangle$

(IV) $\langle W|X(D^{(t)} A - q A D^{(t+1)}) Y'|V\rangle = \lambda_{\|XY'\|+3} \langle W|X A Y'|V\rangle$

(V) $\langle W|X(A E^{(t+1)} - q E^{(t)} A) Y'|V\rangle = \lambda_{\|XY'\|+3} \langle W|X A Y'|V\rangle$

Let $Z_{N,r} = \sum_X \langle W|X|V\rangle$, where the sum is over all A-compatible words X such that $|X| = N$ and $|X|_A = r$. Then in the two-species ASEP on a lattice of N sites with precisely r light particles, the steady state probability of state $\tau = (\tau_1, \ldots, \tau_N)$ is equal to

$$\frac{\langle W|X(\tau)|V\rangle}{Z_{N,r}}.$$

For example, the steady state probability of state $(2, 0, 1, 1, 2, 2)$ is equal to $\frac{\langle W|D^{(0)} E^{(0)} A A D^{(2)} D^{(2)}|V\rangle}{Z_{6,2}}$.

Proof. Let $f_N(\tau_1, \ldots, \tau_N)$ denote the (unnormalized) steady state probability of state $\tau = (\tau_1, \ldots, \tau_N)$ of the two-species ASEP, where $\tau \in \{0, 1, 2\}^N$ and τ contains precisely r 1's. We are in the steady state if the net rate of entering each state τ is

0, or in other words, the following expression equals 0:

(4)
$$\mathbb{1}(\tau_1 = 0)[-\alpha f_N(0, \tau_2, \ldots, \tau_N) + \gamma f_N(2, \tau_2, \ldots, \tau_N)]$$

(5)
$$+\mathbb{1}(\tau_1 = 2)[\alpha f_N(0, \tau_2, \ldots, \tau_N) - \gamma f_N(2, \tau_2, \ldots, \tau_N)]$$

(6)
$$+\sum_{i=1}^{N-1} \mathbb{1}(\tau_i = 2 \text{ and } \tau_{i+1} = 0)[-f_N(\tau_1, \ldots, 2, 0, \ldots, \tau_N) + q f_N(\tau_1, \ldots, 0, 2, \ldots, \tau_N)]$$

(7)
$$+\sum_{i=1}^{N-1} \mathbb{1}(\tau_i = 0 \text{ and } \tau_{i+1} = 2)[f_N(\tau_1, \ldots, 2, 0, \ldots, \tau_N) - q f_N(\tau_1, \ldots, 0, 2, \ldots, \tau_N)]$$

(8)
$$+\sum_{i=1}^{N-1} \mathbb{1}(\tau_i = 1 \text{ and } \tau_{i+1} = 0)[-f_N(\tau_1, \ldots, 1, 0, \ldots, \tau_N) + q f_N(\tau_1, \ldots, 0, 1, \ldots, \tau_N)]$$

(9)
$$+\sum_{i=1}^{N-1} \mathbb{1}(\tau_i = 0 \text{ and } \tau_{i+1} = 1)[f_N(\tau_1, \ldots, 1, 0, \ldots, \tau_N) - q f_N(\tau_1, \ldots, 0, 1, \ldots, \tau_N)]$$

(10)
$$+\sum_{i=1}^{N-1} \mathbb{1}(\tau_i = 2 \text{ and } \tau_{i+1} = 1)[-f_N(\tau_1, \ldots, 2, 1, \ldots, \tau_N) + q f_N(\tau_1, \ldots, 1, 2, \ldots, \tau_N)]$$

(11)
$$+\sum_{i=1}^{N-1} \mathbb{1}(\tau_i = 1 \text{ and } \tau_{i+1} = 2)[f_N(\tau_1, \ldots, 2, 1, \ldots, \tau_N) - q f_N(\tau_1, \ldots, 1, 2, \ldots, \tau_N)]$$

(12)
$$+\mathbb{1}(\tau_N = 0)[\beta f_N(\tau_1, \ldots, \tau_{N-1}, 2) - \delta f_N(\tau_1, \ldots, \tau_{N-1}, 0)]$$

(13)
$$+\mathbb{1}(\tau_N = 2)[-\beta f_N(\tau_1, \ldots, \tau_{N-1}, 2) + \delta f_N(\tau_1, \ldots, \tau_{N-1}, 0)].$$

Above, we use $\mathbb{1}$ to denote the boolean function taking value 1 or 0 based on whether its argument is true or false. And in (6) through (11) above, the arguments $2, 0$ and $0, 2$; $1, 0$ and $0, 1$; and $2, 1$ and $1, 2$, are in positions i and $i + 1$.

We need to prove that if we substitute the quantity $\langle W | X(\tau_1, \ldots, \tau_n) | V \rangle$ for $f_N(\tau_1, \ldots, \tau_N)$, the sum of the expressions in (4) through (13) is 0. After making this substitution, the assumptions of the theorem imply the following:

• The sum of (4) and (5) is only nonzero if $\tau_1 = 0$ or 2, and in that case, is equal to $\mp \lambda_{N+r} f_{N-1}(\tau_2, \ldots, \tau_N)$, where the sign is chosen based on whether τ_1 is 0 or 1.

- The sum of the ith terms of (6) and (7) is only nonzero if $\{\tau_i, \tau_{i+1}\} = \{0, 2\}$, and in that case, equals $\mp\lambda_{N+r}[f_{N-1}(\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_N) + f_{N-1}(\tau_1, \ldots, \tau_i, \tau_{i+2}, \ldots, \tau_N)]$, based on whether τ_i equals 2 or 0.
- The sum of the ith terms of (8) and (9) is only nonzero if $\{\tau_i, \tau_{i+1}\} = \{0, 1\}$, and in that case, is equal to $\mp\lambda_{N+r}[f_{N-1}(\tau_1, \ldots, \tau_{i-1}, 1, \tau_{i+2}, \ldots, \tau_N)]$, based on whether τ_i equals 1 or 0.
- The sum of the ith terms of (10) and (11) is only nonzero if $\{\tau_i, \tau_{i+1}\} = \{1, 2\}$, and in that case, is equal to $\mp\lambda_{N+r}[f_{N-1}(\tau_1, \ldots, \tau_{i-1}, 1, \tau_{i+2}, \ldots, \tau_N)]$, based on whether τ_i equals 2 or 1.
- The sum of (12) and (13) is only nonzero if $\tau_N = 0$ or 2, and in that case, is equal to $\pm\lambda_{N+r}f_{N-1}(\tau_1, \ldots, \tau_{N-1})$ based on whether τ_N is 0 or 2.

Then using these conditions, it is easy to verify that the sum of (4) through (13) is 0, since all the terms involving f_{N-1} cancel out. For example, if $(\tau_1, \ldots, \tau_N) = (0, 2, 1, 0, 2)$ (so that $N = 5$ and $r = 1$), we get that the sum of (4) through (13) equals:

$$
\begin{aligned}
-\lambda_6 f_4(2, 1, 0, 2) &\qquad \text{(from (4) and (5))} \\
+\lambda_6[f_4(2, 1, 0, 2) + f_4(0, 1, 0, 2)] &\qquad \text{(from the } i = 1 \text{ terms of (6) and (7))} \\
-\lambda_6 f_4(0, 1, 0, 2) &\qquad \text{(from the } i = 2 \text{ terms of (10) and (11))} \\
-\lambda_6 f_4(0, 2, 1, 2) &\qquad \text{(from the } i = 3 \text{ terms of (8) and (9))} \\
+\lambda_6[f_4(0, 2, 1, 2) + f_4(0, 2, 1, 0)] &\qquad \text{(from the } i = 4 \text{ terms of (6) and (7))} \\
-\lambda_6 f_4(0, 2, 1, 0) &\qquad \text{(from (12) and (13))}
\end{aligned}
$$

Clearly the sum of these terms is 0. $\qquad\square$

6. The matrices describing rhombic staircase tableaux

In this section we will define a row vector $\langle W|$, a column vector $|V\rangle$, a distinguished matrix A, and two families of matrices $\{D^{(t)} \mid t \in \mathbb{N}\}$ and $\{E^{(t)} \mid t \in \mathbb{N}\}$. We will then explain how appropriate matrix products compute generating functions for rhombic staircase tableaux.

6.1. The definition of the matrices.

Definition 6.1. In particular, our matrices and vectors are not finite. We define "row" and "column" vectors $\langle W| = (W_{ik})$ and $|V\rangle = (V_{j\ell})$, where the indices i, j, k, ℓ range over the non-negative integers $\mathbb{N}$. For each $t \in \mathbb{N}$, we also define matrices $D^{(t)} = (D^{(t)}_{i,j,k,\ell})$ and $E^{(t)} = (E^{(t)}_{i,j,k,\ell})$, as well as the matrix $A = (A_{i,j,k,\ell})$ (where again the indices i, j, k, ℓ range over $\mathbb{N}$) by the following:

$$
W_{ik} = \begin{cases} 1 & \text{if } i = k = 0, \\ 0 & \text{otherwise,} \end{cases}
$$

$$
V_{j\ell} = 1 \text{ always.}
$$

$$
D^{(t)}_{i,j,k,\ell} = \begin{cases} 0 & \text{if } j < i \text{ or } \ell > k+1 \\ \alpha q^i & \text{if } k = 0, \ell = 1, \text{ and } i = j \\ \delta q^i(q^t + (\alpha + \gamma q^t)[t]_q) & \text{if } k = 0, \ell = 0, \text{ and } j = i+1 \\ \delta(D^{(t)} + E^{(t)})_{i,j-1,k-1,\ell} + D^{(t)}_{i,j,k-1,\ell-1} & \text{otherwise.} \end{cases}
$$

$$E_{i,j,k,\ell}^{(t)} = \begin{cases} 0 & \text{if } j < i \text{ or } \ell > k+1 \\ \gamma q^{2t+i} & \text{if } k = 0, \ell = 1, \text{ and } i = j \\ \beta q^i(q^t + (\alpha + \gamma q^t)[t]_q) & \text{if } k = 0, \ell = 0, \text{ and } i = j \\ \beta(D^{(t)} + E^{(t)})_{i,j,k-1,\ell} + qE_{i,j,k-1,\ell-1}^{(t)} & \text{otherwise.} \end{cases}$$

$$A_{i,j,k,\ell} = \begin{cases} 0 & \text{if } \ell > k \text{ or } j - i > k - \ell \\ q^{2i} & \text{if } i = j \text{ and } k = \ell = 0 \\ \beta A_{i,j,k-1,\ell} + \delta q A_{i,j-1,k-1,\ell} + q A_{i,j,k-1,\ell-1} & \text{otherwise.} \end{cases}$$

In the formulas above we use $[t]_q$ to denote the q-analogue $1 + q + \cdots + q^{t-1} = \frac{q^t - 1}{q - 1}$. And by convention, if any subscript i, j, k or ℓ is negative, then the matrix entry $M_{ijk\ell} = 0$.

When we write $M_{i,j,k,\ell}$, we are thinking of the two coordinates i and k as specifying a "row" of a matrix, and the two coordinates j and ℓ as specifying a "column" of a matrix. Therefore matrix multiplication is defined by

$$(MM')_{i,j,k,\ell} = \sum_{a,b} M_{i,a,k,b} M'_{a,j,b,\ell},$$

where a and b vary over $\mathbb{N}$. Note that when M and M' are products of $D^{(t)}$'s, $E^{(t)}$'s, and A's, the sum on the right-hand-side is finite. Specifically, if M is a product of s factors as above, then $M_{i,a,k,b}$ is 0 unless $a + b \leq i + k + s$. This can be shown by induction from the definition of the matrices, or by using the combinatorial interpretation given in Lemma 6.5.

Although we don't need it in this paper, we note that one can use the recursive definition of A to prove the following explicit formula for $A_{i,j,k,\ell}$.

Remark 6.2.
$$A_{i,j,k,\ell} = \delta^{j-i} \beta^{k-\ell-(j-i)} q^{\ell+i+j} \binom{k}{\ell} \binom{k-\ell}{j-i}$$

6.2. **The combinatorial interpretation of our matrices in terms of tableaux.** We say that a horizontal strip of a rhombic staircase tableau T is a horizontal β-*strip* if the leftmost tile in that strip containing a Greek letter contains a β (or βu). Note that every tile to the left of that β must contain a u or u^2. Similarly we will talk about horizontal δ-*strips*; in this case, every tile to the left of that δ must contain a q or q^2. We will also talk about horizontal α/γ-strips, which is shorthand for horizontal strips which are α-strips *or* γ-strips.

Recall the notions of A-compatible and (A, t)-compatible from Definition 5.1.

Theorem 6.3. *Suppose that X is an (A, t)-compatible word in $\{A, D^{(i)}, E^{(j)} \mid i, j \in \mathbb{N}\}$. Set $u = 1$. Then we have that:*
  (1) $X_{ijk\ell}$ is the (weight) generating function for all ways of adding $|X|$ new columns of type $\tau(X)$ to a rhombic staircase tableau with precisely t 1-columns, i horizontal δ-strips, and k horizontal α/γ-strips, so as to obtain a new tableau with j horizontal δ-strips ℓ horizontal α/γ-strips;
  (2) if $t = 0$, then $(WX)_{j\ell}$ is the generating function for rhombic staircase tableaux of type $\tau(X)$ which have j horizontal δ-strips and ℓ horizontal α/γ-strips;

(3) if $t = 0$, then $\langle W|X|V \rangle$ is the generating function for all rhombic staircase tableaux of type $\tau(X)$.

Recall the definition of $\mathbf{Z}_{N,r}(\xi)$ from Definition 3.14.

Corollary 6.4. *We have that $\mathbf{Z}_{N,r}(\xi) = \sum_X \xi^{|X|_D} \langle W|X|V \rangle$, where the sum is over all A-compatible words X such that $|X| = N$ and $|X|_A = r$.*

The main step in proving Theorem 6.3 is the following lemma, which says that the matrices A, $D^{(i)}$ and $E^{(j)}$ are "transfer matrices" for building rhombic staircase tableaux.

Lemma 6.5. *$A_{i,j,k,\ell}$ is the generating function for the weights of all possible new 1-columns that we could add to the left of a rhombic staircase tableau with i horizontal δ-strips and k horizontal α/γ-strips, so as to obtain a new tableau which has j horizontal δ-strips and ℓ horizontal α/γ-strips.*

$D^{(t)}_{i,j,k,\ell}$ is the generating function for the weights of all possible new 02-columns with an α or δ in the bottom box that we could add to the left of a rhombic staircase tableau with precisely t 1-columns, i horizontal δ-strips, and k horizontal α/γ-strips, so as to obtain a tableau which has j horizontal δ-strips and ℓ horizontal α/γ-strips. Similarly for $E^{(t)}_{i,j,k,\ell}$, where the new 02-column has a β or γ in the bottom box.

Proof. Let $\tilde{A}_{ijk\ell}$ denote the generating function for the weights of all possible new 1-columns that we could add to the left of a rhombic staircase tableau with i horizontal δ-strips and k horizontal α/γ-strips, so as to obtain a new tableau which has j horizontal δ-strips and ℓ horizontal α/γ-strips. We will show that $\tilde{A}_{ijk\ell} = A_{ijk\ell}$ by showing that $\tilde{A}$ satisfies the same recurrences.

Note that $\tilde{A}_{ijk\ell} = 0$ if $\ell > k$ because a 1-column consists of tall rhombi, and a tall rhombus may never contain an α or γ. Now consider a potential 1-column that contributes to $\tilde{A}_{ijk\ell}$, and let h be the number of its tiles containing a δ. Then $h = j - i$, but also $k - \ell \geq h$. It follows that $k - l \geq j - i$, and hence $\tilde{A}_{ijk\ell} = 0$ if $j - i > k - \ell$.

Now suppose that $i = j$ and $k = \ell = 0$. Then the horizontal strips of the 1-columns contributing to $\tilde{A}_{ijk\ell}$ are all β-strips or δ-strips. So in particular, each tile must either contain a $u^2 = 1$ or a q^2, based on whether its horizontal strip is a β-strip or a δ-strip. It follows that there is only one column, and it has weight q^{2i}.

Finally suppose that $k > 0$. (Note that if $k = 0$ then this implies that $\ell = 0$ and $j \geq i$, in which case we are in one of the cases considered previously.) Let us consider the different ways that we can fill in the lowest tile B of a 1-column contributing to $\tilde{A}_{ijk\ell}$, such that B belongs to an α/γ-strip. (Since $k > 0$, such a tile exists.) Either B contains a βu or a δq or is empty, in which case Figure 11 implies that it gets assigned a qu. If B contains a βu, then the process of filling in the other tiles has a generating function of $\tilde{A}_{i,j,k-1,\ell}$, so we get a contribution of $\beta \tilde{A}_{i,j,k-1,\ell}$ (recall that we are setting $u = 1$). If B contains a δq, then filling in the other tiles has generating function $\tilde{A}_{i,j-1,k-1,\ell}$, so we get a contribution of $\delta q \tilde{A}_{i,j-1,k-1,\ell}$. And if B contains a qu, then filling in the other tiles has generating function $\tilde{A}_{i,j,k-1,\ell-1}$, so we get a contribution of $q \tilde{A}_{i,j,k-1,\ell-1}$.

This shows that if $k > 0$, then $\tilde{A}_{i,j,k,\ell} = \beta \tilde{A}_{i,j,k-1,\ell} + \delta q \tilde{A}_{i,j-1,k-1,\ell} + q \tilde{A}_{i,j,k-1,\ell-1}$.

Let $\tilde{D}_{ijk\ell}^{(t)}$ denote the generating function for 02-columns described in the second sentence of Lemma 6.5. As before, we want to show that $\tilde{D}^{(t)}$ satisfies the defining recurrence for $D^{(t)}$.

Note that $\tilde{D}_{ijk\ell}^{(t)} = 0$ if $j < i$ because adding a new column to a rhombic staircase tableau never decreases the number of δ-strips. Also $\tilde{D}_{ijk\ell}^{(t)} = 0$ if $\ell > k + 1$ because when we add a new column we can never increase the number of α/γ-strips by more than 1.

Now suppose that $k = 0$. Since we are starting from a tableau with 0 horizontal α/γ-strips, it follows that any new 02-column that we can add to the tableau will have either an α or δ in the bottom square, with all squares above that one empty, and with t short rhombi on top of the squares. If we put an α in the bottom square, then the short rhombi in that column must also be empty, so the weight of the resulting column will be αq^i. The resulting tableau will have $\ell = 1$ horizontal α/γ-strips and $j = i$ horizontal δ-strips. On the other hand, if we put a δ in the bottom square of the new column, the weight of the *squares* in the column will be δq^i. As for the short rhombi, we can either leave them all empty (so that they get weight $(qu)^t = q^t$), or we can put an αu or $\gamma q u$ into one of them. There are t short rhombi, and the weight of the t different ways of putting an αu into one of them is $\alpha(1 + q + \cdots + q^{t-1})$. The weight of the t different ways of putting a γq into one of them is $\gamma q^t(1 + q + \cdots + q^{t-1})$. Therefore the weight of the various columns we can add with a δ in the bottom square is $\delta q^i(q^t + (\alpha + \gamma q^t)[t]_q)$. And any resulting tableau will have $\ell = 0$ horizontal α/γ-strips, and $j = i + 1$ horizontal δ-strips.

From this discussion it follows that $\tilde{D}_{ijk\ell}^{(t)} = \alpha q^i$ when $k = 0$, $\ell = 1$, and $j = i$, and $\tilde{D}_{ijk\ell}^{(t)} = \delta q^i(q^t + (\alpha + \gamma q^t)[t]_q)$ when $k = \ell = 0$ and $j = i + 1$.

In all other situations, we can assume that $k \geq 1$. Suppose that we are adding a new column C with an α or δ at the bottom to the left of a rhombic staircase tableau with i horizontal δ-strips and k horizontal α/γ-strips, so as to create a new tableau T. Consider the lowest square B of C which belongs to an α/γ-strip in T (such a square exists since $k \geq 1$). If we fill B with an α, β, γ or δ, then the bottom square of C must contain a δ (not an α). In this case, if we ignore that bottom δ, then our choices for C are exactly the same as our choices would be for adding a new column to the left of a tableau with i horizontal δ-strips and $k - 1$ horizontal α/γ-strips. Therefore, filling B with an α, β, γ or δ gives us a contribution of $\delta(\tilde{D}^{(t)} + \tilde{E}^{(t)})_{i,j-1,k-1,\ell}$ to our generating function.

On the other hand, if we leave B empty, then this square will get a weight $u = 1$. Filling the rest of the column C is like adding a new column to a tableau with i horizontal δ-strips and $k-1$ horizontal α/γ-strips. Therefore leaving B empty gives us a contribution of $\tilde{D}_{i,j,k-1,\ell-1}^{(t)}$ to our generating function.

It follows that when $k \geq 1$, $\tilde{D}_{i,j,k,\ell}^{(t)} = \delta(\tilde{D}^{(t)} + \tilde{E}^{(t)})_{i,j-1,k-1,\ell} + \tilde{D}_{i,j,k-1,\ell-1}^{(t)}$.

The proof of the statement of the lemma concerning $E_{i,j,k,\ell}^{(t)}$ is analogous to the proof we gave for $D_{i,j,k,\ell}^{(t)}$. □

Proof of Theorem 6.3. The first item follows from Lemma 6.5 and the definition of matrix multiplication. Assuming that $t = 0$, multiplying at the left by $\langle W|$ has the

effect that we start with the empty tableau and then add columns according to X: so $(WX)_{j\ell}$ is the generating function for rhombic staircase tableaux of type $\tau(X)$ which have j horizontal δ-strips and ℓ horizontal α/γ-strips. Finally, multiplying $\langle W|X$ on the right by $|V\rangle$ has the effect of summing over all δ and ℓ, so $\langle W|X|V\rangle$ is the generating function for all rhombic staircase tableaux of type $\tau(X)$. $\square$

7. The proof that our matrices satisfy the Matrix Ansatz

Recall that in Section 6 we defined some matrices and vectors, and showed in Theorem 6.3 (3) that appropriate matrix products compute generating functions for rhombic staircase tableaux. In this section, we will show in Theorem 7.1 that these matrix products also satisfy the Matrix Ansatz relations of Theorem 5.4. This implies that these matrix products also compute steady state probabilities for the two-species ASEP. Our main theorem, Theorem 3.11, will therefore follow once we have proved Theorem 7.1.

Theorem 7.1. *The matrices and vectors from Definition 6.1 satisfy relations (I)-(V) of Theorem 5.4, with $\lambda_n = \alpha\beta - \gamma\delta q^{n-1}$.*

The proof of Theorem 7.1 is long and technical, but we will try to make the structure of the proof transparent from the titles of the following subsections.

7.1. **Reducing Theorem 7.1 to (2) and (4) from Proposition 7.4.** In this section we show that Theorem 7.1 is a consequence of equations (2) and (4) from Proposition 7.4.

Lemma 7.2. *Equation (III) of Theorem 5.4 holds.*

Proof. We use a simple combinatorial argument to show (III), namely that for any A-compatible word X, we have that

$$\alpha\langle W|E^{(0)}X|V\rangle - \gamma\langle W|D^{(0)}X|V\rangle = \lambda_{\|X\|+1}\langle W|X|V\rangle.$$

By Theorem 6.3, we can interpret each of the terms in this equation in terms of rhombic staircase tableaux, for example $\langle W|E^{(0)}X|V\rangle$ (respectively, $\langle W|D^{(0)}X|V\rangle$) is the generating function for tableaux of type $\tau(E^{(0)}X)$ (resp. $\tau(D^{(0)}X)$). Such a tableau has a β or γ (resp. α or δ) in its upper-right corner box.

See the first four terms of Figure 13 for a pictorial representation of the quantity $\alpha\langle W|E^{(0)}X|V\rangle - \gamma\langle W|D^{(0)}X|V\rangle$. Note that the u's in the first term, and the q's in the fourth term, are forced by the presence of the β and δ, respectively. All tiles left blank can be filled arbitrarily, so as to create a rhombic staircase tableaux of the appropriate type. So e.g. the first term represents the sum of the weights of all tableaux of type $\tau(E^{(0)}X)$ which can be obtained by filling in its blank tiles.

It is easy to see that the second and third terms in Figure 13 are equal to each other and hence cancel out. Moreover, one may delete the topmost horizontal strip from each of the first and fourth terms, so as to obtain the fifth and sixth terms in Figure 13. (Remember that we set $u = 1$). To complete the proof, we simply observe that the sum of the fifth and sixth terms in Figure 13 is equal to $\lambda_{\|X\|+1}\langle W|X|V\rangle$. $\square$

To prove the other equations from Theorem 5.4, we reduce them to the more refined statements of Proposition 7.4. First we prove the following useful lemma.

$$\alpha \;\begin{array}{c}\text{(tableau)}\end{array} + \alpha \;\begin{array}{c}\text{(tableau)}\end{array} - \gamma \;\begin{array}{c}\text{(tableau)}\end{array} - \gamma \;\begin{array}{c}\text{(tableau)}\end{array} = \alpha\beta \;\begin{array}{c}\text{(tableau)}\end{array} - \gamma\delta q^{\|X\|} \begin{array}{c}\text{(tableau)}\end{array}$$

FIGURE 13. The combinatorial proof of (III)

Lemma 7.3. *For any (A,t)-compatible word Y in $\{A, D^{(i)}, E^{(j)} \mid i, j \in \mathbb{N}\}$, we have that $Y_{i,j,k,\ell} = q^{\|Y\|} Y_{i-1,j-1,k,\ell}$.*

Proof. To prove the lemma we use the combinatorial interpretation given by Lemma 6.3. Note that both $Y_{i,j,k,\ell}$ and $Y_{i-1,j-1,k,\ell}$ enumerate the ways of adding $|Y|$ new columns to a fixed tableau T so as to increase by $j - i$ the number of horizontal δ-strips, and to increase by $\ell - k$ the number of horizontal α/γ-strips. The only difference is the initial number of horizontal δ-strips. Since $Y_{i,j,k,\ell}$ has one extra initial horizontal δ-strips, this will contribute $|Y|$ extra empty tiles which each get a weight of q or q^2. More specifically, there will be $|Y| - |Y|_A$ square tiles which each get a q, and $|Y|_A$ tall rhombi which each get a q^2. Therefore, the difference in weight will be $q^{\|Y\|}$. $\qquad\square$

Proposition 7.4. *Let r be a non-negative integer and let X be any A-compatible word in $\{A, D^{(i)}, E^{(j)} \mid i, j \in \mathbb{N}\}$ containing precisely r A's. To prove equations (I), (II), (IV), and (V) from Theorem 5.4, it suffices to prove the following:*

(1) $(WXD^{(r)}E^{(r)})_{j,\ell} = q(WXE^{(r)}D^{(r)})_{j,\ell} + \alpha\beta(WX(D^{(r)} + E^{(r)}))_{j,\ell}$
$$\qquad\qquad\qquad - \gamma\delta q^{\|X\|+1}(WX(D^{(r)} + E^{(r)}))_{j-1,\ell}$$

(2) $\beta(WXD^{(r)})_{j,\ell} = \delta(WXE^{(r)})_{j-1,\ell} + \alpha\beta(WX)_{j,\ell-1} - \gamma\delta q^{\|X\|}(WX)_{j-1,\ell-1}$

(4) $(WXD^{(r)}A)_{j,\ell} = q(WXAD^{(r+1)})_{j,\ell} + \alpha\beta(WXA)_{j,\ell} - \gamma\delta q^{\|X\|+2}(WXA)_{j-1,\ell}$

(5) $(WXAE^{(r+1)})_{j,\ell} = q(WXE^{(r)}A)_{j,\ell} + \alpha\beta(WXA)_{j,\ell} - \gamma\delta q^{\|X\|+2}(WXA)_{j-1,\ell}$

Proof. We will show that (1) implies (I), (2) implies (II), (4) implies (IV), and (5) implies (V).

First note that if we take (2) and sum over all non-negative j and ℓ, we immediately obtain (II), because multiplication on the right by $|V\rangle$ has the effect of summing over all indices.

To show that (1) implies (I), we first claim that if (1) is true, then for any (A, r)-compatible word Y, $(WXD^{(r)}E^{(r)}Y)_{j,\ell}$ is equal to

$$q(WXE^{(r)}D^{(r)}Y)_{j,\ell} + \alpha\beta(WX(D^{(r)} + E^{(r)})Y)_{j,\ell} - \gamma\delta q^{\|X\|+\|Y\|+1}(WX(D^{(r)} + E^{(r)})Y)_{j-1,\ell}.$$

To prove the claim, note that: $(WXD^{(r)}E^{(r)}Y)_{j\ell}$ is equal to:

$$\sum_{i,k}(WXD^{(r)}E^{(r)})_{i,k}Y_{i,j,k,\ell}$$

$$= \sum_{i,k}q(WXE^{(r)}D^{(r)})_{i,k}Y_{i,j,k,\ell} + \alpha\beta\sum_{i,k}(WX(D^{(r)}+E^{(r)}))_{i,k}Y_{i,j,k,\ell}$$

$$-\gamma\delta q^{\|X\|+1}\sum_{i,k}(WX(D^{(r)}+E^{(r)}))_{i-1,k}Y_{i,j,k,l}$$

$$= q(WXE^{(r)}D^{(r)}Y)_{j,\ell} + \alpha\beta(WX(D^{(r)}+E^{(r)})Y)_{j\ell} - \gamma\delta q^{\|X\|+\|Y\|+1}(WX(D^{(r)}+E^{(r)})Y)_{j-1,\ell}.$$

To deduce the final equality above, we applied Lemma 7.3 to the last term.

Now note that if we take the equation of the claim, and sum over all j and ℓ, then we get precisely (I).

The proof that (4) implies (IV) and the proof that (5) implies (V) are completely analogous. For example, to prove the former, we first show that (4) implies that for any $(A, r+1)$-compatible word Y', we have that

$$(WXD^{(r)}AY')_{j,\ell} = q(WXAD^{(r+1)}Y')_{j,\ell}+\alpha\beta(WXAY')_{j,\ell}-\gamma\delta q^{\|X\|+\|Y\|+2}(WXAY')_{j-1,\ell}.$$

We then sum this equation over all j and ℓ to obtain (IV). $\qquad\square$

To complete the proof of Theorem 7.1, we need to prove that equations (1), (2), (4), and (5) from Proposition 7.4 hold. In Lemmas 7.5 and 7.6 below, we will show that (2) and (4) imply (5), and also that (2) implies (1). The rest of the proof will be devoted to proving equations (2) and (4).

Lemma 7.5. *If equations (2) and (4) from Proposition 7.4 hold, then so does equation (5).*

Proof. We use the definition of matrix multiplication to write

$$q(WXE^{(r)}A)_{j,\ell} = q\sum_{i,k}(WXE^{(r)})_{i,k}A_{i,j,k,\ell}.$$

Using (2) to expand $(WXE^{(r)})_{i,k}$, we get the following expression for the right-hand side:

$$q\sum_{i,k}\left(\frac{\beta}{\delta}(WXD^{(r)})_{i+1,k} - \frac{\alpha\beta}{\delta}(WX)_{i+1,k-1} + \gamma q^{\|X\|}(WX)_{i,k-1}\right)A_{i,j,k,\ell}$$

$$= q\sum_{i,k}\left(\frac{\beta}{\delta q^2}(WXD^{(r)})_{i+1,k}A_{i+1,j+1,k,\ell} - \frac{\alpha\beta}{\delta q^2}(WX)_{i+1,k-1}A_{i+1,j+1,k,\ell} + \gamma q^{\|X\|}(WX)_{i,k-1}A_{i,j,k,\ell}\right).$$

Note that to get the second line above from the first, we used Lemma 7.3 to replace $A_{i,j,k,\ell}$ by $q^{-2}A_{i+1,j+1,k,\ell}$ in the first two terms. We now use the recurrence for $A_{i,j,k,\ell}$ (see Definition 6.1) on the second and third terms, obtaining the following:

$$q\sum_{i,k}\left(\frac{\beta}{\delta q^2}(WXD^{(r)})_{i+1,k}A_{i+1,j+1,k,\ell}-\frac{\alpha\beta}{\delta q^2}(WX)_{i+1,k-1}(\beta A_{i+1,j+1,k-1,\ell}+\delta q A_{i+1,j,k-1,\ell}+q A_{i+1,j+1,k-1,\ell-1})\right.$$

$$\left.+\gamma q^{\|X\|}(WX)_{i,k-1}(\beta A_{i,j,k-1,\ell}+\delta q A_{i,j-1,k-1,\ell}+q A_{i,j,k-1,\ell-1})\right).$$

Using the definition of matrix multiplication we can get rid of the sums, obtaining:

$$\frac{\beta}{\delta q}(WXD^{(r)}A)_{j+1,\ell}+\beta\gamma q^{\|X\|+1}(WXA)_{j,\ell}+\gamma\delta q^{\|X\|+2}(WXA)_{j-1,\ell}+\gamma q^{\|X\|+2}(WXA)_{j,\ell-1}$$

$$-\frac{\alpha\beta^2}{\delta q}(WXA)_{j+1,\ell}-\alpha\beta(WXA)_{j,\ell}-\frac{\alpha\beta}{\delta}(WXA)_{j+1,\ell-1}.$$

Now we replace the $(WXD^{(r)}A)_{j+1,\ell}$ in the first term above using (4), obtaining:

$$\frac{\beta}{\delta q}\Big(q(WXAD^{(r+1)})_{j+1,\ell}+\alpha\beta(WXA)_{j+1,\ell}-\gamma\delta q^{\|X\|+2}(WXA)_{j,\ell}\Big)+\beta\gamma q^{\|X\|+1}(WXA)_{j,\ell}$$

$$+\gamma\delta q^{\|X\|+2}(WXA)_{j-1,\ell}+\gamma q^{\|X\|+2}(WXA)_{j,\ell-1}-\frac{\alpha\beta^2}{\delta q}(WXA)_{j+1,\ell}-\alpha\beta(WXA)_{j,\ell}-\frac{\alpha\beta}{\delta}(WXA)_{j+1,\ell-1}.$$

The second and third terms above cancel with the seventh and fourth terms, and we apply (2) to the $(WXAD^{(r+1)})_{j+1,\ell}$ in the first term, to obtain:

$$\frac{1}{\delta}\Big(\delta(WXAE^{(r+1)})_{j,\ell}+\alpha\beta(WXA)_{j+1,\ell-1}-\gamma\delta q^{\|X\|+2}(WXA)_{j,\ell-1}\Big)$$

$$+\gamma\delta q^{\|X\|+2}(WXA)_{j-1,\ell}+\gamma q^{\|X\|+2}(WXA)_{j,\ell-1}-\alpha\beta(WXA)_{j,\ell}-\frac{\alpha\beta}{\delta}(WXA)_{j+1,\ell-1}.$$

After cancellation, we obtain, as desired, that

$$q(WXE^{(r)}A)_{j,\ell}=(WXAE^{(r+1)})_{j,\ell}-\alpha\beta(WXA)_{j,\ell}+\gamma\delta q^{\|X\|+2}(WXA)_{j-1,\ell}.$$

$$\square$$

Lemma 7.6. *If equation (2) from Proposition 7.4 holds, then so does equation (1).*

Proof. We use that $(WXD^{(r)}E^{(r)})_{j,\ell}=\sum_{i,k}(WXD^{(r)})_{i,k}E^{(r)}_{i,j,k,\ell}$, and apply (2) to $(WXD^{(r)})_{i,k}$, to obtain:

$$(WXD^{(r)}E^{(r)})_{j,\ell}=\sum_{i,k}\frac{1}{\beta}\Big(\delta(WXE^{(r)})_{i-1,k}+\alpha\beta(WX)_{i,k-1}-\gamma\delta q^{\|X\|}(WX)_{i-1,k-1}\Big)E^{(r)}_{i,j,k,\ell}.$$

In the first and third term on the right-hand side above, we use Lemma 7.3 to replace $E^{(r)}_{i,j,k,\ell}$ by $qE^{(r)}_{i-1,j-1,k,\ell}$, and in the second term we apply the defining recurrence for E, obtaining:

$$\sum_{i,k}\frac{\delta}{\beta}(WXE^{(r)})_{i-1,k}(qE^{(r)}_{i-1,j-1,k,\ell})+\sum_{i,k}\alpha(WX)_{i,k-1}\Big(\beta(D^{(r)}+E^{(r)})_{i,j,k-1,\ell}+qE^{(r)}_{i,j,k-1,\ell-1}\Big)$$

$$-\sum_{i,k}\frac{\gamma\delta q^{\|X\|}}{\beta}(WX)_{i-1,k-1}(qE^{(r)}_{i-1,j-1,k,\ell}).$$

We interpret the first two sums above as matrix multiplication, and apply the defining recurrence for E to the third term, obtaining:

$$\frac{\delta q}{\beta}(WXE^{(r)}E^{(r)})_{j-1,\ell}+\alpha\beta(WX(D^{(r)}+E^{(r)}))_{j,\ell}+\alpha q(WXE^{(r)})_{j,\ell-1}$$

$$-\sum_{i,k}\frac{\gamma\delta q^{\|X\|+1}}{\beta}(WX)_{i-1,k-1}\Big(\beta(D^{(r)}+E^{(r)})_{i-1,j-1,k-1,\ell}+qE^{(r)}_{i-1,j-1,k-1,\ell-1}\Big).$$

We apply (2) to the $(WXE^{(r)}E^{(r)})_{j-1,\ell}$ in the first term, and interpret the final sum as matrix multiplication, obtaining:

$$\frac{\delta q}{\beta}\frac{1}{\delta}\left(\beta(WXE^{(r)}D^{(r)})_{j,\ell} - \alpha\beta(WXE^{(r)})_{j,\ell-1} + \gamma\delta q^{\|X\|+1}(WXE^{(r)})_{j-1,\ell-1}\right) + \alpha\beta(WX(D^{(r)}+E^{(r)}))_{j,\ell}$$

$$+ \alpha q(WXE^{(r)})_{j,\ell-1} - \gamma\delta q^{\|X\|+1}(WX(D^{(r)}+E^{(r)}))_{j-1,\ell} - \frac{\gamma\delta q^{\|X\|+2}}{\beta}(WXE^{(r)})_{j-1,\ell-1}.$$

After cancellation, we obtain the desired quantity

$$q(WXE^{(r)}D^{(r)})_{j,\ell} + \alpha\beta(WX(D^{(r)}+E^{(r)}))_{j,\ell} - \gamma\delta q^{\|X\|+1}(WX(D^{(r)}+E^{(r)}))_{j-1,\ell}.$$

$$\square$$

7.2. **Reducing (2) and (4) from Proposition 7.4 to Theorem 7.10.** In this section we show that equations (2) and (4) from Proposition 7.4 are a consequence of Theorem 7.10 below. We then give a brief outline of the proof of Theorem 7.10.

Definition 7.7. A vertical strip of *type* $F^{(t)}$ is a vertical strip with exactly t short rhombi and at least two square tiles, where the bottom-most square tile contains a δ, and there is a β as the next Greek letter above the δ in the strip (necessarily in a square tile), as in the left side of Figure 14. Similarly, a vertical strip of *type* $G^{(t)}$ has a β in its bottom-most square tile, and a δ as the next Greek letter above the β, as in the right side of Figure 14.

Definition 7.8. Let $F_{i,j,k,\ell}^{(t)}$ be the weight generating function for all ways of adding a new vertical strip of type $F^{(t)}$ to the left of a rhombic staircase tableau with precisely i horizontal δ-strips and k horizontal α/γ-strips, so as to obtain a new tableau with j horizontal δ-strips and ℓ horizontal α/γ-strips.

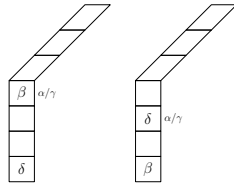
FIGURE 14. Vertical strips of type $F^{(3)}$ and $G^{(3)}$.

Remark 7.9. Since this section contains many long equations, we will try to simplify our notation somewhat. In particular, when X and Y are matrices, we will often write $XY_{i,j,k,\ell}$ instead of $(XY)_{i,j,k,\ell}$, and similarly $XYZ_{i,j,k,\ell}$ instead of $(XYZ)_{i,j,k,\ell}$.

Theorem 7.10. *Let X be (A,t)-compatible, and set $r = |X|_A$. Then we have that*

(14)
$$\beta X D_{0,j,k,\ell}^{(r+t)} - \delta X E_{0,j-1,k,\ell}^{(r+t)} = \alpha\beta X_{0,j,k,\ell-1} - \gamma\delta q^{2t+k+\|X\|}X_{0,j-1,k,\ell-1} + (1-q)F^{(t)}X_{0,j,k,\ell-1},$$

(15)
$$XD^{(r+t)}A_{0,j,k,\ell} - qXAD_{0,j,k,\ell}^{(r+t+1)} = \alpha\beta XA_{0,j,k,\ell} - \gamma\delta q^{2t+k+2+\|X\|}XA_{0,j-1,k,\ell} + (1-q)F^{(t)}XA_{0,j,k,\ell}.$$

Lemma 7.11. *If Theorem 7.10 holds, then so do equations (2) and (4) from Proposition 7.4.*

Proof. Note that equation (2) (respectively, equation (4)) from Proposition 7.4 is precisely the case that $t = 0$ and $k = 0$ from (14) (resp., (15)). Here we are using the fact that a vertical strip of type F always contains a δ, and hence $(F^{(0)}X)_{0,j,0,\ell-1} = (F^{(0)}XA)_{0,j,0,\ell} = 0$. □

The structure of the proof of Theorem 7.10 is as follows. In Section 7.3, we prove some recurrences for $F^{(t)}_{i,j,k,\ell}$ and $G^{(t)}_{i,j,k,\ell}$, and in Section 7.4, we prove Theorem 7.10 when $|X| = 0$. In this case the proof of (14) is very short while the proof of (15) is quite long. In Section 7.5, we prove some useful identities (Proposition 7.16), using Theorem 7.10 in the case that $|X| = 0$. Finally in Section 7.6, we give the inductive proof of (14) and (15). This completes the proof of Theorem 7.10.

Remark 7.12. Note that all the proofs use simple recurrences for D, E and F. Even for $r = t = 0$, they are much simpler than the proof given in [CW13], where generating functions and computer algebra were used. This gives the first human proof of the main result of [CW11].

7.3. Recurrences for $F^{(t)}_{i,j,k,\ell}$ and $G^{(t)}_{i,j,k,\ell}$. Recall from Definition 7.8 the definitions of $F^{(t)}_{i,j,k,\ell}$ and $G^{(t)}_{i,j,k,\ell}$.

Lemma 7.13. *If $j > i \geq 0$ and $k \geq \ell \geq 0$, we have that*
$$F^{(t)}_{i,j,k,\ell} = \delta E^{(t)}_{i,j-1,k-1,\ell} - \gamma\delta q^{2t+i+k-1}\mathbb{1}_{\{k=\ell,\ i=j-1\}} + F^{(t)}_{i,j,k-1,\ell-1}$$
$$G^{(t)}_{i,j,k,\ell} = \beta D^{(t)}_{i,j,k-1,\ell} - \alpha\beta q^i\mathbb{1}_{\{k=\ell,\ i=j\}} + qG^{(t)}_{i,j,k-1,\ell-1}$$
Otherwise we have that $F^{(t)}_{i,j,k,\ell} = G^{(t)}_{i,j,k,\ell} = 0$.

Proof. Note that if $j \leq i$ or $\ell \geq k$, then $F^{(t)}_{i,j,k,\ell} = 0$. This is because any vertical strip of type $F^{(t)}$ contains at least one δ and at least one β, so adding a new vertical strip of this type to a rhombic staircase tableau necessarily increases the number of horizontal δ-strips, and decreases the number of horizontal α/γ-strips by at least 1.

In all other situations, we can assume that $k \geq 1$. Recall the combinatorial definition of $F^{(t)}_{i,j,k,\ell}$, and let C be one of the vertical strips that it is a sum over. Let T denote a representative tableau to which we will append C. Let B be the bottom-most square tile in C which is part of an α/γ-strip from T (such a square tile exists since $k \geq 1$). If we fill B with a β, then filling B and the rest of the vertical strip C is like adding a new vertical strip with a β in the bottom-most square tile to a tableau with i horizontal δ-strips and $k - 1$ horizontal α/γ-strips. This contributes $\delta(E^{(t)}_{i,j-1,k-1,\ell} - \gamma q^{2t+i+k-1}\mathbb{1}_{\{k=\ell,\ i=j-1\}})$ to our generating polynomial.[3]

On the other hand, if we leave B empty, then this square tile will get a weight $u = 1$. The generating function for filling the rest of the vertical strip C is equal to $F^{(t)}_{i,j,k-1,\ell-1}$. This completes the proof of the first recurrence.

---

[3] The set of vertical strips with a β in the bottom-most square tile is obtained by subtracting the set of vertical strips with a γ in the bottom-most square tile from the set of vertical strips of type E. This explains the term containing γ in the expression.

The proof is the same for $G_{i,j,k,\ell}^{(t)}$. The base case where $G_{i,j,k,\ell}^{(t)} = 0$ for $j \le i$ or $\ell \ge k$ is literally the same. For the rest of the argument, the square tile B can either contain a δ, which contributes $\beta(D_{i,j-1,k-1,\ell}^{(t)} - \alpha q^i \mathbb{1}_{\{k=\ell,\ i=j-1\}})$, or a q, which contributes $q G_{i,j,k-1,\ell-1}^{(t)}$ to our generating polynomial. $\square$

Lemma 7.14. *For all i, j, k and ℓ,*

$$F_{i,j,k,\ell}^{(t)} = G_{i,j,k,\ell}^{(t)}.$$

Proof. We give a bijective proof of the lemma. Let C be one of the vertical strips of type $F^{(t)}$ which $F_{i,j,k,\ell}^{(t)}$ is a sum over. Let T be a representative tableau to which we are appending C. Let m be the number of empty square tiles in C which are: adjacent to horizontal α/γ-strips in T; above the bottom-most δ; and below the next Greek letter (which is a β). And let n be the number of empty square tiles which are: adjacent to horizontal α/γ-strips in T; above the bottom-most β in C; and below the next Greek letter in C. It is easy to see this strip has the same weight as a vertical strip of type $G^{(t)}$ in which the roles of m and n are reversed, see Figure 15. This map is clearly a bijection, so this proves the lemma. $\square$
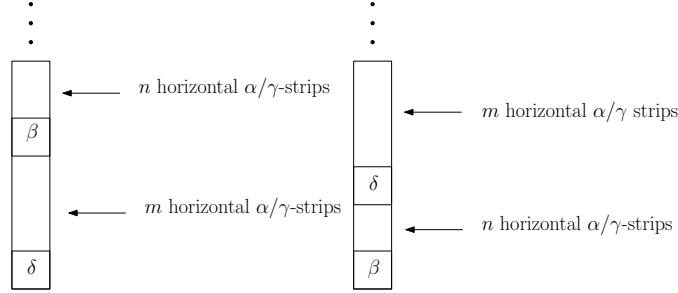

FIGURE 15. The bijection between F and G.

7.4. **The proof of Theorem 7.10 for $|X| = 0$.**

7.4.1. *The proof of* (14) *when $|X| = 0$.* We need to show that

(16) $\beta D_{0,j,k,\ell}^{(t)} - \delta E_{0,j-1,k,\ell}^{(t)} = \alpha\beta \mathbb{1}_{\{j=0,\ k=\ell-1\}} - \gamma\delta q^{2t+k}\mathbb{1}_{\{j=1,\ k=\ell-1\}} + (1-q)F_{0,j,k,\ell-1}^{(t)}.$

First consider the case that $k = 0$. In this case (16) becomes

(17) $\beta D_{0,j,0,\ell}^{(t)} - \delta E_{0,j-1,0,\ell}^{(t)} = \alpha\beta \mathbb{1}_{\{j=0,\ k=0,\ \ell=1\}} - \gamma\delta q^{2t}\mathbb{1}_{\{j=1,\ k=0,\ \ell=1\}} + (1-q)F_{0,j,0,\ell-1}^{(t)}.$

From Definition 6.1 we have that

$$D_{0,j,0,\ell}^{(t)} = \begin{cases} \delta(q^t + (\alpha + \gamma q^t)[t]_q) & \text{if } j = 1, \ell = 0 \\ \alpha & \text{if } j = 0, \ell = 1 \\ 0 & \text{otherwise} \end{cases}$$

and also

$$E_{0,j,0,\ell}^{(t)} = \begin{cases} \beta(q^t + (\alpha + \gamma q^t)[t]_q) & \text{if } j = 0, \ell = 0 \\ \gamma q^{2t} & \text{if } j = 0, \ell = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Using the fact that $F_{0,j,0,\ell-1} = 0$, it is easy to check that we get (17).

We now consider the case that $k > 0$. When $k > 0$, we obtain the following recurrences from Lemma 7.13 and Lemma 7.14. Note that we will omit the superscripts (t) from D, E, and F for readability.

$$
\begin{aligned}
(18) \qquad F_{i,j,k,\ell} &= \delta E_{i,j-1,k-1,\ell} - \gamma \delta q^{2t+i+k-1} \mathbb{1}_{\{j=i+1, \ k=\ell\}} + F_{i,j,k-1,\ell-1} \\
(19) \qquad &= \beta D_{i,j,k-1,\ell} - \alpha \beta q^i \mathbb{1}_{\{j=i, \ k=\ell\}} + q F_{i,j,k-1,\ell-1}
\end{aligned}
$$

Using (18) and (19), we get the desired expression

$$
\beta D_{0,j,k,\ell} - \delta E_{0,j-1,k,\ell} = F_{0,j,k,\ell-1} + \alpha \beta \mathbb{1}_{\{j=0, \ k=\ell-1\}} - q F_{0,j,k,\ell-1} - \gamma \delta q^{2t+k} \mathbb{1}_{\{j=1, k=\ell-1\}}.
$$

7.4.2. *The proof of* (15) *when* $|X| = 0$. Note that Lemma 7.15 directly implies (15) in the case that $|X| = 0$.

Lemma 7.15. *We have that*

$$
\begin{aligned}
D^{(t)} A_{0,j,k,\ell} - q A D^{(t+1)}_{0,j,k,\ell} &= A E^{(t+1)}_{0,j,k,\ell} - q E^{(t)} A_{0,j,k,\ell} \\
&= \alpha \beta A_{0,j,k,\ell} - \gamma \delta q^{2t+k+2} A_{0,j-1,k,\ell} + (1-q) F^{(t)} A_{0,j,k,\ell}.
\end{aligned}
$$

Proof. Once again, we prove this by induction on k. If $k = 0$, then we use the combinatorial interpretation of matrix products from Theorem 6.3(1) to obtain recurrences for each of $A D^{(t+1)}_{0,j,0,\ell}$, $D^{(t)} A_{0,j,0,\ell}$, $A E^{(t+1)}_{0,j,0,\ell}$, and $E^{(t)} A_{0,j,0,\ell}$. For example, the quantity $A D^{(t+1)}_{0,j,0,\ell}$ is the generating function for certain vertical strips consisting of one square box with an α or δ inside it, and with $t+1$ short rhombi above it, see Figure 16.
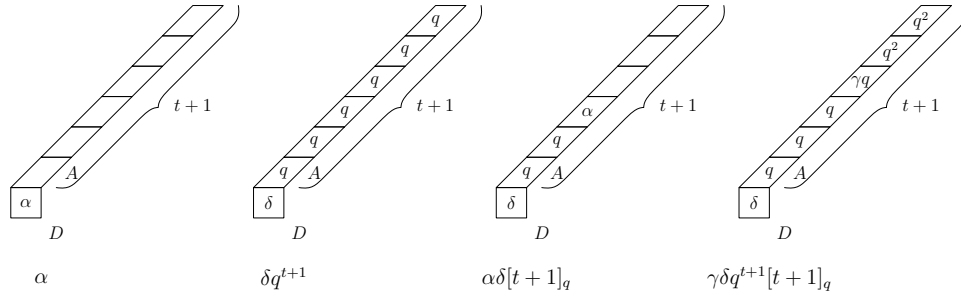

FIGURE 16. The vertical strips contributing to $A D^{(t+1)}_{0,j,0,\ell}$

Note that the third picture should be viewed as shorthand for $(t+1)$ different vertical strips, because there are $(t+1)$ choices for where to place the α. Similarly in the fourth picture there are $(t+1)$ choices for where to place the γq. So we get:

$$AD^{(t+1)}_{0,j,0,\ell} = \begin{cases} \alpha & \text{if } j = 0 \text{ and } \ell = 1 \\ \delta(q^{t+1} + (\alpha + \gamma q^{t+1})[t+1]_q) & \text{if } j = 1 \text{ and } \ell = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$D^{(t)}A_{0,j,0,\ell} = \begin{cases} \alpha q & \text{if } j = 0 \text{ and } \ell = 1 \\ \alpha\delta q + \delta q^2(q^t + (\alpha + \gamma q^t)[t]_q) & \text{if } j = 1 \text{ and } \ell = 0 \\ \alpha\beta & \text{if } j = 0 \text{ and } \ell = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$AE^{(t+1)}_{0,j,0,\ell} = \begin{cases} \gamma q^{2t+2} & \text{if } j = 0 \text{ and } \ell = 1 \\ \beta(q^{t+1} + (\alpha + \gamma q^{t+1})[t+1]_q) & \text{if } j = 0 \text{ and } \ell = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E^{(t)}A_{0,j,0,\ell} = \begin{cases} \gamma q^{2t+1} & \text{if } j = 0 \text{ and } \ell = 1 \\ \beta\gamma q^{2t} + \beta(q^t + (\alpha + \gamma q^t)[t]_q) & \text{if } j = 0 \text{ and } \ell = 0 \\ \gamma\delta q^{2t+1} & \text{if } j = 1 \text{ and } \ell = 0 \\ 0 & \text{otherwise} \end{cases}$$

In what follows, the superscripts (t) and $(t+1)$ are implicit, but we omit them for readability. In each of the various cases ($j = 0$ and $\ell = 0$, etc.), we get

$$AE_{0,j,0,\ell} - qEA_{0,j,0,\ell} = DA_{0,j,0,\ell} - qAD_{0,j,0,\ell} = \alpha\beta\mathbb{1}_{\{j=0,\ell=0\}} - \gamma\delta q^{2t+2}\mathbb{1}_{\{j=1,\ell=0\}}$$
$$= \alpha\beta A_{0,j,0,\ell} - \gamma\delta q^{2t+2}A_{0,j-1,0,\ell} + (1-q)F^{(t)}A_{0,j,0,\ell},$$

as desired (where in the last step we used the fact that $F^{(t)}A_{0,j,0,\ell} = 0$).

Now we suppose that $k > 0$. We will prove the claim by induction. For $k > 0$, we have from Definition 6.1 that

(20) $$D_{i,j,k,\ell} = \delta(D+E)_{i,j-1,k-1,\ell} + D_{i,j,k-1,\ell-1} \text{ and}$$

(21) $$A_{i,j,k,\ell} = \beta A_{i,j,k-1,\ell} + \delta q A_{i,j-1,k-1,\ell} + q A_{i,j,k-1,\ell-1}.$$

Using these, the definition of matrix multiplication, and Lemma 7.3, we get:

$$AD_{0,j,k,\ell} = \sum_{u,v}(\beta A_{0,u,k-1,v} + \delta q A_{0,u-1,k-1,v})D_{u,j,v,\ell} + q A_{0,u,k-1,v-1}(\delta(D+E)_{u,j-1,v-1,\ell} + D_{u,j,v-1,\ell-1})$$

$$DA_{0,j,k,\ell} = \sum_{u,v}\delta(D+E)_{0,u-1,k-1,v}q^2 A_{u-1,j-1,v,l} + D_{0,u,k-1,v-1}(\beta A_{u,j,v-1,\ell} + \delta q A_{u,j-1,v-1,\ell} + q A_{u,j,v-1,\ell-1}).$$

This gives

$$AD_{0,j,k,\ell} = \beta AD_{0,j,k-1,\ell} + \delta q^2 AD_{0,j-1,k-1,\ell} + \delta q A(D+E)_{0,j-1,k-1,\ell} + q AD_{0,j,k-1,\ell-1} \text{ and}$$

$$DA_{0,j,k,\ell} = \beta DA_{0,j,k-1,\ell} + \delta q DA_{0,j-1,k-1,\ell} + \delta q^2(D+E)A_{0,j-1,k-1,\ell} + q DA_{0,j,k-1,\ell-1}.$$

We get

$$
\begin{aligned}
DA_{0,j,k,\ell} - qAD_{0,j,k,\ell} = {} & \delta q^2(DA_{0,j-1,k-1,\ell} - qAD_{0,j-1,k-1,\ell} - AE_{0,j-1,k-1,\ell} + qEA_{0,j-1,k-1,\ell}) \\
& + \delta q(DA_{0,j-1,k-1,\ell} - qAD_{0,j-1,k-1,\ell}) + \beta(DA_{0,j,k-1,\ell} - qAD_{0,j,k-1,\ell} \\
& + q(DA_{0,j,k-1,\ell-1} - qAD_{0,j,k-1,\ell-1}) + \delta q^2(1-q)EA_{0,j-1,k-1,\ell}
\end{aligned}
$$

By the induction hypothesis, we suppose that Lemma 7.15 holds for $k-1$. Substituting, we obtain:

$$
\begin{aligned}
DA_{0,j,k,\ell} - qAD_{0,j,k,\ell} = {} & \delta q \left(\alpha\beta A_{0,j-1,k-1,\ell} - \gamma\delta q^{2t+k+2-1}A_{0,j-2,k-1,\ell} + (1-q)FA_{0,j-1,k-1,\ell} \right) \\
& + \beta \left(\alpha\beta A_{0,j,k-1,\ell} - \gamma\delta q^{2t+k+2-1}A_{0,j-1,k-1,\ell} + (1-q)FA_{0,j,k-1,\ell} \right) \\
& + q \left(\alpha\beta A_{0,j,k-1,\ell-1} - \gamma\delta q^{2t+k+2-1}A_{0,j-1,k-1,\ell-1} + (1-q)FA_{0,j,k-1,\ell-1} \right) \\
& + \delta q^2(1-q)EA_{0,j-1,k-1,\ell} \\
(22) = {} & \alpha\beta(\delta q A_{0,j-1,k-1,\ell} + \beta A_{0,j,k-1,\ell} + q A_{0,j,k-1,\ell-1}) \\
(23) & \quad -\gamma\delta q^{2t+k+1}(\delta q A_{0,j-2,k-1,\ell} + \beta A_{0,j-1,k-1,\ell} + q A_{0,j-1,k-1,\ell-1}) \\
(24) & \quad +(1-q)(\delta q FA_{0,j-1,k-1,\ell} + \beta FA_{0,j,k-1,\ell} + q FA_{0,j,k-1,\ell-1} + \delta q^2 EA_{0,j-1,k-1,\ell}).
\end{aligned}
$$

From (21), it is clear that the quantity in line (22) is equal to $\alpha\beta A_{0,j,k,\ell}$, and the quantity in line (23) is equal to $-\gamma\delta q^{2t+k+2-1}A_{0,j-1,k,\ell}$.

Using (18), (21), the definition of matrix multiplication, and Lemma 7.3, we get

$$
FA_{0,j,k,\ell} = \delta q FA_{0,j-1,k-1,\ell} + \beta FA_{0,j,k-1,\ell} + \delta q^2 EA_{0,j-1,k-1,\ell} - \gamma\delta q^{2t+k+1}A_{0,j-1,k,\ell} + q FA_{0,j,k-1,\ell-1},
$$

which can be rewritten as

$$
\delta q FA_{0,j-1,k-1,\ell} + \beta FA_{0,j,k-1,\ell} + q FA_{0,j,k-1,\ell-1} + \delta q^2 EA_{0,j-1,k-1,\ell} = FA_{0,j,k,\ell} + \gamma\delta q^{2t+k+1}A_{0,j-1,k,\ell}.
$$

Using this to replace line (24), and then simplifying, we obtain the desired expression

$$
DA_{0,j,k,\ell} - qAD_{0,j,k,\ell} = \alpha\beta A_{0,j,k,\ell} - \gamma\delta q^{2t+k+2}A_{0,j-1.k,\ell} + (1-q)FA_{0,j,k,\ell}.
$$

The proof that

$$
AE_{0,j,k,\ell} - qEA_{0,j,k,\ell} = \alpha\beta A_{0,j,k,\ell} - \gamma\delta q^{2t+k+2}A_{0,j-1.k,\ell} + (1-q)FA_{0,j,k,\ell}
$$

is completely analogous. This concludes our proof of Lemma 7.15, which is the base case of $|X| = 0$ in (15). $\qquad\square$

7.5. Commutator identities.

In this section we will prove the three "commutator" identities of Proposition 7.16. Actually we will prove (25) and (27) but not (26), because the proof of (26) is analogous to that of (25). Proposition 7.16 will be a useful tool in the inductive proof of Theorem 7.10.

Proposition 7.16.

(25)
$$(1-q)(E^{(t)}F^{(t)} - F^{(t)}E^{(t)})_{0,j,k,\ell} = (1-q^j)\alpha\beta E^{(t)}_{0,j,k,\ell} + \gamma\delta q^{2t}(q^{\ell+j-1} - q^{k+1})E^{(t)}_{0,j-1,k,\ell},$$

(26)
$$(1-q)(D^{(t)}F^{(t)} - F^{(t)}D^{(t)})_{0,j,k,\ell} = (1-q^j)\alpha\beta D^{(t)}_{0,j,k,\ell} + \gamma\delta q^{2t}(q^{\ell+j-1} - q^{k+1})D^{(t)}_{0,j-1,k,\ell},$$

(27)
$$(1-q)(AF^{(t+1)} - F^{(t)}A)_{0,j,k,\ell} = (1-q^j)\alpha\beta A_{0,j,k,\ell} + \gamma\delta q^{2t}(q^{\ell+j+1} - q^{k+2})A_{0,j-1,k,\ell}.$$

Let us now prove Proposition 7.16 for $k = 0$, and then for general k by induction.

7.5.1. *The case $k = 0$.*

Proof. Note that from the combinatorial interpretation of F, we have that $F^{(t)}E^{(t)}_{i,j,0,\ell} = F^{(t)}D^{(t)}_{i,j,0,\ell} = F^{(t)}A_{i,j,0,\ell} = AF^{(t)}_{i,j,0,\ell} = 0$ for all i, j and ℓ. (This is because when $k = 0$, there are no horizontal α/γ-strips in the tableau we are building, so in particular there is no place to put the β and the δ in the vertical strip of type $F^{(t)}$.)

It is also easy to verify combinatorially (or directly from the matrices) that

$$E^{(t)}F^{(t)}_{0,j,0,\ell} = \begin{cases} \gamma\delta\beta q^{2t}(q^t + (\alpha + \gamma q^t)[t]_q) & rmif \ j = 1 \text{ and } \ell = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Now we use the base cases of the recurrences for A and E:

$$A_{0,j,0,\ell} = \begin{cases} 1 & \text{if } j = 0 \text{ and } \ell = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$E^{(t)}_{0,j,0,\ell} = \begin{cases} \beta(q^t + (\alpha + \gamma q^t)[t]_q) & \text{if } j = \ell = 0 \\ \gamma q^{2t} & \text{if } j = 0 \text{ and } \ell = 1 \\ 0 & \text{otherwise.} \end{cases}$$

A simple computation now yields the case $k = 0$ of (25) and (27). $\square$

7.5.2. *The case $k > 0$.* We prove (25) by induction on k, supposing that (25) and (26) are true for $k - 1$. For simplicity, we again omit the superscripts (t) of E and F. Using Lemma 7.13, and the fact that

(28)
$$E_{i,j,k,\ell} = \beta(D + E)_{i,j,k-1,\ell} + qE_{i,j,k-1,\ell-1},$$

we obtain

$$
\begin{aligned}
EF_{0,j,k,\ell} &= \sum_{u,v} E_{0,u,k,v}F_{u,j,v,\ell} \\
&= \sum_{u,v} \left(\beta(D + E)_{0,u,k-1,v}F_{u,j,v,\ell} + qE_{0,u,k-1,v-1}F_{u,j,v,\ell} \right) \\
&= \sum_{u,v} \Big(\beta(D + E)_{0,u,k-1,v}F_{u,j,v,\ell} + qE_{0,u,k-1,v-1}\delta E_{u,j-1,v-1,\ell} \\
&\qquad - qE_{0,u,k-1,v-1}\mathbb{1}_{\{j=u+1,\ v=\ell\}}\gamma\delta q^{2t+u+v-1} + qE_{0,u,k-1,v-1}F_{u,j,v-1,\ell-1} \Big) \\
&= \beta(D + E)F_{0,j,k-1,\ell} + \delta qEE_{0,j-1,k-1,\ell} - \gamma\delta q^{2t+j+\ell-1}E_{0,j-1,k-1,\ell-1} + qEF_{0,j,k-1,\ell-1}.
\end{aligned}
$$

Similarly we get an expression for FE:

$$FE_{0,j,k,\ell} = \beta F(D+E)_{0,j,k-1,\ell} + \delta q EE_{0,j-1,k-1,\ell} - \gamma \delta q^{2t+k} E_{0,j-1,k,\ell} + q FE_{0,j,k-1,\ell-1}.$$

Putting the expressions for EF and FE together, we obtain

$$(EF-FE)_{0,j,k,\ell} = \beta(EF-FE)_{0,j,k-1,\ell} + \beta(DF-FD)_{0,j,k-1,\ell} + q(EF-FE)_{0,j,k-1,\ell-1}$$
$$+ \gamma \delta q^{2t+k} E_{0,j-1,k,\ell} - \gamma \delta q^{\ell+j+2t-1} E_{0,j-1,k-1,\ell-1}.$$

Using the induction hypothesis (that Proposition 7.16 holds for $k-1$), we obtain:

$$(1-q)(EF - FE)_{0,j,k,\ell} = \beta\Big((1-q^j)\underline{\underline{\alpha\beta E_{0,j,k-1,\ell}}} + q^{2t}\gamma\delta(q^{\ell+j-1} - q^k)E_{0,j-1,k-1,\ell}\Big)$$
$$+ \beta\Big((1-q^j)\underline{\underline{\alpha\beta D_{0,j,k-1,\ell}}} + q^{2t}\gamma\delta(q^{\ell+j-1} - q^k)D_{0,j-1,k-1,\ell}\Big)$$
$$+ q\Big((1-q^j)\underline{\underline{\alpha\beta E_{0,j,k-1,\ell-1}}} + q^{2t}\gamma\delta(q^{\ell+j-2} - q^k)E_{0,j-1,k-1,\ell-1}\Big)$$
$$+ (1-q)\gamma\delta q^{2t+k} E_{0,j-1,k,\ell} - \underline{(1-q)\gamma\delta q^{\ell+j+2t-1} E_{0,j-1,k-1,\ell-1}}.$$

Now we group the terms according to the underlines above, and use the defining recurrences for $D_{i,j,k,\ell}$ and $E_{i,j,k,\ell}$ given in (21) and (28). This gives

$$(1-q)(EF - FE)_{0,j,k,\ell} = \underline{\underline{(1-q^j)\alpha\beta E_{0,j,k,\ell}}} + q^{2t+\ell+j-1}\gamma\delta E_{0,j-1,k,\ell} - \underline{q^{2t+k}\gamma\delta E_{0,j-1,k,\ell}}$$
$$+ (1-q)\gamma\delta q^{2t+k} E_{0,j-1,k,\ell}$$
$$= (1-q^j)\alpha\beta E_{0,j,k,\ell} + q^{2t+\ell+j-1}\gamma\delta E_{0,j-1,k,\ell} - q^{2t+k+1}\gamma\delta E_{0,j-1,k,\ell},$$

which completes the proof of (25).

We now prove the identity for $AF - FA$. Using the recurrences of (21) and (19), we obtain the following:

$$AF_{0,j,k,\ell} = \delta q^2 AF_{0,j-1,k-1,\ell} + \beta AF_{0,j,k-1,\ell} + q^2 AF_{0,j,k-1,\ell-1} + \beta q AD_{0,j,k-1,\ell} - \alpha\beta q^{j+1} A_{0,j,k-1,\ell-1},$$
$$FA_{0,j,k,\ell} = \delta q^2 FA_{0,j-1,k-1,\ell} + \beta q FA_{0,j,k-1,\ell} + q^2 FA_{0,j,k-1,\ell-1} + \beta DA_{0,j,k-1,\ell} - \alpha\beta A_{0,j,k,\ell}.$$

We have that

$$(1-q)(AF - FA)_{0,j,k,\ell} = \delta q^2((1-q)(AF-FA)_{0,j-1,k-1,\ell}) + \beta(1-q)AF_{0,j,k-1,\ell}$$
$$- \beta q(1-q)FA_{0,j,k-1,\ell} + q^2((1-q)(AF-FA)_{0,j,k-1,\ell-1})$$
$$- (1-q)\beta(DA - qAD)_{0,j,k-1,\ell} + (1-q)\alpha\beta(A_{0,j,k,\ell} - q^{j+1}A_{0,j,k-1,\ell-1}).$$

We now use the induction hypothesis along with Lemma 7.15. We then find that $(1-q)(AF - FA)_{0,j,k,\ell}$ is equal to:

$$\delta q^2\Big((1-q^{j-1})\alpha\beta A_{0,j-1,k-1,\ell} + \gamma\delta q^{2t}(q^{\ell+j} - q^{k+1})A_{0,j-2,k-1,\ell}\Big)$$
$$+ \quad \beta(1-q)AF_{0,j,k-1,\ell} - \underline{\beta q(1-q)FA_{0,j,k-1,\ell}}$$
$$+ \quad q^2\Big((1-q^j)\alpha\beta A_{0,j,k-1,\ell-1} + \gamma\delta q^{2t}(q^{\ell+j} - q^{k+1})A_{0,j-1,k-1,\ell-1}\Big)$$
$$- \quad \underline{(1-q)\beta}\Big(\alpha\beta A_{0,j,k-1,\ell} - \gamma\delta q^{2t+k+1}A_{0,j-1,k-1,\ell} + \underline{(1-q)(FA)_{0,j,k-1,\ell}}\Big)$$
$$+ \quad (1-q)\alpha\beta(A_{0,j,k,\ell} - q^{j+1}A_{0,j,k-1,\ell-1}).$$

We group the terms according to the underlines, obtaining:

$$
\begin{aligned}
&\delta q^2 \left((1-q^{j-1})\alpha\beta A_{0,j-1,k-1,\ell} + \gamma\delta q^{2t}(q^{\ell+j}-q^{k+1})A_{0,j-2,k-1,\ell}\right) \\
+\ &\beta((1-q)(AF-FA)_{0,j,k-1,\ell}) \\
+\ &q^2 \left((1-q^j)\alpha\beta A_{0,j,k-1,\ell-1} + \gamma\delta q^{2t}(q^{\ell+j}-q^{k+1})A_{0,j-1,k-1,\ell-1}\right) \\
-\ &(1-q)\beta\left(\alpha\beta A_{0,j,k-1,\ell} - \gamma\delta q^{2t+k+1}A_{0,j-1,k-1,\ell}\right) \\
+\ &(1-q)\alpha\beta(A_{0,j,k,\ell} - q^{j+1}A_{0,j,k-1,\ell-1}).
\end{aligned}
$$

We use again the induction hypothesis to get:

$$
\begin{aligned}
&\delta q^2 \left((1-q^{j-1})\alpha\beta A_{0,j-1,k-1,\ell} + \gamma\delta q^{2t}(q^{\ell+j}-q^{k+1})A_{0,j-2,k-1,\ell}\right) \\
+\ &\beta\left((1-q^j)\alpha\beta A_{0,j,k-1,\ell} + \gamma\delta q^{2t}(q^{\ell+j+1}-q^{k+1})A_{0,j-1,k-1,\ell}\right) \\
+\ &q^2 \left((1-q^j)\alpha\beta A_{0,j,k-1,\ell-1} + \gamma\delta q^{2t}(q^{\ell+j}-q^{k+1})A_{0,j-1,k-1,\ell-1}\right) \\
-\ &(1-q)\beta\left(\alpha\beta A_{0,j,k-1,\ell} - \gamma\delta q^{2t+k+1}A_{0,j-1,k-1,\ell}\right) \\
+\ &(1-q)(\alpha\beta A_{0,j,k,\ell} - \alpha\beta q^{j+1}A_{0,j,k-1,\ell-1}).
\end{aligned}
$$

We combine lines 2 and 4 and then lines 3 and 5 to get:

$$
\begin{aligned}
&\delta q^2 \left((1-q^{j-1})\alpha\beta \underline{A_{0,j-1,k-1,\ell}} + \gamma\delta q^{2t}(q^{\ell+j}-q^{k+1})\underline{A_{0,j-2,k-1,\ell}}\right) \\
+\ &\beta q((1-q^{j-1})\alpha\beta \underline{A_{0,j,k-1,\ell}} + \gamma\delta q^{2t}(q^{\ell+j}-q^{k+1})\underline{A_{0,j-1,k-1,\ell}}) \\
+\ &q^2 \left((1-q^{j-1})\alpha\beta \underline{A_{0,j,k-1,\ell-1}} + \gamma\delta q^{2t}(q^{\ell+j}-q^{k+1})\underline{A_{0,j-1,k-1,\ell-1}}\right) + (1-q)\alpha\beta A_{0,j,k,\ell}
\end{aligned}
$$

We apply (21) to the underlined terms, and obtain:

$$
\begin{aligned}
&\alpha\beta q(1-q^{j-1})\underline{A_{0,j,k,\ell}} + \gamma\delta q^{2t+1}(q^{\ell+j}-q^{k+1})\underline{\underline{A_{0,j-1,k,\ell}}} + (1-q)\alpha\beta A_{0,j,k,\ell} \\
=\ &\alpha\beta(1-q^j)A_{0,j,k,\ell} + \gamma\delta q^{2t+1}(q^{\ell+j}-q^{k+1})A_{0,j-1,k,\ell},
\end{aligned}
$$

as desired. This completes the proof of Proposition 7.16.

7.6. The inductive proof of Theorem 7.10. We now prove Theorem 7.10 by induction on $|X|$. We already verified the base case ($|X| = 0$) in Section 7.4.

7.6.1. *The inductive proof of* (14). We need to prove that for X an (A,t)-compatible word, we have (29) below.
(29)
$$\beta X D_{0,j,k,\ell} - \delta X E_{0,j-1,k,\ell} = \alpha\beta X_{0,j,k,\ell-1} - \gamma\delta q^{2t+k+\|X\|}X_{0,j-1,k,\ell-1} + (1-q)FX_{0,j,k,\ell-1}.$$

Note that by Lemma 7.3, (29) is equivalent to
(30)
$$\beta X D_{i,j,k,\ell} - \delta X E_{i,j-1,k,\ell} = \alpha\beta q^i X_{i,j,k,\ell-1} - \gamma\delta q^{2t+k+i+\|X\|}X_{i,j-1,k,\ell-1} + (1-q)FX_{i,j,k,\ell-1}.$$

For our proof, we will show that if (29) holds for $X = Y$, where Y is (A,t)-compatible, then (29) also holds for $X = DY$, $X = EY$, and $X = AY$.

We start by showing that after plugging $X = DY$ into (29), the left-hand side (LHS) will match the right-hand side (RHS). On the LHS we have:

$$\beta(DYD)_{0,j,k,\ell} - \delta(DYE)_{0,j-1,k,\ell}$$

$$= \sum_{u,v} D_{0,u,k,v} \left(\beta Y D_{u,j,v,\ell} - \delta Y E_{u,j-1,v,\ell}\right)$$

$$= \sum_{u,v} D_{0,u,k,v} \left(\alpha\beta q^u Y_{u,j,v,\ell-1} - \gamma\delta q^{2t+v+u+\|Y\|} Y_{u,j-1,v,\ell-1} + (1-q)FY_{u,j,v,\ell-1}\right)$$

$$= \sum_{u,v} \alpha\beta q^u D_{0,u,k,v} Y_{u,j,v,\ell-1} - \sum_{u,v} \gamma\delta q^{2t+v+u} D_{0,u,k,v} Y_{u+1,j,v,\ell-1} + \sum_{u,v} (1-q) D_{0,u,k,v} FY_{u,j,v,\ell-1}.$$

Here we used the induction hypothesis for (30) to obtain the third line from the second, and we used Lemma 7.3 in the second term to obtain the fourth line from the third. Now in the fourth line, we change our index of summation in the second sum, replacing each u by $u-1$. Also note that $\sum_{u,v} D_{0,u,k,v} FY_{u,j,v,\ell-1} = \sum_{u,v} DF_{0,u,k,v} Y_{u,j,v,\ell-1}$. Thus we obtain that the LHS equals

$$\sum_{u,v} \left(\alpha\beta q^u D_{0,u,k,v} - \gamma\delta q^{2t+v+u-1} D_{0,u-1,k,v} + (1-q)DF_{0,u,k,v}\right) Y_{u,j,v,\ell-1}.$$

The RHS for $X = DY$ is obtained in a similar way:

$$\alpha\beta DY_{0,j,k,\ell-1} - \gamma\delta q^{2t+k+\|DY\|} DY_{0,j-1,k,\ell-1} + (1-q)FDY_{0,j,k,\ell-1}$$

$$= \sum_{u,v} D_{0,u,k,v} \left(\alpha\beta Y_{u,j,v,\ell-1} - \gamma\delta q^{2t+k+\|DY\|} Y_{u,j-1,v,\ell-1}\right) + \sum_{u,v} (1-q)FD_{0,u,k,v} Y_{u,j,v,\ell-1}$$

$$= \sum_{u,v} \left(\alpha\beta D_{0,u,k,v} - \gamma\delta q^{2t+k+\|Y\|+1-\|Y\|} D_{0,u-1,k,v} + (1-q)FD_{0,u,k,v}\right) Y_{u,j,v,\ell-1}.$$

Equation (26) of Proposition 7.16 implies that

$$\alpha\beta q^u D_{0,u,k,v} - \gamma\delta q^{2t+v+u-1} D_{0,u-1,k,v} + (1-q)DF_{0,u,k,v}$$
$$= \alpha\beta D_{0,u,k,v} - \gamma\delta q^{2t+k+1} D_{0,u-1,k,v} + (1-q)FD_{0,u,k,v},$$

which means the LHS equals the RHS of (29) for $X = DY$.

If $X = EY$, the proof is the same. The induction is equivalent to proving that

$$\alpha\beta q^u E_{0,u,k,v} - \gamma\delta q^{2t+v+u-1} E_{0,u-1,k,v} + (1-q)EF_{0,u,k,v}$$
$$= \alpha\beta E_{0,u,k,v} - \gamma\delta q^{2t+k+1} E_{0,u-1,k,v} + (1-q)FE_{0,u,k,v},$$

which is implied by (25). Consequently the LHS equals the RHS of (29) for $X = EY$.

Finally we consider the case $X = AY$, where X is an (A, t)-compatible word (so Y is an $(A, t+1)$-compatible word). The proof is nearly the same as before. We have that the LHS of (29) equals:

$$\beta AYD_{0,j,k,\ell} - \delta AYE_{0,j-1,k,\ell}$$
$$= \sum_{u,v} A_{0,u,k,v}(\beta YD_{u,j,v,\ell} - \delta YE_{u,j-1,v,\ell})$$
$$= \sum_{u,v} A_{0,u,k,v}(\alpha\beta q^u Y_{u,j,v,\ell-1} - \gamma\delta q^{2(t+1)+v+u+\|Y\|}Y_{u,j-1,v,\ell-1} + (1-q)FY_{u,j,v,\ell-1})$$
$$= \sum_{u,v} \alpha\beta q^u A_{0,u,k,v}Y_{u,j,v,\ell-1} - \sum_{u,v} \gamma\delta q^{2t+2+v+u}A_{0,u,k,v}Y_{u+1,j,v,\ell-1} + \sum_{u,v}(1-q)AF_{0,u,k,v}Y_{u,j,v,\ell-1}.$$

In the second sum on the last line, we now change the index of summation, replacing each u by $u-1$. Thus we obtain that the LHS of (29) equals

$$\sum_{u,v}(\alpha\beta q^u A_{0,u,k,v} - \gamma\delta q^{2t+v+u+1}A_{0,u-1,k,v} + (1-q)AF_{0,u,k,v})Y_{u,j,v,\ell-1}.$$

We obtain the RHS of (29) for $X = AY$ in the same way:

$$\alpha\beta AY_{0,j,k,\ell-1} - \gamma\delta q^{2t+k+\|AY\|}AY_{0,j-1,k,\ell-1} + (1-q)FAY_{0,j,k,\ell-1}$$
$$= \sum_{u,v} A_{0,u,k,v}\left(\alpha\beta Y_{u,j,v,\ell-1} - \gamma\delta q^{2t+k+\|AY\|}Y_{u,j-1,v,\ell-1}\right) + \sum_{u,v}(1-q)FA_{0,u,k,v}Y_{u,j,v,\ell-1}$$
$$= \sum_{u,v}\left(\alpha\beta A_{0,u,k,v} - \gamma\delta q^{2t+k+\|Y\|+2-\|Y\|}A_{0,u-1,k,v} + (1-q)FA_{0,u,k,v}\right)Y_{u,j,v,\ell-1}.$$

Equation (27) of Proposition 7.16 implies that

$$\alpha\beta q^u A_{0,u,k,v} - \gamma\delta q^{2t+v+u+1}A_{0,u-1,k,v} + (1-q)AF_{0,u,k,v}$$
$$= \alpha\beta A_{0,u,k,v} - \gamma\delta q^{2t+k+2}A_{0,u-1,k,v} + (1-q)FA_{0,u,k,v},$$

which means the LHS equals the RHS of (29) for $X = AY$.

This completes the proof of (14).

7.6.2. *The inductive proof of* (15). We want to prove that for any X that is (A, t)-compatible,
(31)
$$XDA_{0,j,k,\ell} - qXAD_{0,j,k,\ell} = \alpha\beta XA_{0,j,k,\ell} - \gamma\delta q^{2t+k+2+\|X\|}XA_{0,j-1,k,\ell} + (1-q)FXA_{0,j,k,l}.$$

Note that by Lemma 7.3, (31) is equivalent to
(32)
$$XDA_{i,j,k,\ell} - qXAD_{i,j,k,\ell} = \alpha\beta q^i XA_{i,j,k,\ell} - \gamma\delta q^{2t+i+k+2+\|X\|}XA_{i,j-1,k,\ell} + (1-q)FXA_{i,j,k,l}.$$

We know that (31) holds for $|X| = 0$ by Lemma 7.15. For the inductive step, we will show that if (31) holds for $X = Y$, then (31) also holds for $X = AY$, $X = DY$, and $X = EY$.

When $X = AY$, the LHS of (31) becomes

$$AYDA_{0,j,k,\ell} - qAYAD_{0,j,k,\ell} = \sum_{u,v} A_{0,u,k,v}(YDA_{u,j,v,\ell} - qYAD_{u,j,v,\ell})$$

$$= \sum_{u,v} A_{0,u,k,v}q^{u(\|Y\|+3)}(YDA_{0,j-u,v,\ell} - qYAD_{0,j-u,v,\ell})$$

where we applied Lemma 7.3 to YDA and YAD to get the second equality from the first. Note that since $X = AY$ is (A, t)-compatible, Y must be $(A, t+1)$-compatible. We now use the induction hypothesis (32) to obtain that when $X = AY$, the LHS of (31) equals

$$= \sum_{u,v} A_{0,u,k,v}\left(\alpha\beta q^u YA_{u,j,v,\ell} - \gamma\delta q^{2(t+1)+u+v+2+\|Y\|}YA_{u,j-1,v,\ell} + (1-q)FYA_{u,j,v,\ell}\right)$$

$$= \sum_{u,v}\left(\alpha\beta q^u A_{0,u,k,v}YA_{u,j,v,\ell} - \gamma\delta q^{2t+4+u+v+\|Y\|}A_{0,u,k,v}YA_{u,j-1,v,\ell} + (1-q)AF_{0,u,k,v}YA_{u,j,v,\ell}\right)$$

$$= \sum_{u,v}\left(\alpha\beta q^u A_{0,u,k,v}YA_{u,j,v,\ell} - \gamma\delta q^{2t+4+u+v+\|Y\|-\|YA\|}A_{0,u,k,v}YA_{u+1,j,v,\ell}\right.$$

$$\left. + (1-q)AF_{0,u,k,v}YA_{u,j,v,\ell}\right)$$

$$= \sum_{u,v}\left(\alpha\beta q^u A_{0,u,k,v} - \gamma\delta q^{2t+u+v+1}A_{0,u-1,k,v} + (1-q)AF_{0,u,k,v}\right)YA_{u,j,v,\ell}.$$

To go from the first to the second line above, we used the fact that $\sum_{u,v} A_{0,u,k,v}FYA_{u,j,v,\ell} = \sum_{u,v} AF_{0,u,k,v}YA_{u,j,v,\ell}$. To go from the second to the third line, we used Lemma 7.3 in the second sum. And to go from the third to the fourth line, we changed the index of summation in the second term, replacing each u by $u - 1$.

Similarly, when $X = AY$, the RHS of (31) is

$$\alpha\beta AYA_{0,j,k,\ell} - \gamma\delta q^{2t+k+2+\|Y\|+2}AYA_{0,j-1,k,\ell} + (1-q)FAYA_{0,j,k,l}$$

$$= \sum_{u,v}\left(\alpha\beta A_{0,u,k,v}YA_{u,j,v,\ell} - \gamma\delta q^{2t+k+4+\|Y\|}A_{0,u,k,v}YA_{u,j-1,v,\ell} + (1-q)FA_{0,u,k,v}YA_{u,j,v,\ell}\right)$$

$$= \sum_{u,v}\left(\alpha\beta A_{0,u,k,v}YA_{u,j,v,\ell} - \gamma\delta q^{2t+k+4+\|Y\|-\|YA\|}A_{0,u,k,v}YA_{u+1,j,v,\ell} + (1-q)FA_{0,u,k,v}YA_{u,j,v,\ell}\right)$$

$$= \sum_{u,v}\left(\alpha\beta A_{0,u,k,v} - \gamma\delta q^{2t+k+2}A_{0,u-1,k,v} + (1-q)FA_{0,u,k,v}\right)YA_{u,j,v,\ell}.$$

Consequently, if we can show that

$$\alpha\beta q^u A_{0,u,k,v} - \gamma\delta q^{2t+u+v+1}A_{0,u-1,k,v} + (1-q)AF_{0,u,k,v} = \alpha\beta A_{0,u,k,v} - \gamma\delta q^{2t+k+2}A_{0,u-1,k,v} + (1-q)FA_{0,u,k,v},$$

then the LHS and RHS of (31) will be equal. But the equation above is exactly (27) from Proposition 7.16. Therefore if (31) is true for $X = Y$, it is also true for $X = AY$.

Let us now prove (31) for $X = DY$. The LHS of (31) is

$$DYDA_{0,j,k,\ell} - qDYAD_{0,j,k,\ell} = \sum_{u,v} D_{0,u,k,v}(YDA_{u,j,v,\ell} - qYAD_{u,j,v,\ell}).$$

Using the induction hypothesis and applying (32), we get

$$= \sum_{u,v} D_{0,u,k,v}\left(\alpha\beta q^u Y A_{u,j,v,\ell} - \gamma\delta q^{2t+v+u+2+\|Y\|} Y A_{u,j-1,v,\ell} + (1-q)FY A_{u,j,v,\ell}\right)$$

$$= \sum_{u,v} \left(\alpha\beta q^u D_{0,u,k,v} Y A_{u,j,v,\ell} - \gamma\delta q^{2t+u+v} D_{0,u,k,v} Y A_{u+1,j,v,\ell} + (1-q)DF_{0,u,k,v} Y A_{u,j,v,\ell}\right)$$

$$= \sum_{u,v} \left(\alpha\beta q^u D_{0,u,k,v} - \gamma\delta q^{2t+u-1+v} D_{0,u-1,k,v} + (1-q)DF_{0,u,k,v}\right) Y A_{u,j,v,\ell}.$$

When $X = DY$, the RHS of (31) is

$$\alpha\beta DY A_{0,j,k,\ell} - \gamma\delta q^{2t+k+2+\|Y\|+1} DY A_{0,j-1,k,\ell} + (1-q)FDY A_{0,j,k,\ell}$$

$$= \sum_{u,v} \left(\alpha\beta D_{0,u,k,v} Y A_{u,j,v,\ell} - \gamma\delta q^{2t+k+3+\|Y\|} D_{0,u,k,v} Y A_{u,j-1,v,\ell} + (1-q)FD_{0,u,k,v} Y A_{u,j,v,\ell}\right)$$

$$= \sum_{u,v} \left(\alpha\beta D_{0,u,k,v} Y A_{u,j,v,\ell} - \gamma\delta q^{2t+k+3+\|Y\|-\|YA\|} D_{0,u,k,v} Y A_{u+1,j,v,\ell} + (1-q)FD_{0,u,k,v} Y A_{u,j,v,\ell}\right)$$

$$= \sum_{u,v} \left(\alpha\beta D_{0,u,k,v} - \gamma\delta q^{2t+k+1} D_{0,u-1,k,v} + (1-q)FD_{0,u,k,v}\right) Y A_{u,j,v,\ell}.$$

Thus, if we can show that

$$\alpha\beta q^u D_{0,u,k,v} - \gamma\delta q^{2t+u+v-1} D_{0,u-1,k,v} + (1-q)DF_{0,u,k,v} = \alpha\beta D_{0,u,k,v} - \gamma\delta q^{2t+k+1} D_{0,u-1,k,v} + (1-q)FD_{0,u,k,v},$$

then it follows that for $X = DY$, the LHS of (31) equals the RHS. But the above equation is exactly (26) from Proposition 7.16. Therefore, if (31) is true for $X = Y$, it is also true for $X = DY$.

The proof for $X = EY$ follows the same steps as the proof for $X = DY$. This completes the proof of Theorem 7.10.

8. The proof of our formula for Koornwinder moments

Our goal in this section is to prove Theorem 4.5. Note that it is a direct consequence of Theorem 8.3 and Lemma 8.4 below.

Definition 8.1. Let $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{A}$ be matrices, and $\langle\mathcal{W}|$ and $|\mathcal{V}\rangle$ be vectors, such that the following relations hold:
   (1) $\langle\mathcal{W}|(\alpha\mathcal{E} - \gamma\mathcal{D}) = \langle\mathcal{W}|$
   (2) $(\beta\mathcal{D} - \delta\mathcal{E})|\mathcal{V}\rangle = |\mathcal{V}\rangle$
   (3) $\mathcal{D}\mathcal{E} - q\mathcal{E}\mathcal{D} = \mathcal{D} + \mathcal{E}$
   (4) $\mathcal{D}\mathcal{A} = q\mathcal{A}\mathcal{D} + \mathcal{A}$
   (5) $\mathcal{A}\mathcal{E} = q\mathcal{E}\mathcal{A} + \mathcal{A}$.
(By work of [USW04] and [Uch08] we know that such matrices and vectors exist; see also [CW15, Section 3.4].) We then define $\mathcal{Z}_{N,r}(\xi) = [y^r]\frac{\langle\mathcal{W}|(\xi\mathcal{D}+\mathcal{E}+y\mathcal{A})^N|\mathcal{V}\rangle}{\langle\mathcal{W}|\mathcal{A}^r|\mathcal{V}\rangle}$.

Remark 8.2. Note that if we have $\mathcal{D}, \mathcal{E}, \mathcal{A}, \langle\mathcal{W}|$, and $|\mathcal{V}\rangle$ as above, we can set $D^{(t)} = \mathcal{D}$ and $E^{(t)} = \mathcal{E}$ for all t, and set $A = \mathcal{A}$, $\langle W| = \langle\mathcal{W}|$, and $|V\rangle = |\mathcal{V}\rangle$. Then we have a solution to the relations (I) through (V) of Theorem 5.4, with $\lambda_n = 1$ for all n.

The main result of [CW15] was the following.

Theorem 8.3. [CW15, Theorem 1.1] *The Koornwinder moment* $K_{(N-r,0,0,\dots,0)}(\xi)$ *(where there are precisely r 0's in the partition) is proportional to the fugacity partition function $\mathcal{Z}_{N,r}(\xi)$ for the two-species ASEP on a lattice of N sites with r light particles. More specifically,*

$$K_{(N-r,0,0,\dots,0)}(\xi) = \frac{1}{(1-q)^r} \mathcal{Z}_{N,r}(\xi).$$

We now need to relate $\mathcal{Z}_{N,r}(\xi)$ to $\mathbf{Z}_{N,r}(\xi)$, the generating function for rhombic staircase tableaux.

Lemma 8.4. *Recall that by Theorem 7.1, the matrices $D^{(t)}, E^{(t)}, A$, and vectors $\langle W|$ and $|V\rangle$ satisfy relations (I) through (V) of Theorem 5.4, with $\lambda_n = \alpha\beta - q^{n-1}\gamma\delta$. Let X be an A-compatible word in $D^{(t)}, E^{(t)}$ and A, with $|X| = N$ and $|X|_A = r$. Let $\mathcal{X}$ be the corresponding word in $\mathcal{D}, \mathcal{E}$, and $\mathcal{A}$. For example, if $X = D^{(0)}AAE^{(2)}AD^{(3)}$, then $\mathcal{X} = \mathcal{D}AA\mathcal{E}A\mathcal{D}$. Then*

$$\frac{\langle \mathcal{W}|\mathcal{X}|\mathcal{V}\rangle}{\langle \mathcal{W}|\mathcal{A}^r|\mathcal{V}\rangle} = \frac{\langle W|X|V\rangle}{\langle W|A^r|V\rangle} \prod_{i=0}^{N-r-1} (\alpha\beta - q^{i+2r}\gamma\delta)^{-1}.$$

It follows that

$$\mathcal{Z}_{N,r}(\xi) = \mathbf{Z}_{N,r}(\xi) \prod_{i=0}^{N-r-1} (\alpha\beta - q^{i+2r}\gamma\delta)^{-1}.$$

Proof. We will prove the first statement of the lemma by induction on $N - r$, by comparing relations (1) through (5) to relations (I) through (V). Note that the base case, when $N - r = 0$, is trivial.

Now suppose that the lemma holds for all A-compatible word X, where $|X| - |X|_A = k - 1$ for some fixed positive k. We want to show that it holds for words X' where $|X'| - |X'|_A = k$.

Set $N = |X'|$ and $r = |X'|_A$. By Theorem 5.4, both $\langle W|X'|V\rangle$ and $\langle \mathcal{W}|\mathcal{X}'|\mathcal{V}\rangle$ compute (unnormalized) steady state probabilities of being in the same state τ, where $X' = X(\tau)$. Therefore $\langle W|X'|V\rangle = c_{N,r}\langle \mathcal{W}|\mathcal{X}'|\mathcal{V}\rangle$ for some constant $c_{N,r}$ that depends on N and r but not on X'. We want to show that $c_{N,r} = \frac{\langle W|A^r|V\rangle}{\langle \mathcal{W}|\mathcal{A}^r|\mathcal{V}\rangle} \prod_{i=0}^{N-r-1}(\alpha\beta - q^{i+2r}\gamma\delta)$.

Taking $X' = E^{(0)}X$ and $D^{(0)}X$, respectively, we have that

$$\alpha\langle W|E^{(0)}X|V\rangle - \gamma\langle W|D^{(0)}X|V\rangle = c_{N,r}\alpha\langle \mathcal{W}|\mathcal{E}\mathcal{X}|\mathcal{V}\rangle - c_{N,r}\gamma\langle \mathcal{W}|\mathcal{D}\mathcal{X}|\mathcal{V}\rangle$$
$$= c_{N,r}\langle \mathcal{W}|\mathcal{X}|\mathcal{V}\rangle.$$

But also by (III), we have that

$$\alpha\langle W|E^{(0)}X|V\rangle - \gamma\langle W|D^{(0)}X|V\rangle = \lambda_{N+r}\langle W|X|V\rangle = (\alpha\beta - q^{N+r-1}\gamma\delta)\langle W|X|V\rangle.$$

Therefore we have that

$$c_{N,r} = \frac{(\alpha\beta - q^{N+r-1}\gamma\delta)\langle W|X|V\rangle}{\langle \mathcal{W}|\mathcal{X}|\mathcal{V}\rangle},$$

which by the induction hypothesis is equal to $\frac{\langle W|A^r|V\rangle}{\langle \mathcal{W}|\mathcal{A}^r|\mathcal{V}\rangle} \prod_{i=0}^{N-r-1}(\alpha\beta - q^{i+2r}\gamma\delta)$, as desired.

To prove the second statement of the lemma, note that by Corollary 6.4, we have that $\mathbf{Z}_{N,r}(\xi) = \sum_X \xi^{|X|_D} \langle W|X|V \rangle$, where the sum is over all A-compatible words X such that $|X| = N$ and $|X|_A = r$. Also note that $\langle W|A^r|V \rangle = 1$, since there is only one degenerate tableau of size (r, r), and it has weight 1. The second statement of the lemma now follows from the first. $\qquad\square$

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