Several algebras associated to a (multi)graph
(joint with G. Nenashev, A, Postnikov, and M. Shapiro)

Boris Shapiro, Stockholm University

November 8, 2018
Basic algebra, $SL_n/B$

Trees counting algebras associated to directed graphs

Power algebras associated to undirected graphs

Other analogs

Topics to discuss

1. Basic algebra, $SL_n/B$

2. Trees counting algebras associated to directed graphs

3. Power algebras associated to undirected graphs

4. Other analogs
Main references


Interpret $SL_n/B = \mathcal{U}_n/T^n$ as the space of complete flags in $\mathbb{C}^n$ and take the standard sequence of tautological bundles

$$0 \subset E_1 \subset ... \subset E_n = E$$

(where $E$ is the trivial $\mathbb{C}^n$-bundle over $SL_n/B$) and the corresponding $n$-tuple of quotient line bundles $L_i = E_i/E_{i-1}$.

Fixing some Hermitian metric on the original $\mathbb{C}^n$ one equips every bundle $E_i$, $L_i$ and $E_i/E_j$, $i > j$ with the induced Hermitian metric.
Denote by $w_i$ the curvature form of the above Hermitian metric on $L_i$. (Each $w_i$ is a $\mathcal{U}_n$-invariant 2-form on $SL_n/B$ such that $\frac{\sqrt{-1}w_i}{2\pi}$ represents the first Chern class $c_1(L_i)$ in $H^2(SL_n/B)$.) Setting $x_i = c_1[L_i]$ one has

$$H^*(SL_n/B, \mathbb{Z}) = \frac{\mathbb{Z}[x_1, \ldots, x_n]}{(s_1, s_2, \ldots, s_n)},$$

where $s_i$ stands for the $i$th elementary symmetric functions in variables $x_1, \ldots, x_n$. 
Problem. Study the \( \mathbb{Z} \)-ring \( B_n = \mathbb{Z}(w_1, ..., w_n) \) generated by all \( w_i \)s and compare it to \( H^*(SL_n/B, \mathbb{Z}) \).

Remark. One has the standard surjective ring homomorphism
\[ \pi : B_n \rightarrow H^*(SL_n/B, \mathbb{Z}) \].
\[ B_n \text{ as a subalgebra of a square-free algebra} \]

Using results of Griffiths-Schmid (Acta Math., v.123, 1969) about the curvature forms on the homogeneous spaces, one can present \( w_i \)'s as follows.

**Example of \( B_4 \).**

\[
\begin{pmatrix}
  w_1 \\
  w_2 \\
  w_3 \\
  w_4
\end{pmatrix} = \begin{pmatrix}
  0 & +a & +b & +c \\
  -a & +0 & +d & +e \\
  -b & -d & +0 & +f \\
  -c & -e & -f & +0
\end{pmatrix},
\]

where \( a^2 = b^2 = c^2 = d^2 = e^2 = f^2 = 0 \) with no other relations. Then one has \( w_i^4 = 0; \) \( (w_i + w_j)^5 = 0; \) \( (w_i + w_j + w_k)^4 = 0; \) \( w_1 + w_2 + w_3 + w_4 = 0. \)
Simpler example. The ring $B_3$ is isomorphic to $\frac{\mathbb{Z}[w_1, w_2, w_3]}{I_3}$, where $I_3$ is generated by

$$w_1^3, w_2^3, w_3^3, (w_1 + w_2)^3, (w_1 + w_3)^3, (w_2 + w_3)^3, w_1 + w_2 + w_3.$$ 

The Hilbert polynomial of $B_3$ equals

$$H(t) = 1 + 2t + 3t^2 + t^3.$$ 

(For comparison, the Poincaré polynomial of $SL_3/B$ equals $1 + 2t + 2t^2 + t^3$.)
Proposition 1. \( \mathcal{B}_n \) is a graded ring isomorphic to \( \mathbb{Z}[w_1,...,w_n]/I_n \), where the ideal \( I_n \) is generated by the set of \( 2^n - 1 \) polynomials of the form
\[
g_{i_1,...,i_j}^{(n)} = (w_{i_1} + ... + w_{i_j})^{(n-j)+1},
\]
where \( \{i_1,...,i_j\} \) runs over the set of all nonempty subsets in the set \( \{1,...,n\} \).

Proposition 2. The total dimension of \( \mathcal{B}_n \) equals the number of forests on \( n \) labeled vertices and there exists a natural monomial basis for \( \mathcal{B}_n \) whose monomials are enumerated by the above forests.
Let $G$ be a digraph on the set of vertices $0, 1, \ldots, n$ (with possible multiple edges, but no loops). The vertex $0$ will be the root of $G$. The digraph $G$ is determined by its adjacency matrix $A = (a_{ij})_{0 \leq i,j \leq n}$, where $a_{ij}$ is the number of edges from the vertex $i$ to the vertex $j$. We will regard usual graphs as a special case of digraphs with symmetric adjacency matrix $A$.

An oriented spanning tree $T$ of the digraph $G$ is a subgraph $T \subset G$ such that there exists a unique directed path in $T$ from any vertex $i$ to the root $0$. The number $N_G$ of such trees is given by the Matrix-Tree Theorem:

$$N_G = \det L_G,$$

where $L_G = (l_{ij})_{1 \leq i,j \leq n}$ the truncated Laplace matrix,
$L_G$ is also known as the \textit{Kirkhoff matrix}, given by

$$l_{ij} = \begin{cases} 
\sum_{r \in \{0, \ldots, n\} \setminus \{i\}} a_{ir} & \text{for } i = j, \\
-a_{ij} & \text{for } i \neq j.
\end{cases} \quad (2)$$

If $G$ is a graph, i.e., $A$ is a symmetric matrix, then oriented spanning trees defined above are exactly the usual \textit{spanning trees} of $G$, which are connected subgraphs of $G$ without cycles.

For a subset $I$ in $\{1, \ldots, n\}$ and a vertex $i \in I$, let

$$d_I(i) = \sum_{j \not\in I} a_{ij},$$

i.e., $d_I(i)$ is the number of edges from the vertex $i$ to a vertex outside of the subset $I$. 
A parking function of size $n$ is a sequence $b = (b_1, \ldots, b_n)$ of non-negative integers such that its increasing rearrangement $c_1 \leq \cdots \leq c_n$ satisfies $c_i < i$. Equivalently, we can formulate this condition as $\# \{ i \mid b_i < r \} \geq r$, for $r = 1, \ldots, n$.

The parking functions of size $n$ are known to be in bijective correspondence with trees on $n + 1$ labelled vertices. Thus, according to Cayley’s formula for the number of labelled trees, the total number of parking functions of size $n$ equals $(n + 1)^{n-1}$.
Let us say that a sequence $b = (b_1, \ldots, b_n)$ of non-negative integers is a \textit{G-parking function} if, for any nonempty subset $I \subseteq \{1, \ldots, n\}$, there exists $i \in I$ such that $b_i < d_I(i)$.

If $G = K_{n+1}$ is the complete graph on $n + 1$ vertices then $K_{n+1}$-parking functions are the usual parking functions of size $n$ defined above.

**Theorem**

\textit{The number of G-parking functions equals the number $N_G = \det L_G$ of oriented spanning trees of the digraph $G$.}
We can reformulate the definition of $G$-parking functions in algebraic terms as follows. Throughout this paper we fix a field $K$. Let $\mathcal{I}_G = \langle m_I \rangle$ be the monomial ideal in the polynomial ring $K[x_1, \ldots, x_n]$ generated by the monomials

$$m_I = \prod_{i \in I} x_i^{d_I(i)},$$

where $I$ ranges over all nonempty subsets $I \subseteq \{1, \ldots, n\}$.

Define the algebra $A_G^T$ as the quotient

$$A_G^T = K[x_1, \ldots, x_n]/\mathcal{I}_G.$$
A integer sequence $b = (b_1, \ldots, b_n)$ is a $G$-parking function if and only if the monomial $x^b = x_1^{b_1} \cdots x_n^{b_n}$ is nonvanishing in the algebra $\mathcal{A}_G^T$.

For a monomial ideal $\mathcal{I}$, the set of all monomials that do not belong to $\mathcal{I}$ is a basis of the quotient of the polynomial ring modulo $\mathcal{I}$, called the *standard monomial basis*. Thus the monomials $x^b$, where $b$ ranges over $G$-parking functions, form the standard monomial basis of the algebra $\mathcal{A}_G$.

**Corollary**

$\mathcal{A}_G^T$ is a finite-dimensional linear space over $\mathbb{K}$. Its dimension is equal to the number of oriented spanning trees of the digraph $G$: $\dim \mathcal{A}_G^T = N_G$. 
Let $G$ be an undirected graph on the set of vertices $0, 1, \ldots, n$. In this case the dimension of the algebra $\mathcal{A}_G^T$ is equal to the number of usual spanning trees of $G$.

For a nonempty subset $I$ in $\{1, \ldots, n\}$, let $D_I = \sum_{i \in I, j \notin I} a_{ij} = \sum_{i \in I} d_i(i)$ be the total number of edges that join some vertex in $I$ with a vertex outside of $I$. For any nonempty subset $I \subseteq \{1, \ldots, n\}$, let

$$p_I = \left( \sum_{i \in I} x_i \right)^{D_I}.$$  \hfill (4)
Let $J_G = \langle p_I \rangle$ be the ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by the polynomials $p_I$ for all nonempty subsets $I$. Define the algebra $B^T_G$ as the quotient

$$B^T_G = \mathbb{K}[x_1, \ldots, x_n]/J_G.$$

The algebras $A^T_G$ and $B^T_G$ are graded. For a graded algebra $A^T = A^0 \oplus A^1 \oplus A^2 \oplus \cdots$, the Hilbert series of $A^T$ is the formal power series in $q$ given by

$$\text{Hilb } A^T = \sum_{k \geq 0} q^k \dim A^k.$$
The monomials $x^b$, where $b$ ranges over $G$-parking functions, form a linear basis of the algebra $B_G^T$. Thus the Hilbert series of the algebras $A_G^T$ and $B_G^T$ coincide termwise: $\text{Hilb } A_G^T = \text{Hilb } B_G^T$. In particular, both these algebras are finite-dimensional as linear spaces over $\mathbb{K}$ and

$$\dim A_G^T = \dim B_G^T = N_G$$

is the number of spanning trees of the graph $G$. 
Example

Let $n = 3$ and let $G$ be the graph given by

$$G = \begin{array}{c}
1 \\
2 \\
3 \\
\end{array} \begin{array}{c}
0 \\
\end{array}.$$

Figure: Example of a graph.

The graph $G$ has 8 spanning trees:
The ideals $\mathcal{I}_G$ and $\mathcal{J}_G$ are given by

$$\mathcal{I}_G = \langle x_1^3, x_2^2, x_3^3, x_1^2 x_2, x_1^2 x_3^2, x_2 x_3^2, x_1 x_2^0 x_3 \rangle,$$

$$\mathcal{J}_G = \langle x_1^3, x_2^2, x_3^3, (x_1 + x_2)^3, (x_1 + x_3)^4, (x_2 + x_3)^3, (x_1 + x_2 + x_3)^2 \rangle.$$

The standard monomial basis of the algebra $A_G^T$ is

$$\{1, x_1, x_2, x_3, x_1^2, x_1 x_2, x_2 x_3, x_3^2\}.$$ The corresponding $G$-parking functions are the exponent vectors of the basis elements:

$$(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (2, 0, 0),$$

$$(1, 1, 0), (0, 1, 1), (0, 0, 2).$$

We have $\dim A_G^T = \dim B_G^T = 8$ is the number of spanning trees of $G$, and $\text{Hilb } A_G^T = \text{Hilb } B_G^T = 1 + 3q + 4q^2$. 
We will refine Theorem 3 and interpret dimensions of graded components of the algebras $A^T_G$ and $B^T_G$ in terms of certain statistics on spanning trees. Let us fix a linear ordering of all edges of the graph $G$.

For a spanning tree $T$ of $G$, an edge $e \in G \setminus T$ is called *externally active* if there exists a cycle $C$ in the graph $G$ such that $e$ is the minimal edge of $C$ and $(C \setminus \{e\}) \subset T$. The *external activity* of a spanning tree is the number of externally active edges. Let $N^k_G$ denote the number of spanning trees $T \subset G$ of external activity $k$. Even though the notion of external activity depends on a particular choice of ordering of edges, the numbers $N^k_G$ are known to be invariant on the choice of ordering.
Let $\mathcal{A}_G^k$ and $\mathcal{B}_G^k$ be the $k$-th graded components of the algebras $\mathcal{A}_G^T$ and $\mathcal{B}_G^T$, correspondingly.

**Theorem**

*The dimensions of the $k$-th graded components $\mathcal{A}_G^k$ and $\mathcal{B}_G^k$ are equal to*

$$\dim \mathcal{A}_G^k = \dim \mathcal{B}_G^k = N_G^{\left|G\right|-n-k},$$

*the number of spanning trees of $G$ of external activity $\left|G\right|-n-k$, where $\left|G\right|$ denotes the number of edges of $G$.*
We introduce the following algebra $\mathcal{C}_G^F$ associated to an arbitrary vertex-labeled undirected graph $G$ without loops on the vertex set $[n]$. Let $\Phi_G$ be the graded commutative algebra over $\mathbb{K}$ generated by the variables $\phi_e, e \in G$, with the defining relations: $(\phi_e)^2 = 0$, for every edge $e \in G$. Let $\mathcal{C}_G^F$ be the subalgebra of $\Phi_G$ generated by the elements

$$X_i = \sum_{e \in G} c_{i,e} \phi_e,$$

for $i \in [n]$, where

$$c_{i,e} = \begin{cases} 
1 & \text{if } e = (i, j), \ i < j; \\
-1 & \text{if } e = (i, j), \ i > j; \\
0 & \text{otherwise.}
\end{cases} \quad (5)$$
Observe that we assume that $c^F_G$ contains 1.

To describe the relations between $X_i$, consider the ideal $J_G$ in the ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by

$$p_I = \left( \sum_{i \in I} x_i \right)^{d_I + 1},$$

where $I$ ranges over all nonempty subsets of vertices, and $d_I$ is the total number of edges between vertices in $I$ and vertices outside $I$, i.e., belonging to $V(G) \setminus I$. Define the algebra $B^F_G$ as the quotient

$$\mathbb{K}[x_1, \ldots, x_n]/J_G.$$
Theorem

For any graph $G$, the algebras $B^F_G$ and $C^F_G$ are isomorphic, their total dimension over $\mathbb{K}$ is equal to the number of spanning forests in $G$.

Moreover, the dimension of the $k$-th graded component of these algebras equals the number of spanning forests $F$ of $G$ with external activity $e(G) - e(F) - k$. 
In particular, the Hilbert polynomial of $C^F_G$ is a specialization of the Tutte polynomial of $G$.

**Corollary**

*Given a graph $G$, the Hilbert polynomial $\mathcal{H}_{C^F_G}(t)$ of the algebra $C^F_G$ is given by*

$$\mathcal{H}_{C^F_G}(t) = T_G \left(1 + t, \frac{1}{t} \right) \cdot t^{e(G) - v(G) + c(G)}.$$  

**Theorem (G. Nenashev)**

*Given two graphs $G_1$ and $G_2$, the algebras $C^F_{G_1}$ and $C^F_{G_2}$ are isomorphic if and only if the graphical matroids of $G_1$ and $G_2$ coincide.*
"K-theoretical" analog

In the above notation, our main object here will be the filtered subalgebra \( \mathcal{K}_G \subset \Phi_G \) defined by the generators:

\[
Y_i = \exp(X_i) = \prod_{e \in G} (1 + c_{i,e} \phi_e), \quad i = 0, \ldots, n.
\]

**Remark**

Since \( Y_i \) is obtained by exponentiation of \( X_i \), we call \( \mathcal{K}_G \) the “K-theoretic” analog of \( C^F_G \). The original generators \( X_i \) are similar to the first Chern classes, while their exponentiations \( Y_i \) are similar to the Chern characters which are the main object of \( K \)-theory.
Define the ideal $\mathcal{I}_G$ in $\mathbb{K}[y_0, y_1, \ldots, y_n]$ as generated by the polynomials

$$q_I = \left( \prod_{i \in I} y_i - 1 \right)^{D_I + 1},$$

(6)

where $I$ ranges over all nonempty subsets in $\{0, 1, \ldots, n\}$ and the number $D_I$ is the number of edges connecting the subset $I$ of vertices with its complement. Set

$$\mathcal{D}_G := \mathbb{K}[y_0, \ldots, y_n]/\mathcal{I}_G.$$
Theorem

For any graph $G$, algebras $\mathcal{B}_G^F$, $\mathcal{C}_G^F$, $\mathcal{D}_G$, and $\mathcal{K}_G$ are isomorphic as (non-filtered) algebras.

Moreover, the following stronger statement holds.

Theorem

For any graph $G$, algebras $\mathcal{D}_G$ and $\mathcal{K}_G$ are isomorphic as filtered algebras.
The filtered algebras $\mathcal{D}_G$ and $\mathcal{K}_G$ contain complete information about $G$.

**Theorem**

*Given two graphs $G_1$ and $G_2$ without isolated vertices, $\mathcal{K}_{G_1}$ and $\mathcal{K}_{G_2}$ are isomorphic as filtered algebras if and only if $G_1$ and $G_2$ are isomorphic.*
Further generalizations

Now we consider the Hilbert series of other filtered algebras similar to $\mathcal{K}_G$. (Recall that the Hilbert series of a filtered algebra is, by definition, the Hilbert series of its associated graded algebra.)

Let $f$ be a univariate polynomial or a formal power series over $\mathbb{K}$. We define the subalgebra $\mathcal{F}[f]_G \subset \Phi_G$ as generated by 1 together with

\[
f(X_i) = f \left( \sum_{j} c_{i,j} e^{\phi_e} \right), \quad i = 0, \ldots, n.
\]

**Example**

For $f(x) = x$, $\mathcal{F}[f]_G$ coincides with $\mathcal{C}_G^F$. For $f(x) = \exp(x)$, $\mathcal{F}[f]_G$ coincides with $\mathcal{K}_G$. 
Obviously, the filtered algebra $\mathcal{F}[f]_G$ does not depend on the constant term of $f$. From now on, we assume that $f(x)$ has no constant term, since for any $g$ such that $f - g$ is constant, the filtered algebras $\mathcal{F}[f]_G$ and $\mathcal{F}[g]_G$ are the same.

**Proposition**

*Let $f$ be any polynomial with a non-vanishing linear term. Then the algebras $C^F_G$ and $\mathcal{F}[f]_G$ coincide as subalgebras of $\Phi_G$.***

**Theorem**

*Let $f$ be any polynomial with non-vanishing linear and quadratic terms. Then given two simple graphs $G_1$ and $G_2$ without isolated vertices, $\mathcal{F}[f]_{G_1}$ and $\mathcal{F}[f]_{G_2}$ are isomorphic as filtered algebras if and only if $G_1$ and $G_2$ are isomorphic graphs.*
Since \( X_i^{d_i+1} = 0 \) for any \( i \), we can always truncate any polynomial (or a formal power series) \( f \) at degree \( |G| + 1 \) without changing \( \mathcal{F}[f]_G \). Therefore, for a given graph \( G \), it suffices to consider \( f \) as a polynomial of degrees less than or equal to \( |G| \). To simplify our notation, let us write \( HS_{f,G} \) instead of \( HS_{\mathcal{F}[f]_G} \).

Given a graph \( G \), consider the space of polynomials of degree less than or equal to \( |G| \) and the corresponding Hilbert series.
Proposition

In the above notation, for generic polynomials $f$ of degree at most $|G|$, the Hilbert series $HS_{f,G}$ is the same. This generic Hilbert series (denoted by $HS_G$ below) is maximal in the majorization partial order among all $HS_{g,G}$, where $g$ runs over the set of all formal power series with non-vanishing linear term.

Here (as usual) by generic polynomials of degree at most $|G|$ we mean polynomials belonging to some Zariski open subset in the linear space of all polynomials of degree at most $|G|$.
Recall that, by definition, a sequence \((a_0, a_1, \ldots)\) is \textit{bigger} than \((b_0, b_1, \ldots)\) in the majorization partial order if and only if, for any \(k \geq 0\),

\[
\sum_{i=0}^{k} a_i \geq \sum_{i=0}^{k} b_i.
\]

**Remark**

We know that the Hilbert series of the graded algebra \(C^F_G\) is a specialization of the Tutte polynomial of \(G\). However we can not calculate the Hilbert series of \(K_G\) from the Tutte polynomial of \(G\), because there exists a pair of graphs \((G, G')\) with the same Tutte polynomial and different \(HS_{K_G}\) and \(HS_{K_G'}\), see example on next page.

Additionally, notice that, in general, \(HS_{\exp,G} := HS_{K_G} \neq HS_G\).
Basic algebra, $SL_n/B$
Trees counting algebras associated to directed graphs
Power algebras associated to undirected graphs
Other analogs

Figure: Graphs with the same matroid and different “K-theoretic" and generic Hilbert series.

$G_1$ and $G_2$ have isomorphic matroids and hence, the same Tutte polynomial. Therefore, the Hilbert series of $C^{F}_{G_1}$ and $C^{F}_{G_2}$ coincide. Namely,

$$HS_{C^{F}_{G_1}}(t) = HS_{C^{F}_{G_2}}(t) = 1 + 3t + 6t^2 + 9t^3 + 8t^4 + 4t^5 + t^6.$$
However, the Hilbert series of their “K-theoretic" algebras are distinct. Namely

\[ HS_{K_{G_1}}(t) = 1 + 4t + 10t^2 + 14t^3 + 3t^4, \]

\[ HS_{K_{G_2}}(t) = 1 + 4t + 10t^2 + 15t^3 + 2t^4. \]

Moreover their generic Hilbert series are also distinct and different from their “K-theoretic" Hilbert series. Namely,

\[ HS_{G_1}(t) = 1 + 4t + 10t^2 + 15t^3 + 2t^4, \]

\[ HS_{G_2}(t) = 1 + 4t + 10t^2 + 16t^3 + t^4. \]

Putting our information together we get,

\[ HS_{C^F_{G_1}} = HS_{C^F_{G_2}} \prec HS_{K_{G_1}} \prec HS_{K_{G_2}} = HS_{G_1} \prec HS_{G_2}, \]

where \( \prec \) denotes the majorization partial order.
Q-deformations of Kirillov-Nenashev

Let us define a family of $Q$-deformations of $C^F(G)$ as follows.

For a graph $G$ and parameters $Q = \{q_e \in \mathbb{K} : e \in E(G)\}$, define $\Phi_{G,Q}$ as the commutative algebra generated by the variables $\{u_e : e \in E(G)\}$ satisfying

$$u_e^2 = q_e u_e, \text{ for every edge } e \in G.$$ 

Let $V(G) = [n]$ be the vertex set of a graph $G$. Define the $Q$-deformation $\Psi_{G,Q}$ of $C^F_G$ as the filtered subalgebra of $\Phi_{G,Q}$ generated by the elements:

$$X_i = \sum_{e: i \in e} c_{i,e} u_e, \ i \in [n],$$

where $c_{i,e}$ are the same as always.
The filtered structure on $\Psi_{G,Q}$ is induced by the elements $X_i$, $i \in [n]$. More concrete, the filtered structure is an increasing sequence

$$\mathcal{K} = F_0 \subset F_1 \subset F_2 \ldots \subset F_m = \Psi_{G,Q}$$

of subspaces of $\Psi_{G,Q}$, where $F_k$ is the linear span of all monomials $X_1^{\alpha_1}X_2^{\alpha_2} \cdots X_n^{\alpha_n}$ such that $\alpha_1 + \ldots + \alpha_n \leq k$. Note that algebra $\Phi_{G,Q}$ has a finite dimension, then $\Psi_{G,Q}$ has a finite dimension, which gives that the increasing sequence of subspaces is finite. The Hilbert polynomial of a filtered algebra is the Hilbert polynomial of the associated graded algebra, it has the following formula

$$\mathcal{H}(t) = 1 + \sum_{i=1}^{\infty} (\dim(F_i) - \dim(F_{i-1})) t^i.$$
In case when all parameters coincide, i.e., $q_e = q$, $\forall e \in G$, we denote the corresponding algebras by $\Psi_{G,q}$ and $\Phi_{G,q}$ resp. We refer to $\Psi_{G,q}$ as the Hecke deformation of $C^F_G$.

(i) By definition, the algebra $\Psi_{G,0}$ coincides with $C^F_G$.

(ii) If we change the signs of $q_e$, $e \in E'$ for some subset $E' \subseteq E$ of edges, we obtain an isomorphic algebra.

(iii) It is possible to write relations such as $u_e^2 = \beta_e$ or $u_e^2 = q_e u_e + \beta_e$ where $\beta_e \in \mathbb{K}$.
Example 1. Let $G$ be a graph with two vertices, a pair of (multiple) edges $a$, $b$. Consider the Hecke deformation of its $C^F_G$, i.e., satisfying $q_a = q_b = q$.

The generators are $X_1 = a + b$, $X_2 = -(a + b) = -X_1$. One can easily check that the filtered structure is given by

$$F_0 = \langle 1 \rangle; \quad F_1 = \langle 1, a + b \rangle; \quad F_2 = \langle 1, a + b, ab \rangle.$$

The Hilbert polynomial $\mathcal{H}(t)$ of $\psi_{G,q}$ is given by

$$\mathcal{H}(t) = 1 + t + t^2.$$

The defining relation for $X_1$ is given by

$$X_1(X_1 - q)(X_1 - 2q) = 0.$$
Example 2. For the same graph as before, consider the case when \( Q = \{q_a, q_b\}, \ q_a^2 \neq q_b^2 \).

The generators are the same: \( X_1 = a + b \), \( X_2 = -(a + b) = -X_1 \). Since

\[
X_1^3 = q_a^2 a + q_b^2 b + 3(q_a + q_b)ab = \frac{3(q_a + q_b)}{2} X_1^2 - \frac{q_a^2 + 3q_b^2}{2} a - \frac{3q_a^2 + q_b^2}{2} X_1 + (q_a^2 - q_b^2) a,
\]

we have

\[
F_0 = \langle 1 \rangle; \ F_1 = \langle 1, a+b \rangle; \ F_2 = \langle 1, a+b, q_a a + q_b b + 2ab \rangle; \ F_3 = \langle 1, a+b, q_a a + q_b b + 2ab, (q_a - q_b) a \rangle.
\]

The Hilbert polynomial \( \mathcal{H}(t) \) of \( \Psi_{G,Q} \) is given by

\[
\mathcal{H}(t) = 1 + t + t^2 + t^3.
\]
Observe that in this case the algebra $\Psi_{G,Q}$ coincides with the whole $\Phi_{G,Q}$ as a linear space, but has a different filtration. The defining relation for $X_1$ is given by

$$X_1(X_1 - q_a)(X_1 - q_b)(X_1 - q_a - q_b) = 0.$$  

**Theorem**

For any loopless graph $G$, filtrations of its Hecke deformation $\Psi_{G,q}$ induced by $X_i$ and induced by the algebra $\Phi_{G,q}$ coincide. Furthermore, the Hilbert polynomial $\mathcal{H}_{\Psi_{G,q}}(t)$ of this filtration is given by

$$\mathcal{H}_{\Psi_{G,q}}(t) = T_G \left( 1 + t, \frac{1}{t} \right) \cdot t^{e(G) - v(G) + c(G)},$$

i.e., it coincides with that of $C_F^G$.  

Boris Shapiro, Stockholm University  
Several algebras associated to a (multi)graph
The latter result implies that cases when not all $q_e$ are equal are more interesting than the case of the Hecke deformation. Let us consider weighted graphs, i.e. when each edge $e$ has non-zero $q_e \in \mathbb{K}$, and will simply denote the algebra for a weighted graph $G$ by $\Psi_G$.

**Definition**

For a loopless weighted graph $G$ on $n$ vertices and an orientation $\vec{G}$, define the score vector $D^+_G \in \mathbb{K}^n$ as follows

\[
\left( \sum_{e \in E: \text{end}(\vec{e})=1} q_e, \sum_{e \in E: \text{end}(\vec{e})=2} q_e, \ldots, \sum_{e \in E: \text{end}(\vec{e})=n} q_e \right),
\]

where $\text{end}(\vec{e})$ is the final vertex of oriented edge $\vec{e}$. 
Theorem

For any loopless weighted graph $G$, the dimension of the algebra $\Psi_G$ is equal to the number of distinct score vectors, i.e.

$$\dim(\Psi_G) = \# \{ D \in \mathbb{K}^n : \exists \tilde{G} \text{ such that } D = D^+_G \}.$$

As a consequence of the above theorem, we obtain the following known property.

Corollary

For any graph $G$, the number of its spanning forests is equal to the number of distinct vectors of incoming degrees corresponding to its orientations.
Open problems

1. Is it true that if $HS_{f,G_1} = HS_{f,G_2}$ for any function/polynomial $f$, then the graphs $G_1$ and $G_2$ are isomorphic?
2. One can use the formulas for the curvature forms of all Chern classes for $E_i/E_j$ by P. Griffiths and W. Schmid (Acta Math., v.123, 1969) and ask the following.

**Problem.** For a given $SL_n/P$ study the corresponding algebra $B_P$ generated by its curvature forms. In particular, what is the total dimension of $B_P$ as a vector space? What about its Hilbert series?
Postnikov’s conjecture

Team score sequences

The complete multipartite graph $K_{\bar{n}} = K_{n_1,...,n_k}$ is the graph on vertices $1, ..., n$ with edges $\{i, j\}$, for any $i \in I_a$ and $j \in I_b$ with $a < b$. An orientation of $K_{\bar{n}}$ is a directed graph obtained by orienting each edge of $K_{\bar{n}}$. An orientation is called acyclic if it has no directed cycles. Let us define a weaker notion of semi-acyclic orientations.

Definition. Let us say that a directed cycle $C$ in the multipartite graph $K_{\bar{n}}$ is bipartite if $C$ contains only vertices from $I_a \cup I_b$ for some pair $a, b$. Let us say that an orientation of $K_{\bar{n}}$ is semi-acyclic if it has no bipartite directed cycles.
We can think of an orientation of $K_{\bar{n}}$ as a tournament between $k$ teams with $n_1, \ldots, n_k$ players where each player of each team plays a game with each player of any other team and either wins or looses. If an edge $(i, j)$ in a orientation of $K_{\bar{n}}$ is directed from $i$ to $j$, then the player $i$ wins and the player $j$ looses in the corresponding tournament. The individual score of player $i$ is the number of games the player wins, that is, the individual score of $i$ is the outdegree of vertex $i$ of $K_{\bar{n}}$ in the orientation.

The **team score** of team $l_a$ is the partition

$$\lambda^{(a)} = (\lambda_1^{(a)} \geq \lambda_2^{(a)} \geq \cdots \geq \lambda_{n_a}^{(a)} \geq 0),$$

whose parts $\lambda_i^{(a)}$ are the individual scores of players from $l_a$ arranged in the decreasing order.
Definition. The team score sequence of an orientation of $K_{\bar{n}}$ is the sequence $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(k)})$ of partitions $\lambda^{(a)} = (\lambda_1^{(a)}, \ldots, \lambda_{n_a}^{(a)})$ whose parts $\lambda_i^{(a)}$ are the outdegrees of vertices $j \in I_a$ in the orientation arranged in the decreasing order.

Theorem. The number of team score sequences of acyclic orientations of $K_{\bar{n}}$ equals the multinomial coefficient \( \frac{n!}{n_1! \ldots n_k!} \), which is equal to the dimension of the cohomology ring $H^* (Fl(\bar{n}))$.

Conjecture. The dimension of the algebra of Chern forms $C^* (Fl(\bar{n}))$ equals the number of team score sequences of semi-acyclic orientations of $K_{\bar{n}}$. 
Thank you for your patience